18.937 TOPICS IN GEOMETRY TOPOLOGY: INSTANTON FLOER HOMOLOGY

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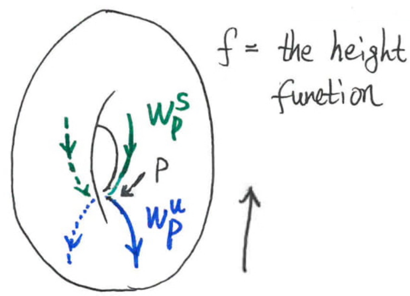
Contents

Lecture 1. An Introduction to Morse Homology

The first a few lectures will be a brief introduction to the Morse theory on finite dimensional (compact) manifolds. This lecture will be an overview. Later, we will generalize this theory to an infinite dimensional setting.



1.0.1. **Flowlines.** Let (X, g) be a smooth Riemannian manifold and $f : X \to \mathbb{R}$ be a smooth Morse function. Let $\mathrm{Crit}(f) = \{p \in X : \nabla_x f = 0\}$ be the set of critical points of f. Then the Hessian of f, defined by $\mathrm{Hess}_p(f) : T_pX \to T_pX, v \mapsto \nabla_v(\nabla f)$, is non-degenerate for any $p \in \mathrm{Crit}(f)$. We investigate negative gradient flows of f:

$$\gamma : \mathbb{R} \to X, \dot{\gamma}(t) = -\nabla_{\gamma(t)} f.$$

Given $p \in \mathrm{Crit}(f)$, let

$$W_p^u = \{x \in X : \exists \gamma, \gamma(0) = x, \dot{\gamma}(t) = -\nabla_{\gamma(t)} f, \lim_{t\to-\infty} \gamma(t) = p\}$$
$$W_p^s = \{x \in X : \exists \gamma, \gamma(0) = x, \dot{\gamma}(t) = -\nabla_{\gamma(t)} f, \lim_{t\to\infty} \gamma(t) = p\}$$

be the stable and unstable sub-manifolds associated to p. The Morse index $:= \mu(p)$ is the number of negative eigenvalues of $\mathrm{Hess}_p(f)$.

For any $p, q \in \mathrm{Crit}(f)$, let $M(p, q) := W_p^u \cap W_q^s$. Then each point of $M(p, q)$ is contained in a flowline connecting p to q:

$$\{x \in X : \exists \gamma, \gamma(0) = x, \lim_{t\to-\infty} \gamma(t) = p, \lim_{t\to\infty} \gamma(t) = q\}.$$

If W_p^u and W_q^s intersect transversely, then $\dim M(p, q) = \mu(p) - \mu(q)$. For any negative flowline γ, we have an energy equation. Indeed, for any $t_1 < t_2$,

$$\begin{aligned} f(\gamma(t_1)) - f(\gamma(t_2)) &= -\int_{t_1}^{t_2} \frac{d}{dt} f(\gamma(t))dt = -\int_{t_1}^{t_2} \langle \nabla_\gamma f, \dot{\gamma} \rangle \\ &= \frac{1}{2}\int_{t_1}^{t_2} |\dot{\gamma}(t)|^2 + |\nabla_\gamma f|^2 - |\dot{\gamma} + \nabla_\gamma f|^2 \\ &= \frac{1}{2}\int_{t_1}^{t_2} |\dot{\gamma}(t)|^2 + |\nabla_\gamma f|^2. \end{aligned}$$

For any flowline γ in $M(p,q)$, the total drop of the Morse function f equals

$$\lim_{t\to-\infty} f(\gamma(t)) - \lim_{t\to\infty} f(\gamma(t)) = f(p) - f(q)$$

and it is independent of the particular flowline. For each n, let $I_n = [n-1, n+1]$. Then for any $\epsilon > 0$, there are finitely many intervals I_n such that

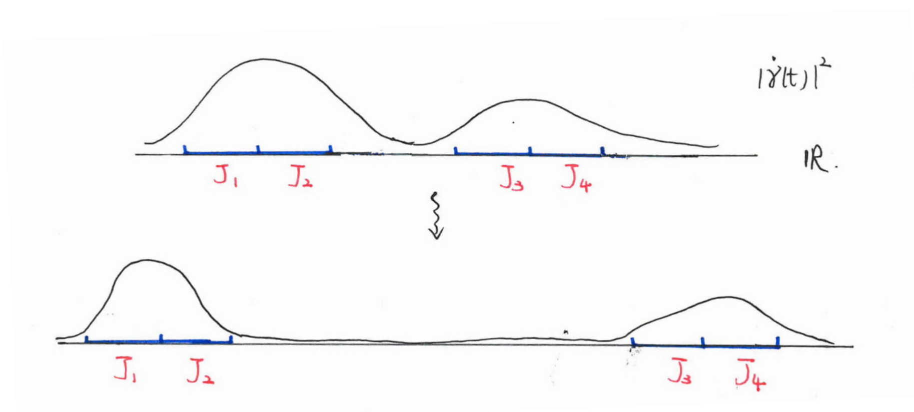
$$\int_{I_n} |\dot{\gamma}(t)|^2 + |\nabla_\gamma f|^2 > \epsilon \tag{1}$$

The number m of such intervals is bounded above by $(f(p) - f(q))/\epsilon$. We choose ϵ small so that when (1) is violated for I_n, then the flow $\gamma|_{I_n}$ is contained in some small neighborhood of $\mathrm{Crit}(f)$.

Now we investigate the compactification of $M(p,q)$ and suppose a sequence of flowlines $\gamma_k \in M(p,q)$ is given. For each γ_k, suppose the property (1) is satisfied for

$$J_i^k = I_{n_i}^k, 1 \leqslant i \leqslant m_k, n_i < n_{i+1}.$$

Since m_k is bounded uniformly, by taking a subsequence, we may assume m_k is the same for each k. There are two potential scenarios:



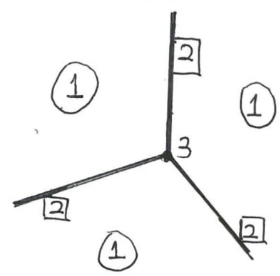
(1) The distance between J_i^k and J_{i+1}^k remains bounded when $k \to \infty$ (for a subsequence). Then we combine J_i^k, J_{i+1}^k and intervals in between.
(2) The distance between J_i^k and J_{i+1}^k diverge to infinity when $k \to \infty$. Then we would have a broken trajectory.

An $\mathbb{R}$ action is defined by $M(p,q)$ by translations of the parameter. Let $\check{M}(p,q) = M(p,q)/\mathbb{R}$ be the quotient space. Then a compactification of $\check{M}(p,q)$ is given by incorporating broken flowlines:

$$\begin{aligned}
\overline{\check{M}}(p,q) = \check{M}(p,q) & \qquad \dim = \mu(p) - \mu(q) - 1 \\
\cup\, \check{M}(p,q_1) \times \check{M}(q_1,q) & \qquad \dim = \mu(p) - \mu(q) - 2 \\
\cup\, \check{M}(p,q_1) \times \check{M}(q_1,q_2) \times \check{M}(q_2,q) & \qquad \dim = \mu(p) - \mu(q) - 3 \\
\cdots \text{ lower dimensional strata} &
\end{aligned}$$

Theorem 1.0.1. *For generic (g, f), $\overline{\check{M}}(p, q)$ is compact and is a compactification of $\check{M}(p, q)$.*

Remark. In the literature, we usually call $\overline{\check{M}}(p, q)$ a manifold with corners. But we do not have a canonical way to produce such an structure. The space $\overline{\check{M}}(p, q)$ could be realized as a topological manifold with corners (depicted in the picture on the right with numbers of breaks indicated). To endow a smooth structure, we could embed $\overline{\check{M}}(p, q)$ into some Euclidean space by choosing some regular values a_i of f and thinking about how $\overline{\check{M}}(p, q)$ intersects with each level set $f^{-1}(a_i)$. However, under this embedding, this corner structure has unpleasant looking. In practice, it is enough to think of $\overline{\check{M}}(p, q)$ as a stratified space and the corner structure is not really useful.



1.0.2. **The Morse Homology.** When $\mu(p) - \mu(q) = 1$, $\overline{\check{M}}(p, q)$ is a compact 0-dim manifold, i.e. a collection of finite points. Let

$$C = C(X, g, f) = \bigoplus_{p \in \mathrm{Crit}(f)} \mathbb{Z}_2\langle p\rangle$$

$$\partial : C \to C,\ p \mapsto \sum_{q|\mu(p)-\mu(q)=1} \#\overline{\check{M}}(p, q) \cdot q$$

When $\mu(p) - \mu(q) = 2$, $\dim \overline{\check{M}}(p, q) = 1$ and its boundary is described as

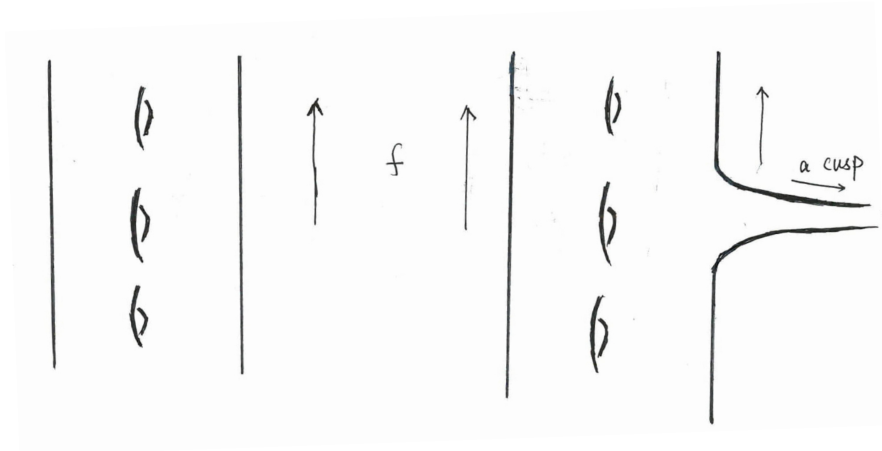
$$\partial\overline{\check{M}}(p, q) = \cup_{q_1} \check{M}(p, q_1) \times \check{M}(q_1, q).$$

which has 0 (mod 2) points. Therefore,

$$\partial^2 p = \sum_{\substack{q,r \\ \mu(p)-\mu(q)=1 \\ \mu(q)-\mu(r)=1}} \#\check{M}(p, q)\#\check{M}(q, r) \cdot r = 0.$$

Finally, the Morse homology is defined by $H(X, f, g) = \mathrm{Ker}(\partial)/\mathrm{Im}(\partial)$. If X is compact, then by the standard argument $H(X, f, g) \cong H_*(X; \mathbb{Z}_2)$ since f induces a CW structure on X and the chain complex (C, ∂) coincides with the CW complex that computes $H_*(X; \mathbb{Z}_2)$.

Remark. When X is non-compact, the way how the function f sees the non-compactness of X is important. To make the whole construction work, we need certain properness: if $\nabla_{x_i} f \to 0$ and $|f(x_i)|$ is uniformly bounded, then $\{x_i\}$ has a converging subsequence. In the picture below, the surface on the right contains infinite holes and a cusp. The hypothesis is violated for the height function because of the cusp.

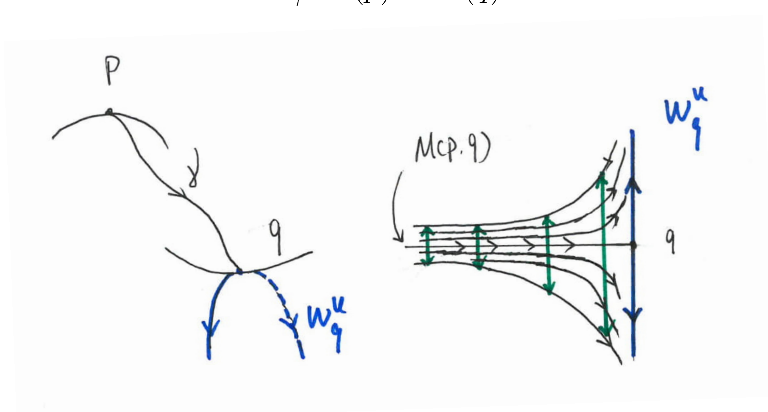


1.0.3. **Orientations.** If $W_p^u \cap W_q^s = M(p,q)$, the flow will stretch the normal bundle of $M(p,q)$ inside W_p^u. Allowing $t \to \infty$, the normal bundle is identified with $T_q W_q^u$. Thus, for any $x \in M(p,q)$, we have a short exact sequence

$$0 \to T_x M(p,q) \to T_x W_p^u \to T_q W_q^u \to 0$$

When $\mu(p) - \mu(q) = 1$, $M(p,q)$ consists of finitely many copies of $\mathbb{R}$ and the flow orients $M(p,q)$ in a natural way. For any $\gamma \in \check{M}(p,q)$, the short exact sequence induces an identification:

$$\epsilon_\gamma : \Lambda(p) \to \Lambda(q)$$



where $\Lambda(p)$ is the 2-element set of orientations of $T_p W_p^u$. Now we can refine our differential by setting $\mathbb{Z}\Lambda(p) = \mathbb{Z} \times_{\mathbb{Z}/2\mathbb{Z}} \Lambda(p)$ and

$$C(X,g,f) = \bigoplus_{p \in \mathrm{Crit}(f)} \mathbb{Z}\Lambda(p),$$

$$\partial = \sum_{q \mid \mu(p)-\mu(q)=1} \sum_{\gamma \in \check{M}(p,q)} \epsilon_\gamma.$$

If we ignore the orientation, it reduces to the previous case.

Lecture 2. The Stable Manifold Theory

We first recall some notations from the last lecture. We investigate the negative gradient flow for a Morse function f:

$$\dot{\gamma} = -\nabla_\gamma f. \tag{2}$$

For any critical point $p \in \mathrm{Crit}(f)$, define the stable manifold as

$$W_p^s = \{x \in X : \exists \gamma, \gamma(0) = x, \dot{\gamma}(t) = -\nabla_{\gamma(t)} f, \lim_{t\to\infty} \gamma(t) = p\}.$$

Theorem 2.0.1. *The stable sub-manifold W_p^s is a smooth manifold of* dim $n - \mu(p)$ *which is the number of positive eigenvalues of* $\mathrm{Hess}_p f$.

We focus on **a local version** of this theorem, i.e.

Theorem 2.0.2. *There exists a neighborhood U of $p \in X$ such that $W_p^s \cap U$ is a smooth manifold of* dim $n - \mu(p)$.

To pass from the local version to the global one, it suffices to note that the time-1 map $\phi_1 : X \to X, \gamma(0) \mapsto \gamma(1)$ is a diffeomorphism on X and

$$W_p^s = \bigcup_{n\in\mathbb{Z}} \phi_1^n(W_p^s \cap U).$$

Therefore, Theorem 2.0.2 implies Theorem 2.0.1.

Now we pick a coordinate chart (V, ψ) near $p \in X$ such that $\psi(V) \subset \mathbb{R}^n$ and $\psi(p) = 0$. For a path $\gamma : [0, \infty) \to \mathbb{R}^n$ with bounded L_1^2 norm, consider the mapping:

$$\begin{aligned} F : L_1^2([0, +\infty), \mathbb{R}^n) &\to L^2([0, \infty), \mathbb{R}^n) \\ \gamma &\mapsto \dot{\gamma} + \nabla_\gamma f. \end{aligned}$$

Since

$$|\gamma(t_2) - \gamma(t_1)| = |\int_{t_1}^{t_2} \dot{\gamma}(t)dt| \leqslant |\int_{t_1}^{t_2} 1^{1/2}|\int_{t_1}^{t_2} |\dot{\gamma}|^2|^{1/2} \leqslant C|t_2 - t_1|^{1/2},$$

we see that $L_1^2([0, +\infty), \mathbb{R}^n) \hookrightarrow C_0^{1/2}([0, \infty), \mathbb{R}^n)$, so the path γ is Hölder continuous and $\lim_{t\to\infty} \gamma(t) = 0$. Because

$$\nabla_x f = \mathrm{Hess}_0(f)(x) + \frac{1}{2} x_i x_j v_{ij}(x)$$

where $v_{ij}(x)$ is a vector-valued symmetric 2-tensor on $\mathbb{R}^n$, one concludes eventually that $\gamma \in F^{-1}(0)$ is indeed infinitely differentiable if f is.

To analyze the equation $F = 0$ in a more concrete way, we consider the property of F as a map between Hilbert spaces. Linearize F at $\gamma \equiv 0$:

$$\begin{aligned} D_0F(\eta) : L_1^2([0, \infty), \mathbb{R}^n) &\to L^2([0, \infty), \mathbb{R}^n) \\ \eta &\mapsto \dot{\eta} + \mathrm{Hess}_0 f(\eta), \end{aligned}$$

then it is a bounded linear operator. If v_λ is an eigenvector of $\operatorname{Hess}_0 f$ with $\lambda > 0$, then

$$\operatorname{Hess}_0 f(v_\lambda) = \lambda v_\lambda, \eta(t) = v_\lambda e^{-\lambda t}$$

is a special solution of $D_0 F = 0$. To study the mapping property of $D_0 F$, we carry out the computation:

$$\begin{aligned}\int_0^\infty |F(\eta)|^2 &= \int_0^\infty |\dot\eta + \operatorname{Hess}_0 f(\eta)|^2 \\ &= \int_0^\infty |\dot\eta|^2 + |\operatorname{Hess}_0 f(\eta)|^2 + 2\langle \dot\eta, \operatorname{Hess}_0 f(\eta)\rangle.\end{aligned}$$

Since $\operatorname{Hess}_0 f$ is a self-adjoint invertible operator, for some $C > 0$

$$|\operatorname{Hess}_0 f(\eta)|^2 \geqslant C|\eta|^2$$

$$2\langle \dot\eta, \operatorname{Hess}_0 f(\eta)\rangle = \frac{d}{dt}\langle \eta, \operatorname{Hess}_0 f(\eta)\rangle.$$

Finally, we obtain

$$\|F(\eta)\|_2^2 \geqslant \int_0^\infty (|\dot\eta|^2 + C|\eta|^2) - \langle \eta(0), \operatorname{Hess}_0 f(\eta(0))\rangle.$$

Let $\Pi_\pm : \mathbb{R}^n \to \mathbb{R}^n$ be the positive (negative) spectral projection for $\operatorname{Hess}_0 f$. Then,

$$\begin{aligned}&\|F(\eta)\|_2^2 + \langle \Pi_+(\eta(0)), \operatorname{Hess}_0 f(\Pi_+(\eta(0)))\rangle \\ \geqslant &\int_0^\infty (|\dot\eta|^2 + C|\eta|^2) - \langle \Pi_-(\eta(0)), \operatorname{Hess}_0 f(\Pi_-(\eta(0)))\rangle \\ \geqslant &\int_0^\infty (|\dot\eta|^2 + C|\eta|^2).\end{aligned}$$

Therefore, the extended operator

$$(D_0 F, \Pi_+ \circ r) : L_1^2([0,\infty), \mathbb{R}^n) \to L^2([0,\infty), \mathbb{R}^n) \oplus \operatorname{Im}\Pi_+$$

is injective with closed range. Here, $r : L_1^2([0,\infty), \mathbb{R}^n) \to \mathbb{R}^n, \eta \mapsto \eta(0)$ is the restriction map at $t = 0$. One can further study the adjoint operator to conclude the cokernel is trivial. A different strategy is to show the equation is solvable for a dense subset of $L^2([0,\infty), \mathbb{R}^n) \oplus \operatorname{Im}\Pi_+$.

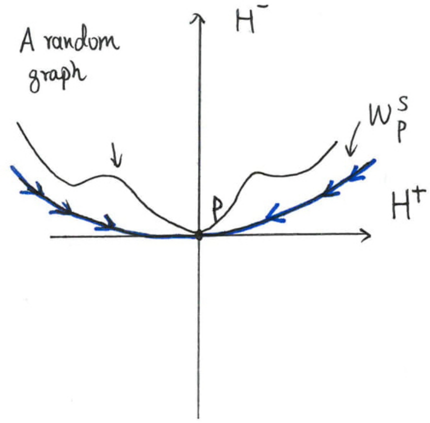
Therefore, this operator is an isomorphism and we have checked the hypothesis for inverse function theorem. As a consequence, the nonlinear operator:

$$(F, \Pi_+ \circ r) : L_1^2([0,\infty), \mathbb{R}^n) \to L^2([0,\infty), \mathbb{R}^n) \oplus \operatorname{Im}\Pi_+$$

is a diffeomorphism from a neighborhood of $0 \in L_1^2([0,\infty), \mathbb{R}^n)$ to another neighborhood of zero in the domain.

This means L_1^2-solution to $F = 0$ with small L_1^2-norm are parametrized by a neighborhood of $0 \in \Pi_+\mathbb{R}^n$. Once we show any solution to (2) with $\lim \gamma(t) = 0$ has enough decay to be in L_1^2, the proof of Theorem 2.0.2 is completed.

It is enlightening to review different proofs of Theorem 2.0.2 in literatures. It was Hadamard who first realized this problem and did a proof. Later Perron gave a different proof by making the stable sub-manifold a graph over $H_+ = \operatorname{Im}\Pi_+$. The gradient flow of f will then deform a random graph. He managed to show this gives a contraction mapping, so a fixed point exists. The next successor was Smale who approached the problem from dynamic systems. For a discrete dynamic system (a diffeomorphism) $\phi : \mathbb{R}^n \to \mathbb{R}^n$ with $\phi(0) = 0$, one constructs a mapping



$$
\begin{aligned}
l^2(\mathbb{R}^n) &\to l^2(\mathbb{R}^n) \\
(x_1, x_2, x_3 \cdots) &\mapsto (\phi^{-1}(x_2), \phi^{-1}(x_3), \cdots).
\end{aligned}
$$

and a fixed point corresponds to a converging orbit. When a unit circle condition is fulfilled, one can prove a version of Theorem 2.0.2 in this case. This condition requires that the tangent map $d_0\phi : T_0\mathbb{R}^n \to T_0\mathbb{R}^n$ has no eigenvalues on $S^1 \subset \mathbb{C}$. As one can imagine, for rotations on the complex plane $\mathbb{C}$, we do not have any non-trivial converging orbit and the proof might break down for similar examples. For the continuous version,

$$\frac{d}{dt}\phi_t = -V(\phi_t)$$

for a vector field $V : \mathbb{R}^n \to \mathbb{R}^n$ with $V(0) = 0$, we require that d_0V has no purely imaginary eigenvalues.

It was also Smale who first realized Theorem 2.0.2 implies the global version Theorem 2.0.1. He envisioned stable and unstable sub-manifolds sitting inside X and they should intersect transversely. This was the first time that the Morse-Smale condition had been proposed.

Finally, we address the problem of exponential decay. For a Morse function $f : \mathbb{R}^n \to \mathbb{R}$ with $f(0) = \nabla_0 f = 0$, $\operatorname{Hess}_0 f$ is invertible. Then $x \mapsto \nabla_x f$ is a local diffeomorphism. For some $c, C > 0$,

$$
\begin{aligned}
c|\nabla_x f|^2 \leqslant |x|^2 &\leqslant C|\nabla_x f|^2 \\
|f(x)| \leqslant C|x|^2 &\leqslant C^2|\nabla_x f|^2.
\end{aligned}
$$

These imply that

$$\frac{d}{dt}f(\gamma(t)) = -|\nabla_\gamma f|^2 \leqslant -C_1|f(\gamma(t))|.$$

If $f(\gamma(t)) > 0$, then $f(\gamma(t)) < f(\gamma(t_0))e^{-C_1(t-t_0)}$ for $t_0 < t$ (so we have exponential decay). If $f(\gamma(t)) < 0$, then $0 > f(\gamma(t)) \geqslant f(\gamma(t_0))e^{C_1(t-t_0)}$. We conclude that (some computation is omitted here)

$$|f(t)| \leqslant C_2 \cosh(C_1(t-t_0))|f(t_0)|.$$

This also shows spikes can't appear in the profile of $|\dot{\gamma}|^2$.

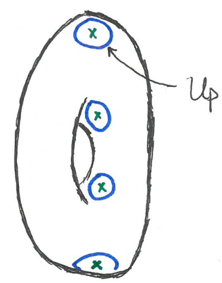
Lecture 3. Morse Homology and its variants

3.1. Morse Homology Continued

For each $p \in \mathrm{Crit}(f)$, choose a neighborhood $U_p \subset X$ of p such that they are disjoint and

$$|\nabla_x f| < \epsilon$$

whenever $x \in U_p$ for some critical point p. This means whenever the flow γ enters these neighborhoods, the velocity will be small. In fact, we can say it more concretely:



Theorem 3.1.1. *The length of the flowline $\gamma(t)$ when it is contained in $\bigcup U_p$ is independent of the length of the time.*

Proof. Suppose $\gamma(t) \in U_p$ for any $t \in [t_0, t_2]$ and when $t = t_1 \in [t_0, t_2]$, $f(\gamma(t)) = f(p)$. The intermediate time t_1 might not exist and one of t_0 and t_2 could be $\pm\infty$, but let us assume that they are all finite. It suffices to show

$$\text{the length of } \gamma([t_0, t_1])$$

has a uniform bound that is independent of $|t_1 - t_0|$. The same estimate would apply to the interval $[t_1, t_2]$. Recall from the last lecture that when $s_0 \in [t_0, t_1]$, the value of f has exponential decay:

$$0 \leqslant f(\gamma(s_0)) - f(p) \leqslant |f(\gamma(t_0)) - f(p)| e^{-\mu(t_0 - s_0)}$$

where $\mu > 0$ is a positive number.

Note that for $s_0, s_1 \in [t_0, t_1]$,

$$\begin{aligned} l(\gamma(s_1), \gamma(s_0)) &= \int_{s_0}^{s_1} |\dot\gamma(t)| dt \leqslant \int_{s_0}^{s_1} |\nabla_{\gamma(t)} f| dt \\ \text{(Hölder's inequality)} &\leqslant \left(\int_{s_0}^{s_1} |\nabla_{\gamma(t)} f|^2 dt\right)^{\frac{1}{2}} \left(\int_{s_0}^{s_1} 1 \cdot dt\right)^{\frac{1}{2}} \\ \text{(Energy Identity)} &\leqslant (f(\gamma(s_1)) - f(\gamma(s_0)))^{\frac{1}{2}} (s_1 - s_0)^{\frac{1}{2}}. \end{aligned}$$

At the last step, we used the energy identity derived in the first lecture. Since $|t_0 - t_1|$ could be any positive number, above estimate is not useful if we take $s_0 = t_0$

and $s_1 = t_1$. Instead, let $s_0 = t_0 + i$ and $s_1 = t_1 + i + 1$ for each $i \in \mathbb{Z}_{\geqslant 0}$, so we obtain

$$\begin{aligned} l(\gamma(t_0), \gamma(t_0+n)) &= \sum_{i=0}^{n-1} l(\gamma(t_0+i), \gamma(t_0+i+1)) \\ &\leqslant \sum_{i=0}^{n-1} [f(\gamma(t_0+i)) - f(\gamma(t_0+i+1))]^{\frac{1}{2}} \\ &\leqslant \sum_{i=0}^{n-1} [f(\gamma(t_0+i)) - f(p))]^{\frac{1}{2}} + [f(\gamma(t_0+i+1)) - f(p)]^{\frac{1}{2}} \\ &\leqslant 2[f(\gamma(t_0)) - f(p)]^{\frac{1}{2}} \sum_{i=0}^{\infty} e^{-\mu i/2} \\ &\leqslant \frac{C}{1-e^{-\mu/2}}. \end{aligned}$$

In the middle, we used the elementary inequality $(a+b)^{\frac{1}{2}} \leqslant a^{\frac{1}{2}} + b^{\frac{1}{2}}$ for a, b positive. □

Let's summarize what properties we have used to define the Morse Homology. There are three important principles:

- The length of flowlines where $|\nabla_x f|$ is large (bounded away from 0) is bounded by the energy.
- On uniformly bounded intervals, flowlines with small energy are short (contained in U_p for some $p \in \mathrm{Crit}(f)$).
- The length of flowlines where $|\nabla_x f|$ is small is independent of the time spent in the region $(\bigcup U_p)$.

3.2. Homology with Local Coefficients

Let $\Gamma \to X$ be a flat bundle over X with fiber a module M over a commutative ring R. It is a bundle equipped a preferred flat connection. This means that for any path $\gamma : [a, b] \to X$, we can define by parallel transportation the holonomy map

$$h_\gamma : \Gamma_{\gamma(a)} \to \Gamma_{\gamma(b)}$$

and it depends only on the homotopy class (rel boundaries) of γ. Now given a Morse function f and a Riemannian metric g, define the chain group by

$$C(f, g, \Gamma) = \bigoplus_{c \in \mathrm{Crit}(f)} \Gamma_c \cdot \Lambda(c)$$

and differentials by

$$\partial : C(f,g,\Gamma) \to C(f,g,\Gamma)$$
$$\partial = \sum_{\substack{a,b\in \mathrm{Crit}(f)\\ \mu(a)-\mu(b)=1}} \sum_{\gamma \in M(a,b)} h_\gamma \otimes \epsilon_\gamma.$$

We can prove $\partial^2 = 0$ as usual.

Example 3.2.1. Suppose X is not orientable.

Consider the real line bundle $\bigwedge^n(T^*X)$ where $n = \dim X$. It admits an $\mathbb{R}^+$ action. The quotient bundle $\lambda \to X$ is called the orientation bundle and its fiber $\lambda_x \cong \{\pm 1\}$. It is not a trivial bundle because X is not orientable.

Where does Poincaré duality come from in terms of Morse functions? We can simply reverse the direction of the flow by taking the negative of f:

$$(X, f, g) \mapsto (X, -f, g)$$

but this duality is not canonical because of λ. Recall that

$$\begin{aligned} C(f,g) &= \bigoplus_{c\in \mathrm{Crit}(f)} \mathbb{Z}\Lambda(c) && \text{Orientations of } \textbf{unstable submanifolds} \text{ of } f \\ C(-f,g) &= \bigoplus_{c\in \mathrm{Crit}(f)} \mathbb{Z}\Lambda'(c) && \text{Orientations of unstable submanifolds of } -f \\ & && = \text{Orientations of } \textbf{stable submanifolds} \text{ of } f \end{aligned}$$

To identify $\Lambda(c)$ with $\Lambda(c')$, we need a section of λ, but it doesn't exist if λ is not trivial. This could be resolved if we use local coefficients:

$$C(-f,g,\lambda) = \bigoplus_{c\in \mathrm{Crit}(f)} \mathbb{Z}\Lambda'(c) \otimes \lambda_c$$

and we use the fact

$$\Lambda(c) \otimes \Lambda'(c) \cong \lambda_c.$$

In this case, the Poincaré duality is formulated as

$$H^*(X;\ \mathbb{Z}) \cong H_*(X;\ \underline{\mathbb{Z}}_\lambda).$$

where $\underline{\mathbb{Z}}_\lambda$ denotes the local coefficient system determined by λ.

Note that a flat bundle (connection) is in bijection with isomorphism classes of representations of $\pi_1(X) \to R$. In the example above, $R \cong \mathbb{Z}/2\mathbb{Z}$ and $M \cong \mathbb{Z}$. The action of R on M is given by sending 1 to the reflection of $\mathbb{Z}$.

3.3. Novikov-Morse Homology

Let θ be a closed 1-form and $\theta = \theta^i dx_i$ in a local coordinate chart. The dual vector field is $\theta^\# = g^{ij}\theta_j\partial_i$. We investigate the flowlines driven by $\theta^\#$:

$$\dot\gamma = -\theta^\#_{\gamma(t)}.$$

Since θ is closed, we can locally write $\theta = df$ for some real-valued function f, so γ is a negative gradient flow within a local coordinate chart. But this is not true globally, if θ is not exact.

The beginning of the story goes through *mutatis mutandis*. The main difference comes from the energy identity. For $\gamma : [a, b] \to X$ a flowline,

$$\int_a^b |\dot\gamma|^2 = \int_a^b |\theta^\#_{\gamma(t)}|^2 = -\int_a^b \langle \theta^\#_{\gamma(t)}, \dot\gamma\rangle = -\int_a^b \gamma^*\theta.$$

If $[\theta] \neq 0 \in H^1(X, \mathbb{R})$, then $\int_a^b \gamma^*\theta$ depends on the homotopy class of γ. We refine the moduli space $M(a, b)$ according to $\pi_1(X, a, b)$:

$$M(a, b) = \bigcup_{[\gamma]\in\pi_1(X,a,b)} M_{[\gamma]}(a, b)$$

and subdivide the unparameterized space $\check M(a, b)$:

$$\check M(a, b) = \bigcup_{[\gamma]\in\pi_1(X,a,b)} \check M_{[\gamma]}(a, b) = \bigcup_{[\gamma]\in\pi_1(X,a,b)} M_{[\gamma]}(a, b)/\mathbb{R}.$$

Define

$$\overline{\check M_{[\gamma]}}(a, b) = \check M_{[\gamma]}(a, b) \bigcup_{c\in \mathrm{Crit}(f)} \bigcup_{\substack{[\gamma_1]\in\pi_1(a,c)\\ [\gamma_2]\in\pi_1(c,b)\\ [\gamma_1]+[\gamma_2]=[\gamma]}} \check M_{[\gamma_1]}(a, c) \times \check M_{[\gamma_2]}(c, b).$$

Then each $\overline{\check M_{[\gamma]}}(a, b)$ is compact as before.

Example 3.3.1. Let $X = S^1$ and $\theta = d\theta^0$.

In this case, the Novikov-Morse homology is zero because there is no critical point at all.

To define the Novikov-Morse homology with local coefficients in general, we note that the differential

$$\partial = \sum_{\substack{a,b\in\mathrm{Crit}(f)\\ \mu(a)-\mu(b)=1}} \sum_{\gamma\in M(a,b)} h_\gamma \otimes \epsilon_\gamma.$$

may involve an infinite sum. To make it work, we need the fiber of Γ to have topology so that we are able to speak of convergence of the sum. The computation above shows that

$$E(\gamma) \to \infty \Rightarrow \int_a^b \gamma^*\theta \to -\infty.$$

Now we introduce the Novikov ring $\mathcal{R}$:

$$\mathcal{R} := \{f : \mathbb{R} \to \mathbb{R} : \mathrm{supp} f|_{[n,+\infty)} \text{ is finite}\}.$$

An element f is identified with a formal power series:

$$f \mapsto F(t) = \sum_{\lambda \in \mathrm{supp} f} f(\lambda) t^{\lambda}.$$

It is like a Laurent series with finitely powers in positive degrees (like $\mathbb{R}[[t^{-1}, t])$, but exponents are allowed to be any real numbers. Choose a base point x_0, then the $\pi_1(X, x_0)$-representation

$$\gamma \mapsto t^{\int_\gamma \theta} \in \mathrm{Aut}(\mathcal{R}, \mathcal{R})$$

determines the local coefficient system. To specify the flat bundle, let

$$X \times \mathcal{R}$$

be the underlying bundle and for each $\gamma \in \pi_1(a, b)$, define parallel transportation by

$$\Gamma_\gamma = t^{\int_\gamma \theta}.$$

The connection 1-form is given by

$$\omega = d - \theta \otimes \ln(t) \in \Gamma(X, T^*X \otimes \mathrm{End}(\mathcal{R}, \mathcal{R})).$$

The operator $\ln(t)$ is only a formal element in $\mathrm{End}(\mathcal{R}, \mathcal{R})$. Now define the differential of the Novikov-Morse chain complex by

$$\partial = \sum_{\substack{a,b \in \mathrm{Crit}(f) \\ \mu(a)-\mu(b)=1}} \sum_{\gamma \in M(a,b)} t^{\int_\gamma \theta} \otimes \epsilon_\gamma.$$

and it is well defined in $\mathrm{End}(\mathcal{R}, \mathcal{R})$.

Lecture 4. Floer Homology: An Overview

4.1. Functionals

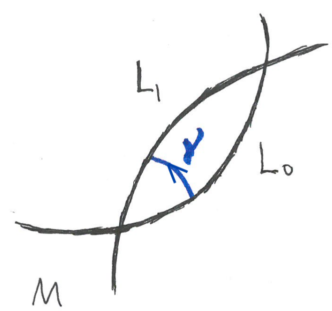
It was Floer who first realized that Morse Homology would work for some infinite dimensional settings. In this lecture, we briefly discuss two setups. The symplectic case is included only for illustrations. Later, we will focus on the Chern-Simons functional and explain the necessary analytic tools to define the theory.

4.1.1. **The Symplectic Action Functional.** Suppose (M, ω) is a closed symplectic manifold and (L_0, L_1) is a pair of Lagrangian sub-manifolds. Let

$$X = \mathcal{P}(M; L_0, L_1) := \mathcal{C}^\infty(([0,1], 0, 1), (M, L_0, L_1)).$$

An element in X is a path $\sigma : [0,1] \to M$ with end points contained in L_0 and L_1 respectively:

$$\sigma : [0,1] \to M, \sigma(0) \in L_0, \sigma(1) \in L_1.$$

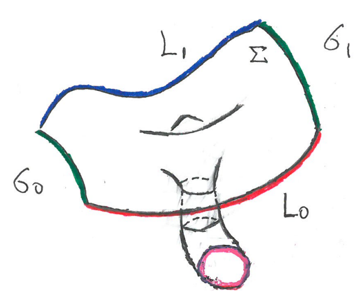


To define the action functional, suppose $\sigma_0, \sigma_1 \in X$ are homotopic, or in general, homologous in (M, L_0, L_1). This means there is a surface:

$$f : \Sigma \to M$$

with $\partial\Sigma = [0,1]_0 \cup [0,1]_1 \cup \gamma_0 \cup \gamma_1$ and

$$f|_{[0,1]_0} = \sigma_0, f|_{[0,1]_1} = \sigma_1, f(\gamma_0) \subset L_0, f(\gamma_1) \subset L_1.$$



Fix σ_0 and allow σ_1 to vary. Define

$$a(\sigma_1) = -\int_\Sigma f^*\omega.$$

Implicitly, this functional a depends on the choice of σ_0 and the homotopy class of the surface Σ.

4.1.2. **The Chern-Simons Functional.** Let G be a compact Lie group and

$$c : \mathfrak{g} \to \mathbb{R}$$

be a symmetric G-invariant homogeneous polynomial of degree 2. Suppose P is a principle G-bundle over Y^3:

$$\begin{array}{ccc} G & \longrightarrow & P \\ & & \big\downarrow \\ & & Y^3 \end{array}$$

Let A_0, A_1 be two connections of P and write $a = A_1 - A_0 \in \Gamma(Y, \mathfrak{g}_P)$. Here, $\mathfrak{g}_P$ is the vector bundle induced by the adjoint representation of G on its Lie algebra $\mathfrak{g}$:

$$\mathfrak{g}_P = P \times_{ad} \mathfrak{g}.$$

The pair (A_0, A_1) fits into a family of connections

$$A_t = (1-t)A_0 + tA_1.$$

The Chern-Simons functional is then defined by the formula

$$\mathcal{CS}(A_1) = \int_Y c(F_{A_0} + \frac{1}{2} d_{A_0} a + \frac{1}{6}[a \wedge a], a)$$

where c acts on the Lie algebra parts of F_{A_0} and a and takes the wedge product for the form part. It is a real valued 3-form to be integrated. This formula has an unpleasant looking and its value depends on the choice of the background connection A_0. In fact, this functional originates from the Chern-Weil form in dimension 4 associated to the connection

$$\frac{d}{dt} + A_t$$

on $[0,1] \times Y$ and it is more natural to think of $\mathcal{CS}$ in that way. More details will be included in later lectures.

4.2. Formal Gradients

Now we have the area functional on the path space:

$$a : \mathcal{P}(M; L_0, L_1) \dashrightarrow \mathbb{R}$$

and the Chern-Simons functional on the space of connections

$$\mathcal{CS} : \mathscr{A} \to \mathbb{R}$$

To pursue the analogue of Morse Theory, we need to define the formal gradients of a and $\mathcal{CS}$ and a "Riemannian" metric on $\mathcal{P}(M; L_0, L_1)$ and $\mathscr{A}$ is required.

In the symplectic case, we choose a metric g on (M, ω) compatible with the symplectic form. This means we can find an almost complex structure

$$J : TM \to TM, J^2 = -\operatorname{Id}$$

such that

$$\omega(X, Y) = g(JX, Y), g(JX, JY) = g(X, Y)$$

for any vectors fields $X, Y \in \Gamma(M, TM)$. For any $\sigma \in \mathcal{P}$, its tangent space is given by

$$T_\sigma \mathcal{P} = \{v \in \Gamma(\sigma^* TM) : v(0) \in T_{\sigma(0)} L_0, v(1) \in T_{\sigma(1)} L_1\}.$$

and define the metric by the formula:

$$\langle v_1, v_2 \rangle = \int_{[0,1]} g(v_1, v_2) dt.$$

Exercise 4.2.1. *Show that the L^2 gradient of the area functional is given by*

$$\nabla_\sigma a = J \sigma_* (\frac{\partial}{\partial t})$$

Here, the vector field $\frac{\partial}{\partial t}$ is the canonical one on the interval $[0, 1]$.

Proof. Let $v(t) \in \Gamma(\sigma^* TM)$ such that $\operatorname{supp}(v)$ is contained in $(0, 1)$. Then

$$\begin{aligned} f : [0,1] \times [0,1] &\to M \\ (s,t) &\mapsto \exp_{\sigma(t)}(sv(t)) \end{aligned}$$

is a family of variations of σ. The restriction $f : [0, s] \times [0, 1] \to M$ gives a homotopy between σ and

$$\sigma_s = f|_{\{s\} \times [0,1]}$$

Therefore,

$$a(\sigma_s) - a(\sigma) = -\int_{[0,s]\times[0,1]} \omega(f_*(\frac{\partial}{\partial s}), f_*(\frac{\partial}{\partial t})) ds \wedge dt.$$

Taking derivative with respect to s gives

$$\begin{aligned} \langle \nabla_\sigma a, v \rangle &= -\int_{[0,1]} \omega(v(t), f_*(\frac{\partial}{\partial t})) = -\int_{[0,1]} g(Jv(t), f_*(\frac{\partial}{\partial t})) \\ &= \int_{[0,1]} g(v(t), Jf_*(\frac{\partial}{\partial t})). \end{aligned}$$

For this example, we may need

$$\sigma_*(\frac{\partial}{\partial t})\big|_{t=0} \perp T_{\sigma(0)} L_0, \sigma_*(\frac{\partial}{\partial t})\big|_{t=1} \perp T_{\sigma(1)} L_1,$$

at the boundary of $I = [0,1]$ to ensure $\nabla_\sigma a \in T_\sigma \mathcal{P}$. In general, $\nabla_\sigma a$ might not satisfy the boundary condition. The tangent space $T_\sigma \mathcal{P}$ is L^2-dense in

$$L^2([0,1], \mathbb{C}^n),$$

and one is not allowed to speak of the boundary value of a L^2-section. $\square$

For **the Chern-Simons Functional**, we have $\nabla_A \mathcal{CS} = *_Y F_A$ for a connection $A \in \mathscr{A}$.

4.3. Hessians

Suppose $M = \mathbb{C}^n$ with the standard metric g and symplectic form ω. By the computation in the previous section,

$$\nabla_\sigma a = i\sigma_*(\frac{\partial}{\partial t})$$

for any path $\sigma : [0,1] \to M$. A tangent vector $v \in T_\sigma \mathcal{P}$ is acted on by the Hessian:

$$\begin{aligned} \mathrm{Hess}_\sigma : \mathcal{C}^\infty([0,1], \mathbb{C}^n) &\to \mathcal{C}^\infty([0,1], \mathbb{C}^n) \\ v &\mapsto \nabla_v(i\sigma_*(\frac{\partial}{\partial t})) = i\frac{\partial}{\partial t}v(t). \end{aligned}$$

It is a self-adjoint operator on $L^2([0,1]; \mathbb{C}^n)$ with domain

$$L^2_1([0,1], \mathbb{C}^n; L_0, L_1) = \{v \in L^2_1([0,1]) : v(0) \in T_{\sigma(0)}L_0, v(1) \in T_{\sigma(1)}L_1\},$$

and it has infinitely many positive and negative eigenvalues. We do not have the Morse index in this case. Instead, **spectral flows** will allow us to speak of relative indices which are all we need to develop a theory.

Example 4.3.1. Consider $i\frac{\partial}{\partial t}$ acting on $L^2_1(S^1, \mathbb{C})$.

We treat S^1 as $\mathbb{R}/2\pi\mathbb{Z}$. Note that for any $\alpha \in [0,1]$,

$$\mathrm{Spec}(i\frac{\partial}{\partial t} + \alpha) = \alpha + \mathbb{Z}.$$

Eigenvectors are e^{-int} and

$$(i\frac{\partial}{\partial t} + \alpha)e^{-int} = (\alpha + n)e^{-int}.$$

When $\alpha = 1$, the operator is unitarily equivalent to $i\frac{\partial}{\partial t}$:

$$(i\frac{\partial}{\partial t} + 1) = e^{it}(i\frac{\partial}{\partial t})e^{-it}.$$

This is a new phenomenon for infinite dimensional spaces.

Note that the space of bounded self-adjoint Fredholm operators on an infinitely dimensional separable Hilbert space H has 3-connected components:

$$\mathrm{SA}_{Fred} = \mathrm{SA}_+ \bigcup \mathrm{SA}_0 \bigcup \mathrm{SA}_- .$$

where $\mathrm{SA}_{+(-)}$ denotes the space of operators with essentially positive (negative) spectrum. If an operator A is Fredholm, then it only has discrete spectrum near the origin and it makes sense to talk about the negative (positive) subspace $H_-(H_+)$ of A by functional calculus of self-adjoint operators. Then A is called essentially positive (negative) if

$$\dim H_- \ (\dim H_+) < \infty.$$

Components SA_+ and SA_- are contractible. The middle part SA_0 is the complement of SA_+ and SA_- and it has non-trivial topology. In fact,

$$\begin{aligned}\Omega\,\mathrm{SA}_0 &\sim \mathbb{Z}\times BU \text{ if } H \text{ is over } \mathbb{C}\\ &\sim \mathbb{Z}\times BO \text{ if } H \text{ is over } \mathbb{R}.\end{aligned}$$

and the $\mathbb{Z}$-component represents the spectrum flow. Here, BU is the classifying space of the complex vector bundles and BO is the classifying space of the real ones. The example above actually represents a non-trivial loop in SA_0 (though they are not bounded operators):

$$S^1 \to (L^2_1(S^1,\mathbb{C}) \to L^2(S^1,\mathbb{C})).$$

Morse index doesn't make sense anymore. Nevertheless, it is the difference $\mu(a) - \mu(b)$ of Morse indices that matters and it is replaced by the spectrum flow.

Suppose $\sigma_0, \sigma_1 \in \mathcal{P}$ fits into a path $\sigma_t \in \mathcal{P}$ for $t \in [0,1]$ and assume

$$0 \notin \mathrm{Spec}(\mathrm{Hess}_{\sigma_0} a) \cup \mathrm{Spec}(\mathrm{Hess}_{\sigma_1} a).$$

Then the spectrum flow of σ_t simply counts the signed number of eigenvalues that come across 0 as t varies and we define

$$\mu(\sigma_0) - \mu(\sigma_1) = \mathrm{SF}(\mathrm{Hess}_{\sigma_t}(a)).$$

even though the value of $\mu(\sigma_0)$ is not well-defined.

In the finite dimensional setting, $\mathrm{Hess}_p f : T_pX \to T_pX$ is a self-adjoint operator and the spectrum flow could be defined along a path $\gamma : [0,1] \to X$.

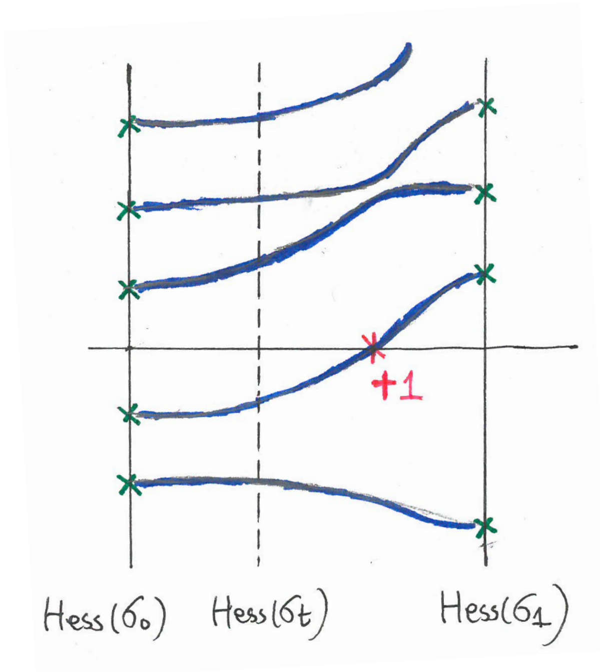
4.4. Novikov Rings

Both a and $\mathcal{CS}$ can be viewed as refinement of the period map and the Chern-Weil integral:

$$\begin{aligned} a \Leftarrow \theta_\omega &= \int_\bullet \omega : H_2(M,\mathbb{Z}) \to \mathbb{R}\\ \mathcal{CS} \Leftarrow \theta_c &= \int_X c(F_A, F_A) : \{\text{Isom. cls. of princ. } G\text{-bundle}\} \to \mathbb{R}.\end{aligned}$$

Maps on the right gives obstructions of functionals to be real valued:

$$a : \mathcal{P}(M; L_0, L_1) \to \mathbb{R}/\operatorname{Im}\theta_\omega.$$



$$\begin{array}{ccc} \mathcal{CS} : \mathscr{A}_P & \longrightarrow & \mathbb{R} \\ \downarrow & & \downarrow \\ \mathscr{A}_P/\mathscr{G} & \longrightarrow & \mathbb{R}/\operatorname{Im}\theta_c \end{array}$$

For the Chern-Simons functional, it is real valued on $\mathscr{A}_P$, but we need to quotient out gauge transformations to do Morse theory and it becomes circle valued on the quotient space. So we need to work with Novikov-Morse theory.

Note that both functionals and spectrum flows defines maps on π_1 of the target space:

$$\begin{aligned} \mathrm{SF} &: \Omega\mathcal{P}(M; L_0, L_1) \to \pi_1(\mathcal{P}) \to \mathbb{Z} \\ \mathrm{SF} &: \Omega(\mathscr{A}/\mathscr{G}) \to \mathbb{Z} \\ \Delta a &: \pi_1(\mathcal{P}) \to \operatorname{Im}\theta_\omega \\ \Delta\mathcal{CS} &: \pi_1(\mathscr{A}/\mathscr{G}) \to \operatorname{Im}\theta_c. \end{aligned}$$

If they are proportional, which means for some $\lambda \in \mathbb{R}$, $\lambda\,\mathrm{SF}_a = \Delta a$, then the ordinary Morse homology still works. If we pin down the spectral flow, then we have uniform control for the drop of the functional and hence the energy of the flowline. However, if the theory is not monotone, then a modification of the Novikov-Morse homology will be needed here.

Lecture 5. Connections and Curvatures

5.1. PRINCIPAL BUNDLES

For this lecture, we will quickly go over some basics on principal bundles. It was back in 1983 when Tom first learned this stuff from Raoul Bott who had an interesting character. Tom felt it necessary to tell the story in the same way Raoul told him.

For a compact Lie group G, a principle G-bundle is a smooth manifold P with a free G-action acting on the right:

$$P \times G \to P$$
$$(p, g) \mapsto pg.$$

Let $X = P/G$ be the quotient space and P becomes a G-bundle over X. Two principle G-bundles P, P' over X are isomorphic, if there is a bundle map η

$$\begin{tikzcd} P \arrow[rr, "\eta"] \arrow[dr] & & P' \arrow[dl] \\ & X & \end{tikzcd}$$

that covers the identity map on X and commutes with the right G-action.

Besides this highbrow point of view, a lowbrow point of view characterizes a principal bundle in terms of transition functions.

A principle G-bundle is equivalent to an open over $\{U_\alpha\}$ of X with transition functions defined on each intersection:

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to G.$$

such that the co-cycle condition is satisfied:

$$g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha} = e \in G \quad \text{on } U_\alpha \cap U_\beta \cap U_\gamma.$$

for any α, β and γ. Two bundles $\{g_{\alpha\beta}\}$ and $\{g'_{\alpha\beta}\}$ are equivalent if one can find intertwining maps:

$$\eta_\alpha : U_\alpha \to G$$

such that

$$g_{\alpha\beta}\eta_\beta = \eta_\alpha g'_{\alpha\beta} \text{ on } U_\alpha \cap U_\beta.$$

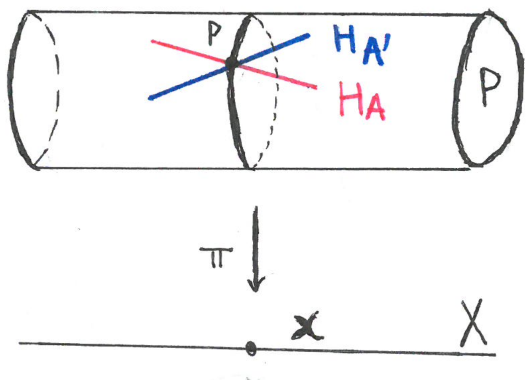
One also need to define equivalence with respect to refinements of open covers $\{U_\alpha\}$, but we omit it here.

5.2. CONNECTIONS

We give five definitions of connections on a principal bundle P:

(1) A connection A is a G-invariant sub-bundle of TP called $H_A \subset TP$ such that under the projection map $\pi : P \to X$,

$$d\pi|_{H_{A,p}} : H_{A,p} \xrightarrow{\cong} T_{\pi(p)}X.$$



(2) The projection map π induces a short exact sequence of bundles:

$$\begin{tikzcd}
0 \arrow[r] & \ker(d\pi) \arrow[r] & TP \arrow[r, "d\pi"] \arrow[l, bend right, "A"'] & \pi^*TX \arrow[r] & 0 \\
& & H_A \arrow[u, hook] \arrow[ul, dashed, "0"] & &
\end{tikzcd}$$

A connection $A : TP \to \ker(d\pi)$ is a left inverse to $\ker(d\pi)$ such that the sequence splits G-equivariantly. The horizontal space H_A is $\ker(A)$. For fixed $p \in P$, the inclusion map

$$\begin{aligned} \iota_p : G &\to P \\ g &\mapsto p \cdot g \end{aligned}$$

identifies $\mathfrak{g} = T_eG$ with $\ker(d\pi_p) \subset T_pP$ by taking the tangent map.

(3) Covariant derivative:

$$\nabla_A : \mathcal{C}^\infty(X, \operatorname{ad} P) \to \mathcal{C}^\infty(X, T^*X \otimes \operatorname{ad} P).$$

Recall that $\operatorname{ad} P = P \times_{\mathrm{ad}} \mathfrak{g}$.

(4) Holonomy (Parallel Transportation). Given a C^1-path $\gamma : [a, b] \to X$ and $p \in P$ with $\pi(p) = \gamma(a)$, we can find a unique path lifting γ upstairs:

$$\tilde{\gamma}_{A,p} : [a, b] \to P$$

such that $\pi \circ \tilde{\gamma} = \gamma$ and $\tilde{\gamma}_{A,p}(a) = p$. We also require this lift to be equivariant: if $p' = p \cdot g$ for some $g \in G$, then

$$\tilde{\gamma}_{A,p'}(t) = \tilde{\gamma}_{A,p}(t) \cdot g.$$

for any $t \in [a, b]$.

(5) This is the lowbrow point of view. A connection is also equivalent to a collection of 1-forms:

$$a_\alpha \in \Gamma(T^*U_\alpha \otimes \mathfrak{g})$$

such that the overlapping condition is satisfied

$$a_\beta = g_{\alpha\beta}^{-1} a_\alpha g_{\alpha\beta} + g_{\alpha\beta}^{-1} dg_{\alpha\beta}$$

over the intersection $U_\alpha \cap U_\beta$.

These definitions are equivalent to each other.

Recall that $\operatorname{ad} P$ is a bundle of Lie algebras and it is the vertical tangent bundle of P. On the other hand,

$$\operatorname{Ad} P = P \times_{\operatorname{Ad}} G$$

is a bundle of Lie groups. Any automorphism η of P,

$$\begin{tikzcd} P \arrow[rr, "\eta"] \arrow[dr] & & P \arrow[dl] \\ & X & \end{tikzcd}$$

determines a section $\tilde{\eta}$ of $\operatorname{Ad} P$ and vice versa. Indeed, for any $p \in P$, we can find $\tilde{\eta}(p) \in G$ such that

$$\eta(p) = p \cdot \tilde{\eta}(p).$$

Since η preserves the right-action of G, $\eta(p \cdot g) = \eta(p) \cdot g$ and hence,

$$(p \cdot g) \cdot \tilde{\eta}(p \cdot g) = \eta(p \cdot g) = \eta(p) \cdot g = (p \cdot \tilde{\eta}(p)) \cdot g.$$

This shows $\tilde{\eta}(p \cdot g) = g^{-1} \tilde{\eta}(p) g$, so

$$p \mapsto (p, \tilde{\eta}(p)) \in P \times G$$

descends to a section $X \to \operatorname{Ad} P$.

Just as $\mathfrak{g} \cong T_e G$, sections of $\operatorname{ad} P$ are identified with infinitesimal transformations of P.

The bundle $\bigwedge^*(T^*X) \otimes \operatorname{ad} P$ inherits a super Lie algebra structure. For any two sections ω, η, the super Lie bracket

$$[\omega \wedge \eta]$$

takes the wedge of forms and the Lie product on $\operatorname{ad} P$. It is essentially a fiber-wise definition. A grading is assigned by the degree of the form. A super Lie algebra satisfies

(1) (Super Commutativity)

$$[\omega^p \wedge \eta^q] = (-1)^{pq+1} [\eta^q \wedge \omega^p].$$

(2) (Super Jacobi Identity)

$$(-1)^{pr} [\omega^p \wedge [\omega^q \wedge \omega^r]] + (-1)^{qp} [\omega^q \wedge [\omega^r \wedge \omega^p]] + (-1)^{rq} [\omega^r \wedge [\omega^p \wedge \omega^q]] = 0.$$

In terms of covariant derivatives, it is easy to see two connections A, A' of P differ by a Lie algebra valued 1-form:

$$A' - A = a \in \Gamma(X, T^*X \otimes \operatorname{ad} P).$$

The second definition of connections also allows us to think of A as an 1-form on P taking value in $\mathfrak{g}$ (since $\ker(d\pi) = P \times \mathfrak{g}$ is trivial). Recall that a form ω (maybe Lie algebra valued) on P is called **basic** if for any vertical vector field v:

(1) (Contraction) $\iota_v(\omega) = 0$.
(2) (Lie derivative) $\mathcal{L}_v\omega = 0$.

and a basic form ω is necessarily pulled back from $\pi : P \to X$. One can verify that $A - A'$ satisfies (1) and

(2′) $\mathcal{L}_v\omega = -\operatorname{ad}(v)\omega$

for any v vertical, so it is pulled back from $\Omega^1(X, \operatorname{ad} P)$.

5.3. Curvatures

The curvature of A is the obstruction of H_A being integrable in the sense of Frobenious (tangent spaces of foliations) and it has several equivalent definitions.

(1) For $X, Y \in T_pP$, let $\tilde{X}, \tilde{Y}$ be local extensions of vector fields and $h_A : TP \to H_A$ be the horizontal projection, so $A + h_A = \mathrm{Id}$. Define

$$F_A(X, Y) = -A([h_A\tilde{X}, h_A\tilde{Y}]).$$

It is checked that F_A is basic 2-form, so $F_A \in \Gamma(X, \bigwedge^2 T^*X \otimes \operatorname{ad} P)$. We could think of F_A as a (basic) 2-form either upstairs or downstairs.

(2) Since A is a 1-form on P, we can take exterior differential, so

$$F_A = dA + \frac{1}{2}[A \wedge A]$$

(3) On the adjoint bundle $\operatorname{ad} P$, A determines an exterior differential

$$d_A : \mathcal{C}^\infty(X, \Omega^* \otimes \operatorname{ad} P) \to \mathcal{C}^\infty(X, \Omega^{*+1} \otimes \operatorname{ad} P).$$

In general, $d_A^2 \neq 0$, but it is tensorial and

$$d_A^2 s = [F_A \wedge s].$$

In particular, if $s \in \Gamma(X, \operatorname{ad} P)$, this relation determines F_A.

(4) Limit of holonomy around a square.

(5) (The lowbrow point of view) Using local trivialization of P,

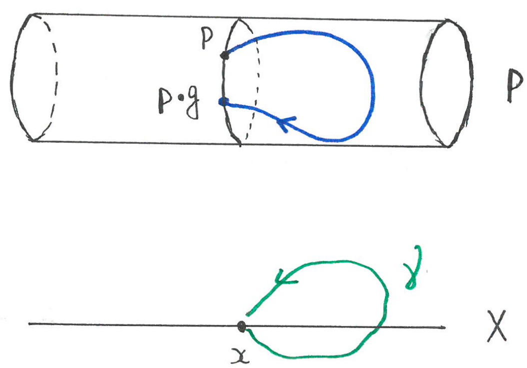
$$(F_A)_\alpha = da_\alpha + \frac{1}{2}[a_\alpha \wedge a_\alpha]$$

The connection A is called **flat** if $F_A \equiv 0$, in which case $d_A^2 = 0$ (by the third definition) and

$$\Omega^0(\operatorname{ad} P) \to \Omega^1(\operatorname{ad} P) \to \Omega^2(\operatorname{ad} P) \to \cdots$$

is a complex. There are others equivalent definitions:

(1) $g_{\alpha\beta}$ can be chosen be locally constant.
(2) The end point of path lifting depends on the homotopy class of the path in the base manifold X only.



The last property allows us to define the holonomy map. For a base point $x \in X$, choose $p \in \pi^{-1}(x)$. Define

$$\begin{aligned} \operatorname{hol}_{A,p} : \pi_1(X, x) &\to G \\ \gamma &\mapsto g_\gamma \end{aligned}$$

if the end point of $\tilde{\gamma}_{A,p}$ is $p \cdot g_\gamma$. Changing p to $p \cdot g$ will change the image of γ into

$$\operatorname{hol}_{A,pg}(\gamma) = g^{-1} \operatorname{hol}_{A,p}(\gamma) g.$$

So a flat connection determines a representation of $\pi_1(X)$ up to conjugacy.

On the other hand, the universal cover $\tilde{X}$ of X can be viewed as a principle $\pi_1(X)$-bundle. The right action of $\pi_1(X)$ is obtained from Deck transformations (which is a left action) by the usual trick. Thus, a representation

$$\rho : \pi_1(X, x) \to G,$$

induces a principle G-bundle

$$\begin{array}{ccc} G & \longrightarrow & \tilde{X} \times_\rho G \\ & & \downarrow \\ & & X, \end{array}$$

and we declare $(\tilde{\gamma}, e) \subset \tilde{X} \times_\rho G$ to be the horizontal lift of $\gamma = \pi(\tilde{\gamma})$. This makes it a flat bundle. These two constructions are inverse to each other, but we need to be careful about the statement. The correct one is:

$$\begin{aligned}&\{\text{Flat connections up to isomorphism}\} \Leftrightarrow\\&\quad\{\pi_1\text{-representations up to conjugacy}\}\end{aligned}$$

5.4. Deformations of connections

Now we study the variation of curvatures under deformations of connections. If $A' = A + a$ for $a \in \Omega^1(\operatorname{ad} P)$, then

$$F_{A'} = F_A + d_A a + \frac{1}{2}[a \wedge a].$$

Indeed, for a section $s \in \Omega^0(\operatorname{ad} P)$,

$$d_{A+a} s = d_A s + [a \wedge s].$$

Apply it twice:

$$\begin{aligned}
d^2_{A+a} s &= (d_A + a) \circ (d_A + a) s \\
&= d_A^2 s + [a \wedge d_A s] + d_A[a \wedge s] + [a \wedge [a \wedge s]] \\
&= d_A^2 s + [d_A a \wedge s] + \frac{1}{2}[[a \wedge a] \wedge s] \\
&= [(F_A + d_A a + \frac{1}{2}[a \wedge a]) \wedge s].
\end{aligned}$$

In the middle, we used the super Jacobi identity to write:

$$[a \wedge [a \wedge s]] = \frac{1}{2}[[a \wedge a] \wedge s].$$

Let $c : \mathfrak{g} \to \mathbb{R}$ be an Ad-invariant polynomial function. Then

$$c(F_A) \in \Omega^*(X)$$

is a closed real-valued form on X and its cohomology class

$$[c(F_A)] \in H^*(X, \mathbb{R})$$

is independent of the connection A. This assignment

$$c \mapsto [c(F_A)] \in H^*(X, \mathbb{R})$$

is the so-called Chern-Weil homomorphism. If $A' = A + a$, then

$$c(F_{A'}) - c(F_A) = dG(a)$$

for some form $G(a)$. The Chern-Simons functional is to find a canonical expression of $G(a)$ in terms of the difference $a = A' - A$. Suppose c is homogeneous of degree k. This means

$$c(\xi) = \tilde{c}(\xi, \xi, \cdots, \xi)$$

for a symmetric Ad-invariant multi-linear function $\tilde{c} : \bigotimes_k \mathfrak{g} \to \mathbb{R}$. Write

$$A_t = A + ta,$$

then the derivative $\frac{\partial}{\partial t} c(F_{A_t})$ equals

$$\begin{aligned}
\frac{\partial}{\partial t}\tilde{c}(F_{A_t}, \cdots, F_{A_t}) &= k\tilde{c}(\frac{\partial}{\partial t}F_{A_t}, F_{A_t}, \cdots, F_{A_t}) \text{ (because } \tilde{c} \text{ is symmetric)} \\
&= k\tilde{c}(d_{A_t}a, F_{A_t}, \cdots, F_{A_t}) \\
&= kd(\tilde{c}(a, F_{A_t}, \cdots, F_{A_t})).
\end{aligned}$$

At the last step, we used the Bianchi identity $d_{A_t}F_{A_t} = 0$. Finally, we set

$$G(a) = k\int_0^1 (\tilde{c}(a, F_{A_t}, \cdots, F_{A_t}))dt.$$

Exercise 5.4.1. *Fill in details in the computation of $G(a)$.*

Proof. Recall that the defining property of d_A on $\Omega^p(\operatorname{ad} P)$ is

$$d_A(\nu^p \otimes b) = d\nu^p \otimes b + (-1)^p \nu^p \wedge d_A b, \tag{3}$$

and it is compatible with the super Lie bracket structure:

$$d_A[\omega^p \wedge \omega^q] = [d_A\omega^p \wedge \omega^q] + (-1)^p[\omega^p \wedge d_A\omega^q].$$

For the derivative of F_{A_t}, we have

$$\begin{aligned}
\frac{\partial}{\partial t}F_{A_t} &= \frac{\partial}{\partial t}(F_A + td_Aa + \frac{1}{2}t^2[a \wedge a]) \\
&= d_Aa + t[a \wedge a] \\
&= d_{A_t}a.
\end{aligned}$$

The remaining step is to prove the Leibniz rule:

$$d(\tilde{c}(\omega^{p_1}, \cdots, \omega^{p_k})) = \sum_{i=1}^{k}(-1)^{p_1+\cdots p_{i-1}}\tilde{c}(\omega^{p_1}, \cdots, d_A\omega^{p_i}, \cdots, \omega^{p_k}) \tag{4}$$

in $\Omega^{|p|+1}(X, \mathbb{R})$. The base case when $p_1 = \cdots = p_k = 0$ is because $\tilde{c}$ is a parallel section in

$$P \times_\rho \operatorname{Hom}(\bigotimes_k \mathfrak{g}, \mathbb{R}).$$

To deal with the general case, use the defining property of $\tilde{c}$ in local coordinate charts: if $\omega^{p_i} = \nu^{p_i} \otimes b_i$, then

$$\tilde{c}(\omega^{p_1}, \cdots, \omega^{p_k}) = v^{p_1} \wedge \cdots \wedge v^{p_k} \otimes \tilde{c}(b_1, \cdots, b_k).$$

Now apply (3) and the base case of (4) to conclude. $\square$

Lecture 6. The Chern-Simons Functional

6.1. The Chern-Simons Functional

Let $c : \mathfrak{g} \to \mathbb{R}$ be a homogeneous polynomial of degree k. For any pair of connections (A_0, A_1), there is a canonical way to assign a form $\mathfrak{cs}_c(A_0, A_1)$ so that

$$c(F_{A_1}, \cdots, F_{A_1}) - c(F_{A_0}, \cdots, F_{A_0}) = d\mathfrak{cs}_c(A_0, A_1)$$

and it is explicitly computed by

$$\mathfrak{cs}_c(A_0, A_1) = k \int_0^1 \tilde{c}(a, F_{A_t}, \cdots, F_{A_t}) dt.$$

We specialize to dimension 3 and 4 and let $\deg c = 2$. For semi-simple Lie groups, all homogeneous polynomials of degree 2 are multiplies of the Killing form

$$B(\xi, \eta) = -\operatorname{tr}(\operatorname{ad}\xi \operatorname{ad}\eta).$$

Usually, we take the structure group G to be $SU(n)$, $SO(n)$, $Sp(n)$ and $PU(n) = SU(n)/\mathbb{Z}_n$. For the first two examples, we have fundamental representations and define

$$B'(\xi, \eta) = -\operatorname{tr}(\xi\eta)$$

for $\xi, \eta \in \mathfrak{g}$. This polynomial function is related to the Killing form by the dual Coxeter number $h^\vee$:

$$B = h^\vee B'$$

From now on, fix $c = -\frac{1}{2}\operatorname{tr}$ in the fundamental representation. Then

$$\mathfrak{cs}_c(A_0, A_1) = -\int_0^1 \operatorname{tr}(a \wedge F_{A_t}) dt \tag{5}$$

Using the relation $F_{A_t} = F_{A_0} + t d_{A_0} a + \frac{t^2}{2}[a \wedge a]$, we get

$$\begin{aligned}\mathfrak{cs}_c(A_0, A_1) &= -\int_0^1 \operatorname{tr}(a \wedge (F_{A_0} + t d_{A_0} a + \frac{t^2}{2}[a \wedge a])) dt \\ &= -\operatorname{tr}(a \wedge (F_{A_0} + \frac{1}{2} d_{A_0} a + \frac{1}{6}[a \wedge a])).\end{aligned}$$

On Y^3, define the Chern-Simons functional by

$$\begin{aligned}\mathcal{CS}(A_0, A_1) &= \int_Y \mathfrak{cs}_c(A_0, A_1) \\ &= -\int_Y \operatorname{tr}(a \wedge (F_{A_0} + \frac{1}{2} d_{A_0} a + \frac{1}{6}[a \wedge a]))\end{aligned}$$

How does $\mathcal{CS}$ change under gauge transformations? An automorphism of $P \to Y^3$ is also a section $g : Y \to \operatorname{Ad} P$. Its action on $\mathscr{A}_P$ is defined by pulling back connections:

$$A \mapsto gAg^{-1}.$$

We wish to compute

$$\mathcal{CS}(A_0, gA_0).$$

We form a new bundle over $S^1 \times Y$ by taking the quotient

$$P_g = [0,1] \times P/(0,p) \sim (1, g(p)).$$

Then the connection

$$\tilde{A} = \frac{\partial}{\partial t} + (1-t)A_0 + tg(A_0)$$

thought of as a connection on $[0,1] \times Y$ descends to a connection on P_g. Therefore,

$$\begin{aligned}\mathcal{CS}(A_0, g(A_0)) &= -\frac{1}{2}\int_{S^1\times Y} \operatorname{tr}(F_{\tilde{A}} \wedge F_{\tilde{A}}) \\ &= \langle \text{A characteristice class of } P_g, [S^1 \times Y]\rangle \\ &\in \alpha\mathbb{Z} \subset \mathbb{R}.\end{aligned}$$

for some $\alpha \in \mathbb{R}$.

This shows $\mathcal{CS}$ is not fully gauge invariant under the action of $\mathscr{G} = \Gamma(X, \operatorname{Ad} P)$.

Theorem 6.1.1. *Let*

$$\mathscr{A}_P = \textit{connections on } P \to Y^3.$$
$$\mathscr{G}_P = \textit{gauge transformations of } P.$$

Then the Chern Simons functional is well-defined on the quotient space as an S^1-valued function:

$$\mathcal{CS} : \mathscr{A}_P/\mathscr{G}_P \to \mathbb{R}/\alpha\mathbb{Z}.$$

So we are in the Novikov-Morse setting. If $G = SU(2)$, then $\alpha = 4\pi^2$ and

$$[\frac{1}{8\pi^2}\operatorname{tr}(F_A \wedge F_A)] \in H^4(X;\mathbb{Z})$$

is the second Chern class $c_2(E)$. Here, E is the complex 2-plane bundle induced by the fundamental representation:

$$E = P \times_\rho \mathbb{C}^2 \to X.$$

6.2. Classifications of $SU(n)$-bundles on Y^3

Since the first cell of $BSU(n)$ is in dimension 4:

$$BSU(n) = e^4 \cup \text{higher dimensional cells},$$

each map $Y^3 \to BSU(n)$ is null-homotopic by cellular approximation. This implies all $SU(n)$ bundles are trivial over Y^3.

Here is a lowbrow proof. Suppose $E \to Y^3$ is the induced n-plane bundle. Since the zero section

$$s_0 : Y \to E$$

has co-dimension $2n > 3$, we can find a perturbed section s so that $s \cap s_0 = \varnothing$. We assume s has unit length. This section reduces the structure group to $SU(n-1) \oplus \{e\}$. When $n = 2$, the structure group is $SU(2)$, so there is a canonical section in $\{s\}^{\perp}$ and E is a trivial bundle.

Similarly, we can classify $U(n)$-bundle and $PU(n)$-bundles:

(1) $U(n)$-bundles. The lowbrow argument above allows us to write:

$$E \cong \mathbb{C}^{n-1} \oplus \wedge^n E.$$

Since line bundles are classified by their first Chern classes c_1, P is classified by $c_1(\wedge^n E) = c_1(E)$. The highbrow point of view is that

$$\det : U(n) \to S^1$$

induces an isomorphism.

$$\wedge^n : [Y, BU(n)] \to [Y, BU(1)].$$

The embedding $\iota : S^1 \to U(n), e^{i\theta} \mapsto diag\{e^{i\theta}, 1, \cdots\}$ gives a map $BU(1) \to BU(n)$ that induces a right inverse to $\wedge^n$, so $\wedge^n$ is surjective. Since each map $f : Y \to BU(n)$ is homotopic to a map whose image is contained in the 2-skeleton $BU(n)^{(2)} \subset BU(1)$, $\wedge^n$ is also injective.

(2) $PU(n)$-bundles. A principal bundle is also classified by the first Čech cohomology group

$$\check{H}^1(Y, PU(n)).$$

and it fits into a long exact sequence:

$$\to \check{H}^1(Y, SU(n)) \to \check{H}^1(Y, PU(n)) \xrightarrow{\iota} \check{H}^2(Y, \mathbb{Z}_n)$$

which is induced by the short exact sequence

$$0 \to \mathbb{Z}_n \to SU(n) \to PU(n) \to 0.$$

Since $\check{H}^1(Y, SU(n)) = 0$, ι is injective. To show it is surjective, we need to solve the lifting problem:

$$\begin{tikzcd} & BSU(n) \arrow[r] & BPU(n) \arrow[d] \\ & Y^3 \arrow[r] \arrow[ur, dashed] & K(\mathbb{Z}_n, 2) \end{tikzcd}$$

Since $\pi_i(BSU(n)) = 0$ for $0 \leqslant i \leqslant 3$, the first obstruction class lies in $H^5(Y^3, \pi_4(BSU(n)))$ which is zero for dimension reasons.

Note that a $PU(n)$-bundle P doesn't necessarily come from a $U(n)$-bundle. This is determined by whether $\iota([P]) \in H^2(Y, \mathbb{Z}_n)$ has an integral lift in $H^2(Y, \mathbb{Z})$.

6.3. The gradient

We study the first variation of $\mathcal{CS}(A) = \mathcal{CS}(A_0, A)$:

$$\frac{\partial}{\partial t}\mathcal{CS}(A + ta)\big|_{t=0} = -\int_Y \operatorname{tr}(a \wedge F_A). \tag{6}$$

for any $a \in \Omega^1(\operatorname{ad} P)$. Therefore, the first variation is zero if and only if

$$F_A \equiv 0$$

which means A is flat. We get the bijection:

$$\{\text{Critical points in } \mathscr{A}_P/\mathscr{G}_P\} \Leftrightarrow$$
$$\{\text{Representations of } \pi_1(Y) \to G \text{ modulo conjugacy}\}$$

In higher dimensions, it will be a wedge product of F_A and we do not have good geometric interpretations for critical points. This is a small miracle in dimension 3.

By choosing a Riemannian metric on Y, we endow the tangent space

$$T\mathscr{A}_P = \mathscr{A}_P \times \Omega^1(\operatorname{ad} Y).$$

with a metric. Given $a, b \in T_A\mathscr{A}_P = \Omega^1(\operatorname{ad} P)$, let

$$\langle a, b\rangle = -\int_Y \operatorname{tr}(a \wedge *b)$$

where $*$ is the Hodge $*$-operator on forms. Therefore, (up to normalization of $\mathcal{CS}$)

$$\nabla_A \mathcal{CS} = *F_A.$$

The downward gradient flow is

$$\frac{\partial}{\partial t}A = - * F_A. \tag{7}$$

A big miracle occurs here. The initial value problem of this evolutionary equation (7) is not well-posed, just like the backward heat equation. This is because the Hessian $- * d_A$ (acting on $T_{[A]}(\mathscr{A}_P/\mathscr{G}_P)$) has infinitely many positive and negative eigenvalues. However, we could translate the problem in dimension 4 and it will give an elliptic theory, as we explain now.

Let X^4 be an oriented Riemannian manifold. The Hodge $*$-operator acting on Λ^2 satisfies $*^2 = 1$, so Λ^2 splits as

$$\Lambda^2 = \Lambda^+ \oplus \Lambda^-$$

where $* = \pm 1$ on $\Lambda^\pm$. For an orthogonal framing $\{e_1, e_2, e_3, e_4\}$,

$$e_1 \wedge e_2 \pm e_3 \wedge e_4, e_1 \wedge e_3 \pm e_4 \wedge e_2, e_1 \wedge e_4 \pm e_2 \wedge e_3,$$

gives a framing for $\Lambda^\pm$. Given a principal G-bundle $P \to X$ and a connection A, we impose the (anti)-self dual Yang-Mills equations:

$$F_A = \pm *_4 F_A \tag{8}$$

Let $X = \mathbb{R} \times Y$ and $P = \mathbb{R} \times Q$ with $Q \to Y$ a principal G-bundle. A connection A of P can be written as

$$A = \tilde{B} + dt \otimes c(t)$$

where $c(t) \in \Omega^0(\operatorname{ad} Q)$ and $B(t)$ is a family of connections on Q such that

$$\tilde{B} = \frac{\partial}{\partial t} + B(t).$$

Therefore,

$$\begin{aligned} F_A &= F_{\tilde{B}} + d_{\tilde{B}}(dt \otimes c(t)) + \underbrace{\frac{1}{2}[dt \otimes c(t) \wedge dt \otimes c(t)]}_{=0} \\ &= F^{(3)}_{B(t)} + dt \wedge \frac{\partial}{\partial t} B(t) - dt \wedge d_{B(t)} c(t). \end{aligned} \tag{9}$$

The orientation convention is $dt \wedge dvol_Y = dvol_X$. Let ω be a 2-form on Y and ν be a 1-form, then

$$*_4\omega = dt \wedge *_3\omega,\ \ *_4(dt \wedge \nu) = *_3\nu.$$

Apply $*_4$ to F_A:

$$*_4F_A = dt \wedge *_3F^{(3)}_{B(t)} + *_3(\frac{\partial}{\partial t}B(t) - d_{B(t)}c(t)).$$

The equation (8) becomes

$$\frac{\partial}{\partial t}B(t) = \underbrace{\pm *_3 F^{(3)}_{B(t)}}_{\pm\nabla_B \mathcal{CS}} + d^{(3)}_{B(t)}c(t)$$

We recognize that the first part is the gradient of the Chern-Simons functional. Comparing with the downward gradient flow equation (7), the second part is new and it is tangential to the gauge group orbit at $B(t)$. Equation (7) is only invariant under the 3-dimensional gauge group while the (anti)-dual equation (8) admit a larger symmetric group (invariant under the 4-dimensional gauge group). This will allows us to produce an elliptic theory and it will be the task for the next lecture.

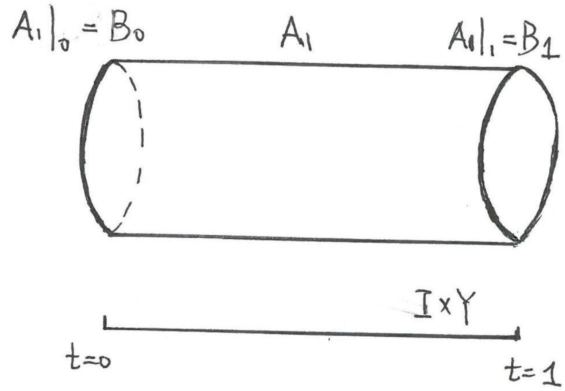
Exercise 6.3.1. *Compute the first variation* (6) *of* $\mathcal{CS}$.

Proof. Let (B_0, B_1) be a pair of connections on Y and

$$A_0 = \frac{\partial}{\partial t} + B_0$$

be the product connection on $P = I \times Q \to I \times Y$. Suppose A_1 is any connection of P such that

$$A_1|_{t=0} = B_0, A_1|_{t=1} = B_1.$$



Let $c = -\frac{1}{2}\operatorname{tr} : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$. Since F_{A_0} does not have dt-component, $c(F_{A_0}) = 0$. Therefore,

$$\begin{aligned}
\int_{I\times Y} c(F_{A_1}) &= \int_{I\times Y} c(F_{A_1}) - \int_{I\times Y} c(F_{A_0}) \\
&= \int_{I\times Y} d\mathfrak{cs}_c(A_0, A_1) \\
&= \int_{\{1\}\times Y} \mathfrak{cs}_c(B_0, B_1) - \int_{\{0\}\times Y} \mathfrak{cs}_c(B_0, B_0) \\
&= \int_{\{1\}\times Y} \mathfrak{cs}_c(B_0, B_1) = \mathcal{CS}(B_0, B_1).
\end{aligned}$$

In the middle, we used the fact $\mathfrak{cs}_c(A_0, A_1) = \mathfrak{cs}_c(B_0, B_1)$ on $\{1\} \times Y$. This is due to formula (9), because the normal component $c(t)$ does not contribute to $F_A|_{\{t\}\times Y}$. This shows the functional $\mathcal{CS}(B_0, B_1)$ is additive:

$$\mathcal{CS}(B_0, B_2) = \mathcal{CS}(B_0, B_1) + \mathcal{CS}(B_1, B_2).$$

Let $B_s = B + sb$. Using formula (5), we get

$$\begin{aligned}
\frac{\partial}{\partial t}\mathcal{CS}(B_0, B + tb)\big|_{t=0} &= \lim_{t\to 0} \frac{1}{t}\mathcal{CS}(B, B + tb) \\
&= -\lim_{t\to 0} \int_Y \int_0^1 \operatorname{tr}(b \wedge F_{B_{st}}) ds \\
&= -\int_Y \operatorname{tr}(b \wedge F_B).
\end{aligned}$$

□

Lecture 7. Representations

Let Y^3 be a closed 3-manifold and $P = SU(2)\times Y \to Y$ be the trivial $SU(2)$-bundle. Critical points of the Chern-Simons functional

$$\mathcal{CS} : \mathscr{A}_P \to \mathbb{R}$$

are flat connections. We choose the background connection A_0 to be the trivial one. A gauge transformation g is a section of the adjoint bundle $\operatorname{Ad} P \to Y$. In our case, it is simply a map

$$g : Y^3 \to SU(2),$$

because P is a trivial bundle. The value of the functional changes equivariantly under gauge transformations:

$$\mathcal{CS}(g\cdot A) - \mathcal{CS}(A) = \alpha \deg(g),$$

where $\alpha = -4\pi^2$ in our convention, so we get a circle valued functional on the quotient space

$$\mathcal{CS} : \mathscr{A}_P/\mathscr{G}_P \to \mathbb{R}/\alpha\mathbb{Z}.$$

The space of flat connections and values of $\mathcal{CS}$ on them are all invariants of Y^3. In this lecture, we will see computations for some special 3-manifolds.

7.1. Lens Spaces

For (p, q) co-prime, the Lens space $L(p, q)$ is defined as the quotient space

$$S^3/(\mathbb{Z}/p\mathbb{Z})$$

where we view S^3 as the unit sphere of $\mathbb{C}^2$ and the $\mathbb{Z}/p\mathbb{Z}$-action is given by

$$u = \underline{1} \in \mathbb{Z}/p\mathbb{Z} \mapsto d(u) : (z_1, z_2) \mapsto (z_1 \cdot e^{\frac{2\pi i}{p}}, z_2 \cdot e^{\frac{2\pi i q}{p}}).$$

It is easy to see $\pi_1(L(p, q)) = \mathbb{Z}/p\mathbb{Z}$ and any $SU(2)$-representation is conjugate to

$$\begin{aligned} \rho_n : \pi_1(L(p,q)) &\mapsto SU(2) \\ u &\mapsto \begin{pmatrix} e^{2\pi i n/p} & 0 \\ 0 & e^{-2\pi i n/p} \end{pmatrix} \end{aligned}$$

for some $n \in \mathbb{Z}/p\mathbb{Z}$. The value of $\mathcal{CS}/4\pi^2$ on ρ_n is

$$\frac{n^2 r}{p} \mod 1$$

where $rq = 1 \mod p$. These numbers are independent of metrics and will determine $L(p, q)$ up to homotopy equivalence, but not up to homeomorphism. Still, they give more information than just $\pi_1(L(p, q))$. Note that $\rho_n \cong \rho_{p-n}$ since

$$\begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix}, \begin{pmatrix} g^{-1} & 0 \\ 0 & g \end{pmatrix}$$

are conjugate in $SU(2)$.

Exercise 7.1.1. *Compute the value of* $\mathcal{CS}(\rho_n)$.

Proof. Let A_n be the flat connection associated to ρ_n, then the pull-back connection $\pi^* A_n$ on $\pi^* P \to S^3$ is also flat. On S^3, all flat connections are gauge-equivalent to the trivial connection A_0', so

$$\pi^* A_n = g^*(A_0').$$

for some $g : S^3 \to SU(2)$. Let $a = A_n - A_0 \in \Omega^1(L(p,q); \operatorname{Ad} P)$, then

$$\begin{aligned}\mathcal{CS}_{L(p,q)}(A_n) &= \int_{L(p,q)} \mathfrak{cs}_c(A_0, A_n) = \frac{1}{p}\int_{S^3} \mathfrak{cs}_c(\pi^* A_0, \pi^* A_n)\\ &= \frac{1}{p}\mathcal{CS}_{S^3}(A_0', \pi^* A_n) = -\frac{4\pi^2}{p}\deg(g).\end{aligned}$$

If g factorizes through $L(p,q)$, then $\deg(g)$ is divisible by p. We need to determine

$$\deg(g) \mod p.$$

What we need is an explicit $\mathbb{Z}/p\mathbb{Z}$-equivariant map $S^3 \to SU(2) \subset \mathbb{H} \cong \mathbb{C}^2$ intertwining the Deck transformations d and the action of ρ_n. One example is constructed as

$$g(z_1, z_2) = \frac{(z_1^n, \bar{z}_2^{nr})}{(|z_1|^{2n} + |z_2|^{2nr})^{\frac{1}{2}}}.$$

It is easy to see $\deg(g) = -n^2 r$ and our result follows. $\square$

7.2. The three torus

Let $Y = \mathbb{T}^3$. Then $\pi_1(Y) \cong \mathbb{Z}^3$ is abelian. A representation

$$\rho : \pi_1(Y) \to SU(2)$$

gives three commuting matrices A_0, A_1, A_2 and they can be diagonalized simultaneously in $SU(2)$. Hence, we may assume

$$A_k = \begin{pmatrix} e^{2\pi i \alpha_k} & 0 \\ 0 & e^{-2\pi i \alpha_k} \end{pmatrix} \in S^1 \subset SU(2)$$

for $0 \leqslant k \leqslant 2$. Note that

$$\operatorname{Pin}(2) = S^1 \bigcup S^1 \cdot \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

is the largest subgroup of $SU(2)$ such that $S^1 \lhd \operatorname{Pin}(2)$ and it is a semi-direct product

$$\operatorname{Pin}(2) = S^1 \rtimes \mathbb{Z}/2\mathbb{Z}.$$

The $\mathbb{Z}/2\mathbb{Z}$ factor acts on S^1 by taking inverse:

$$u \mapsto u^{-1}.$$

In general, we use

$$\mathcal{R}_G(Y) = \{\rho : \pi_1(Y) \to G\}/\text{conjugations}$$

to denote the representation variety. Then,

$$\mathcal{R}_{SU(2)}(\mathbb{T}^3) = (S^1)^3/(\mathbb{Z}/2\mathbb{Z}).$$

The $\mathbb{Z}/2\mathbb{Z}$-action comes from the conjugation by

$$j = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

It is slightly easier to think about the two torus $\mathbb{T}^2$:

$$\mathcal{R}_{SU(2)}(\mathbb{T}^2) = (S^1)^2/(\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{CP}^1 = S^2.$$

In general, if G is a semi-simple compact Lie group with $\pi_1(G)$ trivial, an element of $\mathcal{R}_G(\mathbb{T}^2)$ is given by a pair of elements in the same maximal torus, so

$$\mathcal{R}_G(\mathbb{T}^2) = \mathbb{T}_G \times \mathbb{T}_G/\text{the action of Weyl group of } G.$$

and this is a weighted projective space by a theorem of [??]. For instance, for $G = SU(n)$,

$$\mathcal{R}_{SU(n)}(\mathbb{T}^2) = \mathbb{T}_{SU(n)} \times \mathbb{T}_{SU(n)}/S_n \cong \mathbb{CP}^{n-1}.$$

In this case, $\dim \mathbb{T}_{SU(n)} = n-1$ and the Weyl group is the permutation group S_n. A representation that is close to the trivial representation A_0 is given by a pair of elements in $\mathbb{R}^{n-1}$ and so an element in $\mathbb{C}^{n-1}$. We embed $\mathbb{C}^{n-1}$ into $\mathbb{C}^n$ as the zero locus of $\sum_{i=1}^n x_i$. The S_n-action is given by permuting $n-$axes, so a point in the quotient space is parametrized by elementary symmetric polynomials:

$$\sigma_1 = \sum x_i = 0, \sigma_2 = \sum_{i \neq j} x_i x_j, \cdots, \sigma_n = \prod x_i.$$

This shows $\mathcal{R}_{SU(n)}(\mathbb{T}^2)$ is a manifold of complex dimension $n-1$ in a neighborhood of A_0.

7.3. Seifert Fibered Spaces

A good resource for what we will discuss is

- Fintushel, Ronald, Stern, Ronald J. Instanton homology of Seifert fibred homology three spheres. Proc. London Math. Soc.(3) 61 (1990), no. 1, 109-137.

Given a triple (p, q, r) of positive integers that are pairwise co-prime, consider the polynomial function:

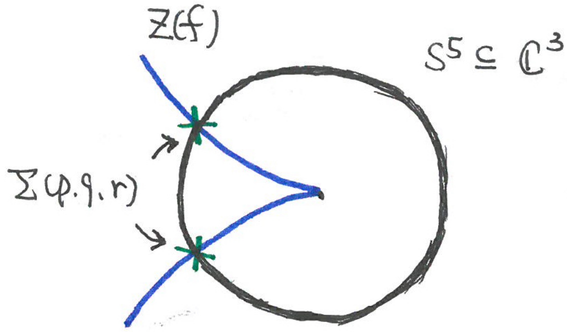
$$f(z_1, z_2, z_3) = z_1^p + z_2^q + z_3^r$$

on $\mathbb{C}^3$. It has a single singular point at the origin. The Brieskorn homology spheres $\Sigma(p, q, r)$ is defined by taking the intersection of the unit sphere with $f^{-1}(0)$:

$$\Sigma(p, q, r) = S^5 \cap \{z_1^p + z_2^q + z_3^r = 0\} \subset \mathbb{C}^3.$$

It admits an S^1-action:

$$u \in S^1 : (z_1, z_2, z_3) \mapsto (u^{qr} z_1, u^{pr} z_2, u^{pq} z_3).$$



The quotient space $\Sigma(p,q,r)/S^1$ is an orbifold S^2 with three orbifold points:

$$S^1 \to \Sigma(p,q,r) \to \Sigma(p,q,r)/S^1 = S^2(p,q,r).$$

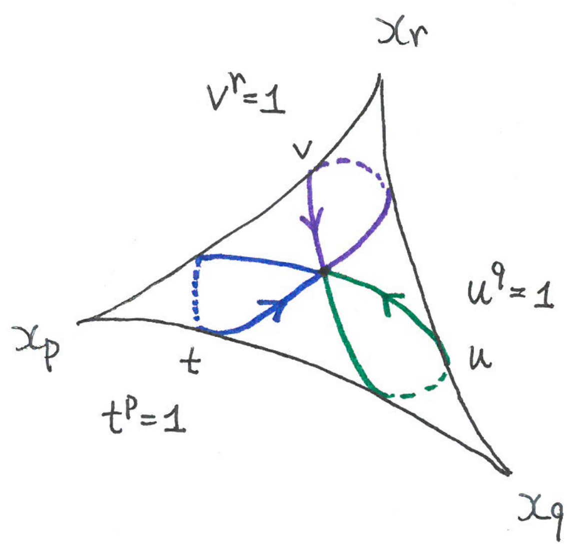
To compute $\pi_1(\Sigma(p,q,r))$, we can still use the long exact sequence of homotopy groups as long as we use the orbifold fundamental group of $S^2(p,q,r)$:

$$1 \to \pi_1(S^1) = \mathbb{Z} \to \pi_1(\Sigma(p,q,r)) \to \pi_1^{orb}(S^2(p,q,r)) \to 1$$

and $\pi_1(S^1)$ is in the center of $\pi_1(\Sigma(p,q,r))$. One presentation of the orbifold fundamental group of $S^2(p,q,r)$ is given by

$$\{t,u,v : t^p = 1, u^q = 1, v^r = 1, tuv = 1\}$$

where t,u,v corresponds three small loops for each of orbifold points.



This is the triangle group T_{pqr}. Therefore, $\pi_1(\Sigma(p,q,r))$ admits a presentation as

$$\{\tilde{t},\tilde{u},\tilde{v} : h \text{ central}, \tilde{t}^p = h^{-b_1}, \tilde{u}^q = h^{-b_2}, \tilde{v}^r = h^{-b_3}, \tilde{t}\tilde{u}\tilde{v} = h^{-b}\}$$

for some $b_i, b \in \mathbb{Z}$. These number could be chosen (with more work required) such that

$$-b + \frac{b_1}{p} + \frac{b_2}{q} + \frac{b_3}{r} = \frac{1}{pqr},$$

from which one can easily deduce that $\Sigma(p,q,r)$ is an integral homology sphere.

A representation of $\pi_1(\Sigma(p,q,r))$ into $SU(2)$ sends its center into $Z(SU(2)) \cong \{\pm 1\}$, so the image of h is either 1 or -1. Let's make it simpler by looking at its image in $SO(3) = SU(2)/Z(SU(2))$:

$$\begin{tikzcd} \pi_1(\Sigma(p,q,r)) \arrow[r, "\rho"] \arrow[rr, bend left, "\bar{\rho}"] & SU(2) \arrow[r, "\pi"] & SO(3). \end{tikzcd}$$

Then $\bar{\rho}(h) = 1$ and $\bar{\rho}(\pi_1(\Sigma(p,q,r)))$ is generated by

$$T = \bar{\rho}(\tilde{t}), U = \bar{\rho}(\tilde{u}), V = \bar{\rho}(\tilde{v}).$$

This is equivalent to the representation of the triangle group T_{pqr} into $SO(3)$. Suppose

$$\begin{aligned} T &= \text{the rotation by } 2\alpha = \frac{2\pi k}{p} \text{ about an axis } l_t \\ U &= \text{the rotations by } 2\beta = \frac{2\pi l}{q} \text{ about an axis } l_u \\ V &= \text{the rotations by } 2\gamma = \frac{2\pi m}{r} \text{ about an axis } l_v. \end{aligned}$$

Then $TUV = 1 \in SO(3)$ gives a constraint for these angles and axes. Assuming $\bar{\rho}$ is irreducible (does not factorize through an S^1), these axes are not coplanar. Each two of them determine a great circle on $S^2 \subset \mathbb{R}^3$:

$$S_{uv}, S_{tv}, S_{tu}.$$

and divide S^2 into eight spherical triangles whose angles are given by (they come in pairs):

$$\{(\alpha,\beta,\gamma), (\alpha, \pi-\beta, \pi-\gamma), (\pi-\alpha, \pi-\beta, \gamma), (\pi-\alpha, \beta, \pi-\gamma)\}.$$

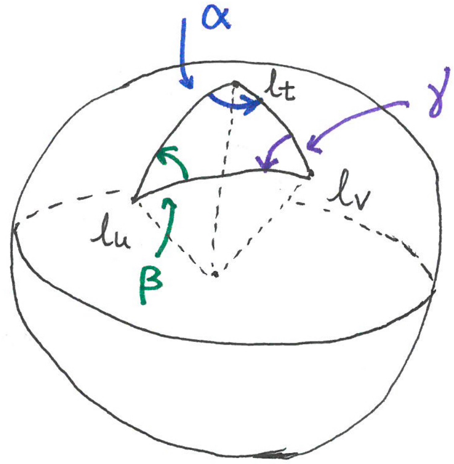
Exercise 7.3.1. *Prove this constraint holds*

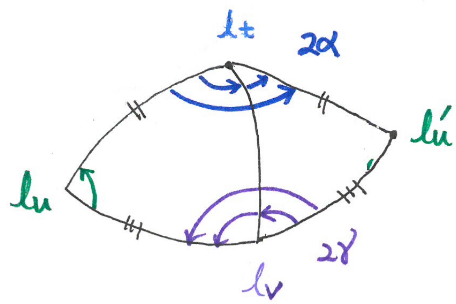
Proof. Without loss of generality, we assume

$$0 \leqslant 2\alpha, 2\beta, 2\gamma < \pi,$$

and rotations are right-handed. We focus on the spherical triangle spanned by $l_t, l_u, l_v \in S^2$. Let $l_u' = V^{-1}(l_u)$. Since $TUV = \mathrm{Id}$ and l_u is a fixed point of U on S^2, we have

$$T(l_u) = TU(l_u) = TUV(l_u') = l_u'.$$





This implies that

$$d(l_t, l_u) = d(l_t, l_u'), d(l_v, l_u) = d(l_v, l_u').$$

Hence, $\triangle l_t l_u l_v \cong \triangle l_t l_u' l_v$ and

$$\angle l_u l_t l_v = \angle l_u' l_t l_v = \frac{1}{2}\angle l_u' l_t l_u = \frac{1}{2}(2\alpha) = \alpha.$$

$\square$

Therefore, we see the space of $SO(3)$ representations are discrete and consist of finitely many points parametrized by suitable triples of integers (k, l, m).

For the quotient map, we could form the mapping cylinder $C\pi$ of

$$\pi : \Sigma(p, q, r) \to S^2(p, q, r).$$

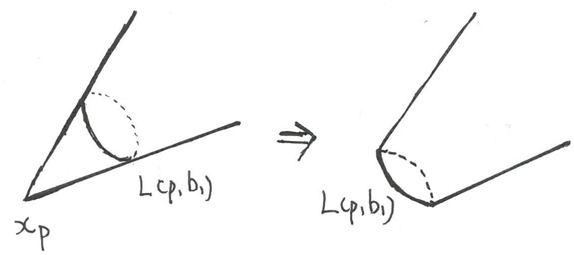
Note that if π is a genuine S^1-fiber bundle, then $C\pi$ is the associated disk bundle. This shows near a smooth point of $S^2(p, q, r)$, $C\pi$ is a smooth manifold. Globally, $C\pi$ is an orbifold with boundary

$$\partial C\pi = \Sigma(p, q, r).$$

Three orbifold points lie in $S^2(p,q,r)$ with links

$$L(p,b_1), L(q,b_2), L(r,b_3),$$

so each of these orbifold points has a neighborhood homeomorphic to a cone over a



Lens space. By removing these neighborhoods, we obtain a cobordism:

$$W = C\pi - (N_p \cup N_q \cup N_r) : L(p,b_1) \cup L(q,b_2) \cup L(r,b_3) \to \Sigma(p,q,r).$$

Take the trivial principal bundle P over W and pick a flat connection A of P. By the defining proper of $\mathcal{CS}$, we have

$$\begin{aligned} 0 &= \int_W c(F_A) - \int_W c(F_{A_0}) \\ &= \int_{\Sigma(p,q,r)} \mathfrak{cs}_c(A_0, A) - \int_{\cup L} \mathfrak{cs}_c(A_0, A) \\ &= \mathcal{CS}_{\Sigma(p,q,r)}(A|_{\Sigma(p,q,r)}) - \sum_L \mathcal{CS}_L(A|_L). \end{aligned}$$

Therefore, the computation is reduced to Lens spaces. This property was used by Furuta to study the homology cobordism group Θ_3^H of oriented homology 3-spheres.

Finally, for Seifert fibered spaces over S^2 with $n \geqslant 4$ singular fibers, we will get a polygon constraint. Triangles are rigid, but spherical squares could have non trivial moduli. When $n = 4$, it will be $S^2 \vee S^2$ and for $n = 5$, different rational manifolds could occur as the moduli space.

As an ending remark, for a Riemann surface Σ of genus g, we have the Narasimhan-Seshadri theorem

$$\begin{aligned} &\{\text{Unitary representations of } \pi_1(\Sigma_g)\} \\ \Leftrightarrow &\{\text{(semi)-stable holomorphic vector bundles of } \deg 0 \text{ over } \Sigma_g\}. \end{aligned}$$

Furthermore, for a Riemann surface $\Sigma_g(p_1, \cdots, p_k)$ with some marked points, we have

$$\begin{aligned} &\{\text{Unitary representations of } \pi_1(\Sigma_g - \{p_1, \cdots, p_k\}) \\ &\quad \text{with fixed conjugacy classes of the monodromy around these points}\} \\ \Leftrightarrow &\{\text{(semi)-stable parabolic holomorphic vector bundles over } \Sigma_g(p_1, \cdots, p_k)\}. \end{aligned}$$

Exercise 7.3.2. *Show that the link around each singular point of $C\pi$ is a Lens space.*

Proof. The S^1-action on $\Sigma(p,q,r)$ is free if coordinates are non-zero. There are three orbits whose stablizers are non-trivial,

$$\begin{gathered}\{(0,z_2,z_3) : z_2^q + z_3^r = 0, |z_2|^2 + |z_3|^2 = 1\} \text{ with stablizer} = \{e^{\frac{2\pi i}{p}}\}\\ \{(z_1,0,z_3) : z_1^p + z_3^r = 0, |z_1|^2 + |z_3|^2 = 1\} \text{ with stablizer} = \{e^{\frac{2\pi i}{q}}\}\\ \{(z_1,z_2,0) : z_1^p + z_2^q = 0, |z_1|^2 + |z_2|^2 = 1\} \text{ with stablizer} = \{e^{\frac{2\pi i}{r}}\}.\end{gathered}$$

We will focus on the first one. Its local model is given by

$$\begin{aligned}u \in S^1 : S^1 \times D^2 &\to S^1 \times D^2\\ u(e^{i\theta}, z) &= (e^{i\theta}u^p, z\cdot u^{qr})\end{aligned}$$

Consider the p-fold cover of $S^1 \times D^2$:

$$\begin{aligned}h : \tilde{X} = S^1 \times D^2 &\to S^1 \times D^2\\ (e^{i\theta}, w) &\to (e^{ip\theta}, w\cdot e^{iqr\theta}).\end{aligned}$$

and an S^1-action on $\tilde{X}$ is defined by setting

$$\begin{aligned}u : \tilde{X} = S^1 \times D^2 &\to S^1 \times D^2\\ (e^{i\theta}, w) &\to (e^{i\theta}u, w).\end{aligned}$$

which makes h an S^1-equivariant covering map. We obtain a commutative diagram by taking quotient maps:

$$\begin{tikzcd} \tilde{X} \arrow[r, "h"] \arrow[d, "\tilde{\pi}"', "/S^1"] & S^1 \times D^2 \arrow[d, "\pi"', "/S^1"] \\ D^2 \arrow[r, "/(\mathbb{Z}/p\mathbb{Z})"] & D^2 \end{tikzcd} \tag{10}$$

where the bottom horizontal maps is given by taking the p-th power:

$$w \mapsto w^p.$$

Note that $\tilde{\pi}$ is a trivial S^1-bundle and its mapping cone is simply a product. It admits a $\mathbb{Z}/p\mathbb{Z}$-action which extends Deck transformations on $\tilde{X}$:

$$\begin{aligned}a = e^{\frac{2\pi i}{p}} : D^2 \times D^2 &\to D^2 \times D^2\\ (\zeta, w) &\mapsto (\zeta\cdot a, w\cdot a^{-qr})\end{aligned}$$

Taking mapping cones for vertical maps in (10), we obtain a map between $C\tilde{\pi}$ and $C\pi$:

$$C\tilde{\pi} = D^2 \times D^2 \to C\pi$$

which is the quotient map by the $\mathbb{Z}/p\mathbb{Z}$-action. The orbifold point x_p comes from the origin and this map gives a orbifold chart. Therefore, the link of x_p is

$$\partial(D^2 \times D^2)/(\mathbb{Z}/p\mathbb{Z}).$$

which is the definition of $L(p, -qr)$. $\square$

Lecture 8. The Energy Identity and Instantons on S^4

8.1. Characteristic Classes

Let $\mathrm{Sym}(\mathfrak{g})$ be the space of polynomial functions on $\mathfrak{g}$ and

$$\mathrm{Sym}_{\mathrm{Ad}}(\mathfrak{g})$$

be the subset of Ad-invariant functions. Take $\mathfrak{g} = \mathfrak{u}(n)$ to be the Lie algebra of $U(n)$ and $\mathfrak{t}_n$ be its Cartan sub-algebra. The Weyl group of $U(n)$ is the permutation group S_n and its action on $\mathfrak{t}_n \cong \mathbb{C}^n$ is permuting coordinates. Then,

$$\begin{aligned}\mathrm{Sym}_{\mathrm{Ad}}(\mathfrak{u}(n)) &= \mathrm{Sym}_{S_n}(\mathfrak{t}_n)\\ &= \{\text{Symmetric polynomials of } n \text{ variables}\}.\end{aligned}$$

Let c_i be the i-th elementary symmetric polynomial σ_i. For a connection A on a principal $U(n)$-bundle, let

$$c_i(A) := c_i(\frac{i}{2\pi}F_A).$$

We have $\mathrm{tr}(F_A \wedge F_A) \in \Omega^4(X)$, and we wish to compare it with

$$c_2(A) \in \sigma_2(\frac{i}{2\pi}F_A) \text{ and } c_1^2(A) = (\mathrm{tr}(\frac{i}{2\pi}F_A))^2.$$

Suppose

$$F_A = \begin{pmatrix} ia & b+ic \\ -b+ic & id \end{pmatrix}$$

for some real-valued 2-forms $a, b, c, d \in \Omega^2(X)$. Then

$$\begin{aligned}\mathrm{tr}(F_A \wedge F_A) &= -(a^2 + d^2 + 2b^2 + 2c^2)\\ c_2(A) &= \det(\frac{i}{2\pi}\begin{pmatrix} ia & b+ic \\ -b+ic & id \end{pmatrix}) = -\frac{1}{4\pi^2}(-ad + b^2 + c^2)\\ c_1(A)^2 &= (\mathrm{tr}(\frac{i}{2\pi}F_A))^2 = \frac{1}{4\pi^2}(a+d)^2.\end{aligned}$$

Therefore,

$$\frac{1}{8\pi^2}\mathrm{tr}(F_A \wedge F_A) = c_2(A) - \frac{1}{2}c_1(A)^2.$$

For an $SU(2)$-bundle, it reduces to

$$\frac{1}{8\pi^2}\mathrm{tr}(F_A \wedge F_A) = c_2(A).$$

Consider $SO(3)$-bundles or more generally,

$$PU(n) = SU(n)/\mathbb{Z}_n$$

-bundles. When $n = 2$, $SU(2)/\mathbb{Z}_2 \cong SO(3)$. Pontryagin classes of a real vector bundle $V \to X$ is defined by

$$p_k = (-1)^k c_{2k}(V \otimes_{\mathbb{R}} \mathbb{C}).$$

Suppose P is an $SO(3)$-bundle induced from an $U(2)$-bundle Q:

$$P = Q/S^1 = Q \times_{Ad} SO(3).$$

and let $V = P \times_{\rho_{std}} \mathbb{R}^3 = Q \times_{Ad} \mathfrak{su}(2)$. On the other hand, we have the complex 2-plane bundle E induced from Q.

$$E = Q \times_{\rho_{std}} \mathbb{C}^2.$$

We claim that

$$V \otimes \mathbb{C} = \mathrm{End}_0(E) = \{\text{Traceless endomorphism of } E\}.$$

Indeed, we are just using the decomposition

$$\mathfrak{u}(2) = \mathbb{R} \oplus \mathfrak{su}(2)$$

as irreducible $U(2)$-representations. We compute $c_k(\mathrm{End}_0(E)) = c_k(\mathrm{End}(E))$ in terms of $c_k(E)$. Using the fact that the Chern character is multiplicative and

$$\mathrm{ch}(E) = n + c_1(E) + \frac{1}{2}(c_1^2(E) - 2c_2(E)), \mathrm{ch}(E^*) = n - c_1(E) + \frac{1}{2}(c_1^2(E) - 2c_2(E)),$$

we obtain,

$$\begin{aligned} c_1(\mathrm{End}(E)) &= 0 \\ c_2(\mathrm{End}(E)) &= -\mathrm{ch}_2(\mathrm{End}(E)) = -n(c_1^2(E) - 2c_2(E)) + c_1^2(E) \\ &= 4c_2(E) - c_1^2(E) \text{ (when } n = 2). \end{aligned}$$

This shows

$$p_1(V) = -4c_2(Q)$$

if P is obtained from an $SU(2)$-bundle Q.

8.2. Energy Identity

Consider the (anti)-self duality equation:

$$F_A = \pm * F_A. \tag{11}$$

Define its energy to be

$$\mathcal{E}(A) = -\int_{X^4} \mathrm{tr}(F_A \wedge *F_A) = \int_{X^4} \|F_A\|^2 = \int_X \|F_A^+\|^2 + \int_X \|F_A^-\|^2.$$

Define the instanton number of $P \to X^4$ to be

$$\kappa(P) = \begin{cases} \langle c_2(P), [X] \rangle & \text{if } G = SU(n) \\ -\frac{1}{4}\langle p_1(P), [X] \rangle & \text{if } G = SO(3). \end{cases}$$

If $G = SU(2)$, then

$$\begin{aligned}8\pi^2\kappa &= \int_X \mathrm{tr}(F_A \wedge F_A) = \int_X \mathrm{tr}(F_A^+ \wedge F_A^+) + \int_X \mathrm{tr}(F_A^- \wedge F_A^-)\\ &= \int_X \mathrm{tr}(F_A^+ \wedge *F_A^+) - \int_X \mathrm{tr}(F_A^- \wedge *F_A^-)\\ &= -\int_X \|F_A^+\|^2 + \int_X \|F_A^-\|^2.\end{aligned}$$

This shows

$$\mathcal{E}(A) \geqslant 8\pi^2|\kappa(P)|$$

with the equality achieved if and only if the (anti)-self duality equation (11) holds. This will be the analogue of the energy identity in the case of the gradient flow equation on $X^4 = \mathbb{R} \times Y^3$.

8.3. Finding (anti-)self duality connections

We will make use of quaternion numbers to construct instantons on S^4.

Let $\mathbb{H} = \mathbb{R} + \mathbb{R}I + \mathbb{R}J + \mathbb{R}K$ with

$$\begin{aligned}&IJ = -JI = K,\ JK = -KJ = I,\ KI = -IK = J\\ &I^2 = J^2 = K^2 = -1.\end{aligned}$$

and let $SU(2) = Sp(1) =$ units of $\mathbb{H} = \{x \in \mathbb{H} : x\bar{x} = 1\}$. Consider the quaternion Hopf bundle. Let

$$S^7 \subset \mathbb{H} \oplus \mathbb{H}$$

be the unit sphere. There are 2 distinct right $SU(2)$-actions on S^7:

$$\begin{aligned}(x, y) *_1 q &= (xq, yq)\\ (x, y) *_2 q &= (\bar{q}x, \bar{q}y).\end{aligned}$$

which makes S^7 a principal $SU(2)$-bundle over S^4 in two different ways. Let us call them P_+ (the right action)and P_- (the left-inverse action) respectively.

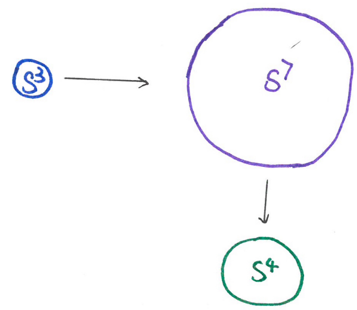
We define a connection A_+ on P_+ by declaring that the horizontal space H_+ is the orthogonal complement of the orbit of $SU(2)$. We wish to show that this connection satisfies the anti-self duality equation:

$$F^+_{A_+} = 0.$$

(Tom had no idea how to draw a better picture except for making one bigger the other.)

The trick is that the compact symplectic group $Sp(2)$ acts on $\mathbb{H}^2$:

$$Sp(2) = \{g \in SO(8) : g(v) *_1 q = g(v *_1 q)\ \forall q \in SU(2)\}.$$



This is the subgroup of linear transformations that preserve the metric and the right action $*_1$ of $SU(2)$. More explicitly,

$$Sp(2) = \{\begin{pmatrix} q_1 \cos\theta & -q_3 \sin\theta \\ q_2 \sin\theta & q_4 \cos\theta \end{pmatrix} : q_1, q_2, q_3, q_4 \in Sp(1)$$
$$q_4 = q_2 \bar{q}_1 q_3\}.$$

This action is transitive on S^7 and descends to S^4. This shows F_{A_+} as a section of

$$\Omega^2(S^+; \mathrm{ad}\, P_+)$$

has to be invariant under the action of $Sp(2)$. To make it simpler, consider the subgroup of $Sp(2)$ that fixes

$$x = (1, 0) \in \mathbb{H}^2.$$

which is a copy of $SU(2)$:

$$C_x = \{\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} : p \in SU(2)\}.$$

It induces an action on

$$\bigwedge^2 T^*_{[x]} S^4 = \Lambda^+_{[x]} \oplus \Lambda^-_{[x]}.$$

We have to show the action of C_x is non-trivial on Λ^+ and is trivial on Λ^-. Note that the action on the Lie algebra $\mathrm{ad}\, P_+|_{[x]}$ is already trivial since C_x commutes with $*_1$. The induced action of C_x on $T_{[x]}S^4$ is the standard left action by $SU(2)$.

Operators I, J, K give rise to different complex structures on $\mathbb{R}^4 \cong \mathbb{H}$ and we get a triple of symplectic forms $\omega_I, \omega_J, \omega_K$:

$$\omega_I(v, w) = \langle Iv, w\rangle, etc.$$

These forms are self dual:

$$\begin{aligned} \omega_I &= dx^0 \wedge dx^1 + dx^2 \wedge dx^3 \\ \omega_J &= dx^0 \wedge dx^2 + dx^3 \wedge dx^1 \\ \omega_K &= dx^0 \wedge dx^3 + dx^1 \wedge dx^2. \end{aligned}$$

If instead we consider $\omega'_I(v,w) = \langle vI, w\rangle$, then they are anti-self dual. It is clear that

$$\omega'_I, \omega'_J, \omega'_K$$

are invariant under the left action of $SU(2)$, while

$$\omega_I, \omega_J, \omega_K$$

give the adjoint-representation. This proves $F^+_{A_+} = 0$.

Similarly, we define A_- on P_-. This time we investigate the group that preserves the action $*_2$:

$$Sp(2)' = \{g \in SO(8) : g(v) *_2 q = g(v *_2 q)\ \forall q \in SU(2)\}.$$

Then the stabilizer C'_x is

$$q \in SU(2) : \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \mapsto \begin{pmatrix} v_1 \\ v_2 \cdot \bar{q} \end{pmatrix}.$$

We look at the induced action on $T_{[x]}S^4$ and this group preserves

$$\omega_I, \omega_J, \omega_K.$$

Hence, $F^-_{A_-} = 0$.

We could compute the connection A_+ and F_{A_+} more explicitly. This will be the task for the next lecture.

Lecture 9. Instantons on S^4 continued

9.1. Instantons on P_+

Recall from the last lecture that the unit sphere $S^7 \subset \mathbb{H}^2$ admits two right actions of $SU(2)$:

$$(x, y) *_1 q = (xq, yq)$$
$$(x, y) *_2 q = (\bar{q}x, \bar{q}y).$$

We will focus on the first action. It makes S^7 into principal bundle over S^4, and we call it P_+. Let $Sp(2)$ be the isometry group of $\mathbb{H} \oplus \mathbb{H}$ that commutes with the right action of $\mathbb{H}$. It is realized as the left multiplication of 2×2 quaternion matrices:

$$Sp(2) = \{\begin{pmatrix} q_1 \cos\theta & -q_3 \sin\theta \\ q_2 \sin\theta & q_4 \cos\theta \end{pmatrix} : \ q_1, q_2, q_3, q_4 \in Sp(1), q_4 = q_2 \bar{q}_1 q_3\}.$$

Then the stabilizer of $w = (0, 1) \in S^7$ is

$$C_w = \begin{pmatrix} Sp(1) & 0 \\ 0 & 1 \end{pmatrix} \subset Sp(2).$$

Remark. $Sp(2) = Spin(5)$ is the universal cover of $SO(5)$. The diagonal inclusion $Sp(1) \hookrightarrow Sp(2)$:

$$q \mapsto \begin{pmatrix} q & 0 \\ 0 & q \end{pmatrix}$$

sends the center $Z(Sp(1))$ to the center of $Sp(2)$, so we get a diagram

$$\begin{array}{ccccc} Sp(1) & \longrightarrow & Sp(2) & \longrightarrow & Sp(2)/Sp(1) \\ \downarrow & & \downarrow & & \| \\ SO(3) & \longrightarrow & SO(5) & \longrightarrow & SO(5)/SO(3) \end{array}$$

Therefore, $Sp(2)/Sp(1) \cong SO(5)/SO(3)$. The embedding

$$SO(3) \hookrightarrow SO(5)$$

is induced by the inclusion $\mathbb{R}^3 \hookrightarrow \mathbb{R}^5$ and its quotient space is a Stiefel manifold $V_{5,2}$. This space parametrizes ordered pairs of orthogonal unit vectors in $\mathbb{R}^5$:

$$\{(v_1, v_2) : |v_1| = |v_2| = 1, v_1 \perp v_2\}$$

which is also the unit sphere bundle of TS^4. Let us call it $\mathbb{S}(TS^4)$, so

$$Sp(2)/Sp(1) \cong SO(5)/SO(3) \cong \mathbb{S}(TS^4).$$

However, we can identify the first and the last spaces without making use of $SO(5)$. In fact, the tangent bundle of S^7 splits orthogonally:

$$TS^7 = V \oplus H$$

by the right action of $SU(2)$. The left action of $Sp(2)$ extends to the sphere bundle of H:

$$\mathbb{S}(H).$$

One verifies that this action is free and transitive. This shows $Sp(2) \cong \mathbb{S}(H)$ as manifolds. This action descends to $\mathbb{S}(TS^4)$ since

$$\mathbb{S}(TS^4) = \mathbb{S}(H)/SU(2)$$

just as $S^4 = S^7/SU(2)$. This copy of $SU(2)$ is precisely the diagonal inclusion.

To see $Sp(2) = Spin(5)$, note that the induced action of $Sp(2)$ on S^4 preserves the round metric, so we obtain a group homomorphism:

$$\rho : Sp(2) \to SO(5).$$

It suffices to compute the kernel and the image. Since the action is transitive, we look at the stabilizer at $[w] \in S^4$:

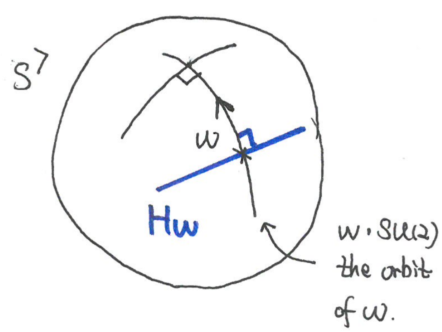
$$\begin{tikzcd} Sp(1) \times Sp(1) \arrow[r, "\rho'"] \arrow[d] & SO(4) \arrow[d] \\ Sp(2) \arrow[r, "\rho"] & SO(5). \end{tikzcd}$$

The first factor $Sp(1) \times \{1\}$ preserves H_w, while $(1, q)$ sends $H_w = H_{(0,1)}$ to $H_{(0,q)}$. These two spaces are identified by the right multiplication of q^{-1}. The tangent map of :

$$q \cdot q^{-1} : \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \mapsto \begin{pmatrix} v_1 q^{-1} \\ q v_2 q^{-1} \end{pmatrix}$$

at H_w is simply q^{-1}. We recognize that the top horizontal map ρ' is the standard two-fold cover, so is ρ.

One can also proceed in the other way: construct the Clifford algebra of $\mathbb{R}^5$ and identify it with the algebra of 2×2 quaternion matrices.



Let us return to the computation of curvatures. As above, we define a connection $A = A_+$ on P_+ via orthogonal projection to the tangent space of the orbit, and we want to understand this connection more concretely. Let x, y be quaternion-valued coordinate functions on $\mathbb{H}^2$, so $dx, dy, d\bar{x}, d\bar{y}$ are well-defined (quaternion-valued) 1-forms on $\mathbb{H}^2$. We claim that as a $\mathfrak{su}(2)$-valued 1-form

$$A = -\operatorname{Im}(d\bar{x}\cdot x + d\bar{y}\cdot y) = \operatorname{Im}(\bar{x}\cdot dx + \bar{y}\cdot dy).$$

where forms are pulled back via $S^7 \hookrightarrow \mathbb{H}^2$.

- A is the identity on vertical vectors. Given $(x, y) \in S^7$, a vertical vector is given by $v_q = (x\cdot q, y\cdot q)$ for $q \in \operatorname{Im}\mathbb{H} = \mathfrak{su}(2)$, so
$$dx(v_q) = x\cdot q, dy(v_q) = y\cdot q.$$
Hence, $A(v_q) = (|x|^2 + |y|^2)q = q$.
- A is invariant under $Sp(2)$. Take $g \in Sp(2)$, then
$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, g^T = \begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix}$$
for some $a, b, c, d \in \mathbb{H}$ and $g^T g = \mathrm{Id}$. If $(x, y)^T$ maps to $g(x, y)^T$, then
$$(\bar{x}, \bar{y}) \mapsto (\bar{x}, \bar{y})g^T, \begin{pmatrix} dx \\ dy \end{pmatrix} \mapsto g\begin{pmatrix} dx \\ dy \end{pmatrix}.$$
Since A is the imaginary part of
$$(\bar{x}, \bar{y})\cdot\begin{pmatrix} dx \\ dy \end{pmatrix},$$
one easily sees that it is invariant under the action of g.
- A kills the horizontal space H. It suffices to check at a point, Take $w = (1, 0) \in S^7$ where H_w is
$$(0, y) \in T_w S^7.$$
Since $\bar{y}(w) = 0$ and $dx = 0$ on H_w, $\ker A = H$.
- $q^*A = \operatorname{Ad}(q^{-1})(A)$ for $q \in Sp(1)$ acting on the right. Indeed,
$$\bar{x}\cdot dx \mapsto \overline{xq}\cdot d(xq) = \bar{q}(\bar{x}\cdot dx)q.$$

One can compare the similar construction for the complex Hopf bundle

$$\begin{array}{ccc} S^1 & \longrightarrow & S^3 \\ & & \downarrow \\ & & S^2. \end{array}$$

The same expression will give a standard connection for $\mathcal{O}(1)$ or $\mathcal{O}(-1)$.

Trivialize P_+ on a coordinate chart:

$$\mathbb{H} \ni x \mapsto s(x) = \frac{(x, 1)}{(1 + |x|^2)^{1/2}} \in S^7.$$

Then the connection form under this trivialization is

$$a = s^*A = \frac{\mathrm{Im}(\bar{x} \cdot dx)}{1 + |x|^2}$$

Let us compute the curvature form:

$$\begin{aligned} s^*F_A &= da + a \wedge a \ (= da + \frac{1}{2}[a \wedge a]) \\ &= \mathrm{Im}(\frac{d\bar{x} \wedge dx}{1+|x|^2} - \frac{(d\bar{x} \cdot x + \bar{x} \cdot dx) \wedge (\bar{x} \cdot dx)}{(1+|x|^2)^2} + \frac{(d\bar{x} \cdot x) \wedge (d\bar{x} \cdot x)}{(1+|x|^2)^2}) \\ &= \mathrm{Im}(\frac{d\bar{x} \wedge dx}{(1+|x|^2)^2}). \end{aligned}$$

where we used the fact $\mathrm{Im}(b \wedge b) = \mathrm{Im}\, b \wedge \mathrm{Im}\, b$ for a quaternion-valued 1-form b.

Finally, we need to compute $d\bar{x} \wedge dx$. Note that

$$d\bar{x} = dx^0 - Idx^1 - Jdx^2 - Kdx^3, dx = dx^0 + Idx^1 + Jdx^2 + Kdx^3.$$

Therefore, by direct computation, we have

$$\begin{aligned} d\bar{x} \wedge dx &= 2I(dx^0 \wedge dx^1 - dx^2 \wedge dx^3) \\ &\quad + 2J(dx^0 \wedge dx^2 - dx^3 \wedge dx^1) \\ &\quad + 2K(dx^0 \wedge dx^3 - dx^1 \wedge dx^2) \\ &= 2I\omega_I' + 2J\omega_J' + 2K\omega_K' \end{aligned}$$

which is anti-self dual. If we take the wedge product in the reverse order, it will be a self-dual form:

$$dx \wedge d\bar{x} = -2I\omega_I - 2J\omega_J - 2K\omega_K.$$

9.2. Instantons on S^4 by conformal transformations

We will exploit the fact that the (anti)-duality equation

$$F_A = \pm * F_A$$

is conformally invariant. The reason is that the Hodge $*$-operator:

$$* : \Lambda^2 \to \Lambda^2$$

depends only on the conformal class of the metric when $\dim X = 4$. Indeed, if $\dim X = n$ and $\tilde{g} = e^{2\sigma}g$ for some $\sigma : X \to \mathbb{R}$,

$$*_{\tilde{g}} = e^{(n-2p)\sigma} *_g : \Lambda^p \to \Lambda^{n-p},$$

so it remains unchanged when $p = n/2$ is the middle dimension.

Let $SO(5,1)$ be the conformal group of S^4:

$$\begin{aligned} SO(5,1) = \{u : S^4 \to S^4 \text{ orientation preserving} :\ & g = \text{the round metric}, \\ & u^*g = e^{2\sigma}g \text{ for some } \sigma : S^4 \to \mathbb{R}\}. \end{aligned}$$

This group is generated by rotations ($SO(5)$) and dilations. It is isomorphic to the special orthogonal group of type $(5,1)$ as the name indicates. Since A_+ is invariant under the $Sp(2)$-action on S^7, it is invariant downstairs under the action of $SO(5)$ on S^4. Therefore, the moduli space of instantons on S^4 produced by applying conformal transformations to A_+ is

$$SO(5,1)/SO(5) \cong \mathbb{R}^5.$$

Let us compute this family explicitly. The dilation $\tau_\lambda(x) = \lambda x$ on $\mathbb{R}^4$ comes from a conformal transformation of S^4. Then,

$$\tau_\lambda^*(a) = \frac{\lambda^2 \operatorname{Im}(\bar{x} \cdot dx)}{1 + \lambda^2 |x|^2} = \frac{\operatorname{Im}(\bar{x} \cdot dx)}{1/\lambda^2 + |x|^2},$$

$$\tau_\lambda^*(F_A) = \operatorname{Im}(\frac{\lambda^2 d\bar{x} \wedge dx}{(1 + \lambda^2|x|^2)^2}).$$

Initially, $\|F_{A_+}\|^2$ distributes evenly on S^4. As $\lambda \to \infty$, $\|\tau_\lambda^* F_{A_+}\|^2$ concentrates at the origin and

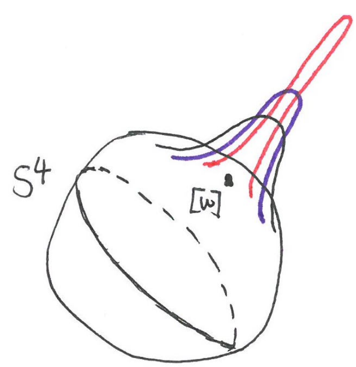
$$\tau_\lambda^*(a) \to a_\infty = \frac{\operatorname{Im}(\bar{x} \cdot dx)}{|x|^2}$$

Consider the singular gauge transformation

$$g(x) = \frac{x}{|x|}.$$

We have $g^* a_\infty = g \cdot (a_\infty - g^{-1} dg) \cdot g^{-1}$ and

$$g^{-1} dg = \frac{\bar{x}}{|x|} \cdot (\frac{dx}{|x|} - \frac{x d|x|^2}{2|x|^3}) = \frac{\bar{x} \cdot dx - d\bar{x} \cdot x}{2|x|^2} = a_\infty.$$



This shows $g^* a_\infty$ is the trivial connection. The path of connections $\tau_\lambda^*(a)$ converges away from the origin to the trivial connection and the limit extends to a connection in a different bundle. Therefore, to do compactification, we have to allow the topology of the bundle to change.

This "bubbling" phenomenon also occurs in symplectic topology. If we study pseudo-holomorphic curves and fix a conformal structure on the Riemann surface, then the bubbling of spheres could occur. It becomes more complicated when allowing the conformal structure to change. It is helpful to briefly review the history:

(1) Sacks & Uhlenbeck. They first discovered this phenomenon when studying the energy equation for a Riemann surface mapping into a Riemannian manifold.
(2) Uhlenbeck & Taubes. They worked out the case for the Yang-Mills equation.
(3) Gromov. Pseudo-holomorphic curves.

9.3. The Atiyah-Drinfeld-Hitchin-Manin (ADHM) Construction

It was back in 1975 when Belavin et al. [1] first wrote down explicit instanton solutions on $\mathbb{R}^4$ where the rotational symmetry was assumed. Later, the ADHM construction came out and gave a finite dimensional parameter space for all solutions on S^4. This construction was discovered by these four mathematicians independently. In this case, some magics happened. We will briefly describe this construction.

(Tom made some corrections in the next lecture. The description below was not accurate.)

Again, take $S^4 = \mathbb{P}(\mathbb{H}^2) = \mathbb{H} \oplus \mathbb{H}/\mathbb{H}^*$ and $SU(2) = Sp(1)$. Pick a pair of vectors

$$C, D \in \mathbb{H}^{k+1}.$$

Let $v(x, y) = Cx + By$ for $(x, y) \in \mathbb{H}^2$ and suppose $v(x, y) = 0$ if and only if $(x, y) = 0$, that is, C, D are linearly independent over $\mathbb{H}$. Then,

$$\tilde{V} = ((x, y), v(x, y)) \in (\mathbb{H}^2 - \{0\}) \times \mathbb{H}^{k+1}$$

descends to a vector bundle over $S^4 = \mathbb{P}(\mathbb{H}^2)$. Let us call it V. The principal bundle P_+ induces a quaternion bundle by the right-inverse representation:

$$E_+ = P_+ \times_\rho \mathbb{H} \text{ where } \rho(q)(v) = v\bar{q}, v \in SU(2), q \in \mathbb{H}.$$

Then E_+ has a canonical connection induced by A_+ and V is a sub-bundle of E_+^{k+1}. Let

$$\Pi : E_+^{k+1} \to V$$

be the orthogonal projection and ∇ be the product connection in E_+^{k+1}. Then

$$\Pi \circ \nabla$$

becomes a connection in V. The obstacle is that $\Pi \circ \nabla$ is not necessarily self-dual or anti-self dual. In order to give solutions to self-dual Yang-Mills equation, vectors C, D have to satisfy a quadratic equation. We will continue this subject in the next lecture.

Lecture 10. The ADHM construction

10.1. The Atiyah-Drinfeld-Hitchin-Manin (ADHM) Construction for SU(2)-instantons

Tom highly recommended this book for supplementary reading:

- Atiyah, M. F, Geometry on Yang-Mills fields. Scuola Normale Superiore Pisa, Pisa, 1979. 99 pp.

Take the unit sphere $S^7 \subset \mathbb{H} \oplus \mathbb{H}$. The right action of $SU(2)$ makes it a principle bundle over S^4. Let $\gamma_+ = P_+ \times_\rho \mathbb{H} \to S^4$ be the tautological bundle:

$$\gamma_+ = \{(x, y, h) \in S^7 \times \mathbb{H}\}/((xq, yq), h) \sim ((x, y), qh).$$

Let C, D be two $(k+1) \times k$ quaternion-valued matrices. For each $(x, y) \in \mathbb{H} \oplus \mathbb{H}$, define a map

$$\begin{aligned} v(x, y) = Cx + Dy : \mathbb{H}^k &\to \mathbb{H}^{k+1} \\ w &\mapsto (Cx + Dy)w. \end{aligned}$$

We require that $v(x, y)$ has rank k whenever $(x, y) \neq (0, 0)$. This homomorphism commutes with the right action of $\mathbb{H}$, so it is $\mathbb{H}$-linear. For any $q \in SU(2)$,

$$v((x, y) \cdot q) = v(xq, yq) = v(x, y) \circ \rho(q) = \rho^*(q^{-1})v(x, y).$$

Therefore, v defines a section of

$$P_+ \times \mathrm{Hom}_{\mathbb{H}}(\mathbb{H}^k, \mathbb{H}^{k+1}).$$

and it descends to a section

$$v : \gamma_+^k \to \mathbb{H}^{k+1}.$$

Here, $\mathbb{H}^{k+1}$ denotes the trivial rank-$(k+1)$ bundle, and it has the trivial connection d.

Remark. One way to convince yourself that v is indeed a bundle map from γ_+^k to $\mathbb{H}^{k+1}$ is to look at the special case when $k = 1$ and

$$C = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, D = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

In this special case, $v(x, y) = (x, y)^T$ sends $h \in \mathbb{H}$ to $(xh, yh)^T \in \mathbb{H} \oplus \mathbb{H}$. The image of v is precisely the quaternion line that passes through (x, y) (with respect to the right multiplication). Therefore, γ_+ is called the tautological quaternion bundle over $S^4 = \mathbb{H}^2 - \{0\}/\mathbb{H}^*$. $\square$

Let $E = (\mathrm{Im}\, v)^\perp$ and

$$\Pi : \mathbb{H}^{k+1} \to \mathbb{H}^{k+1}$$

be the orthogonal projection onto E. Since the sequence

$$0 \to \gamma_+^k \to \mathbb{H}^{k+1} \to E \to 0$$

is exact, we have

$$c_2(E) = -c_2(\gamma_+^k) = -kc_2(\gamma_+) = -ke(\gamma_+) = -k.$$

Remark. It is hard to convince oneself that $e(\gamma_+) = 1$ because in the complex case $e(\gamma_{\mathbb{C}}) = -1$ for the complex tautological bundle $\gamma_{\mathbb{C}} = \mathcal{O}(-1) \to \mathbb{CP}^1$. The issue is that γ_+ is oriented by the right multiplication of I on each fiber, so the basis

$$(1, I, J, J \cdot I = -K).$$

gives the default complex orientation on γ_+. To compute the Euler number of γ_+, cover S^4 by two coordinate charts and choose a clever section of γ_+:

$$p \in \mathbb{H} \mapsto \left(\frac{1}{(1+|p|^2)^{\frac{1}{2}}}, \frac{p}{(1+|p|^2)^{\frac{1}{2}}}\right), w_p = \frac{1}{(1+|p|^2)^{\frac{1}{2}}} \in \mathbb{H}$$
$$q \in \mathbb{H} \mapsto \left(\frac{q}{(1+|q|^2)^{\frac{1}{2}}}, \frac{1}{(1+|q|^2)^{\frac{1}{2}}}\right), w_q = \frac{\bar{q}}{(1+|q|^2)^{\frac{1}{2}}} \in \mathbb{H}$$

The transition map is given by $q = p^{-1}$ and

$$\frac{\bar{q}}{|q|} \cdot w_p = w_q.$$

The section $\{w_p, w_q\}$ has a single zero at $q = 0$. The linearization is $q \mapsto \bar{q}$. In the complex case, this is orientation reversing. In the quaternion case, this is orientation preserving because different conventions are used for the domain and the target as noted above. □

We aim to construct **self-dual** connections on E because $c_2(E) = -k$ is negative. We take it to be

$$\nabla = \Pi \circ d.$$

Exercise 10.1.1. *Prove that* $F_\nabla = \Pi \circ (d\Pi \wedge d\Pi) \circ \Pi$.

Proof. Since $\mathrm{Hom}(\mathbb{H}^{k+1}, \mathbb{H}^{k+1})$ is a trivial bundle, it admits the trivial connection d. We regard

$$\nabla = \Pi \circ d \circ \Pi$$

as a first order differential operator on $\mathbb{H}^{k+1}$. Then

$$F_\nabla(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}.$$

Note that

$$\begin{aligned}\nabla_X \nabla_Y s &= \nabla_X(\Pi(Y(s)) + \Pi \circ Y(\Pi)s) \\ &= \Pi(XY(s)) + \Pi \circ X(\Pi)(Y(s)) \\ &\quad + \Pi \circ X(\Pi) \circ Y(\Pi)s + \Pi \circ XY(\Pi)s + \Pi \circ Y(\Pi)(X(s)).\end{aligned}$$

Therefore, $F_\nabla(X, Y) = \Pi \circ (X(\Pi) \circ Y(\Pi) - Y(\Pi) \circ X(\Pi))$. Since $\nabla_X s = \nabla_X(\Pi(s))$, we can add a copy of Π at the beginning for free. Our result follows. □

Let $\Pi^\perp$ be the orthogonal projection onto $\operatorname{Im} v$. Then $\Pi^\perp + \Pi = \mathrm{Id}$ and

$$\Pi^\perp = v(v^*v)^{-1}v^*$$

where v^* denotes the $\mathbb{H}$-conjugate transpose. Using the relation

$$d\Pi = -d\Pi^\perp,$$

we obtain

$$F_\nabla = \Pi \circ (d\Pi^\perp \wedge d\Pi^\perp) \circ \Pi.$$

Since $\Pi \circ v \equiv 0$ and $v^* \circ \Pi \equiv 0$,

$$\Pi \circ d\Pi^\perp = \Pi(dv)(v^*v)^{-1}v^*, d\Pi^\perp \circ \Pi = v(v^*v)^{-1}d(v^*) \circ \Pi.$$

The curvature is simplified as

$$F_\nabla = \Pi \circ (dv) \wedge (v^*v)^{-1}d(v^*) \circ \Pi.$$

Now take a coordinate chart

$$s : \mathbb{H} \to \mathbb{P}(\mathbb{H}^2), x \mapsto [x : 1]$$

and carry out the computation locally. Then

$$s^*v = Cx + D, s^*dv = Cdx$$
$$s^*F_\nabla = \Pi(Cdx) \wedge (v^*v)^{-1}(d\bar{x}C^*)\Pi.$$

From previous lectures, we have known that $dx \wedge d\bar{x}$ is a self-dual form. With more work, one shows that s^*F_∇ is self-dual if and only if

$$(v^*v)^{-1} \text{ is real.}$$

This condition is equivalent to v^*v being real. Let us make it more explicit. In the construction of v, there is some ambiguity about C and D. Since C has full rank, by changing a quaternion basis of $\mathbb{H}^k$, we assume that

$$C = \begin{pmatrix} z \\ I_k \end{pmatrix}$$

for some row vector $z \in \mathbb{H}^k$. By changing basis in $\mathbb{H}^{k+1}$, we can further eliminate z and get

$$C = \begin{pmatrix} 0 \\ I_k \end{pmatrix}.$$

Now write D into the same block matrix form:

$$D = \begin{pmatrix} \Lambda \\ B \end{pmatrix}.$$

Since $v = Cx + D$, we have

$$v^*v = |x|^2 I_k + \Lambda^*\Lambda + B^*B + (\bar{x}B + B^*x).$$

For this to be real for any $x \in \mathbb{H}$, we must have

(1) $\bar{x}B + B^*x$ is real for any $x \in \mathbb{H}$.

(2) $B^*B + \Lambda^*\Lambda$ is real.

The first condition implies $B = B^T$ is a symmetric $k \times k$ quaternion matrix. The main theorem in the ADHM construction is that all solutions actually arise in this way:

Theorem 10.1.2. *All charge k instantons on S^4 are given by pairs (B, Λ) where B is a symmetric $k \times k$ quaternion matrix and $\Lambda \in \mathbb{H}^k$ is a vector such that*

$$B^*B + \Lambda^*\Lambda \text{ is real.} \tag{12}$$

Two such pairs $(B, \Lambda), (B', \Lambda')$ give gauge equivalent connections if and only if they are related by

$$B' = T^T BT, \Lambda' = q\Lambda T$$

for some $T \in O(k)$ and $q \in Sp(1)$.

Let us formally count the dimension of the parameter space:

$$\begin{aligned} B &\rightsquigarrow 4 \cdot \frac{k(k+1)}{2} \text{ variables} \\ \Lambda &\rightsquigarrow 4k \text{ variables} \\ \text{the constraint (12)} &\rightsquigarrow 3 \cdot \frac{k(k-1)}{2} \text{ equations} \\ O(k) &\rightsquigarrow \frac{k(k-1)}{2} \text{ dimensional symmetry} \\ Sp(1) &\rightsquigarrow 3 \text{ dimensional symmetry.} \end{aligned}$$

Therefore, we get $8k - 3$ parameters as predicted by the index theorem. This construction fits into a more geometric context. The constraint (**??**) can be viewed as a hyper-Kähler moment map. The space of symmetric $\mathbb{H}$-matrices and $\mathbb{H}$-vectors admit an $O(k)$-action

$$(B, \Lambda) \mapsto (T^T BT, \Lambda T) \in \mathrm{Sym}^k(\mathbb{H}) \oplus \mathbb{H}^k.$$

Since this action preserves the Hyper-Kähler structure, we obtain a moment map:

$$\mu : \mathrm{Sym}^k(\mathbb{H}) \oplus \mathbb{H}^k \mapsto \mathfrak{o}(k) \otimes \mathrm{Im}(\mathbb{H}).$$

which is $O(k)$-equivariant. Then (**??**) is saying that (B, Λ) is contained in the fiber $\mu^{-1}(0)$.

Even though S^4 is not a complex manifold, it is closely related to $\mathbb{CP}^3$ where one can use algebraic geometry. The embedding $S^1 \hookrightarrow Sp(1)$ induces a fiber bundle:

$$\begin{tikzcd} S^2 = Sp(1)/S^1 \arrow[r] & \mathbb{CP}^3 = S^7/S^1 \arrow[d] \\ & S^4 = S^7/Sp(1). \end{tikzcd}$$

For each $x \in S^4$, the fiber S^2_x is identified with the space metric-compatible almost complex structures on T_xS^4:

$$\{aI + bJ + cK : a^2 + b^2 + c^2 = 1\}.$$

An anti-self dual connection is pulled back to a holomorphic connection on $\mathbb{CP}^3$. The total space $\mathbb{CP}^3$ is also called the twistor space.

10.2. Inverting ADHM data

Finally, let us explain why the ADHM construction is supposed to work. We have to construct matrix data out of self dual connections. The process of inverting the ADHM construction was discovered independently by Atiyah, Drinfeld, Hitchin and Manin after whom it was named. Some physicists might have different point of views. We come back to the bundle $E \to S^4$ with $c_2(E) = -k$ and choose a $Sp(1)$-connection A. Consider the Dirac operator twisted by E:

$$D_A = D_A^+ : \Gamma(S^+ \otimes_{\mathbb{C}} E) \to \Gamma(S^- \otimes_{\mathbb{C}} E).$$

It is easier to work on $\mathbb{R}^4$. Then the Weitzenböck formula shows

$$D_A^* D_A = \nabla_A^* \nabla_A + \rho(F_A^+)$$
$$D_A D_A^* = \nabla_A^* \nabla_A + \rho(F_A^-).$$

No scalar curvature shows up because $\mathbb{R}^4$ is flat. If $F_A^- \equiv 0$, then the cokernel of D_A is trivial.

By the index theorem, $\mathrm{Ind}_{\mathbb{C}}(D_A) = k$. We get k-dimensional (over $\mathbb{C}$) harmonic spinors:

$$\phi_i, 1 \leqslant i \leqslant k.$$

Each of them decays as $1/r^3$ at the infinity. This means

$$\phi_i = \frac{a_i}{r^3} + \mathcal{O}(\frac{1}{r^4})$$

for some $a_i \in (S^+ \otimes_{\mathbb{C}} E)_\infty \cong \mathbb{C}^4$ as $r \to \infty$. There are two steps to recover the ADHM data,

(1) Pick up the leading coefficients of the spinors at $\infty \rightsquigarrow \Lambda$.
(2) $x_0\phi_i, x_1\phi_i, x_2\phi_i, x_3\phi_i \rightsquigarrow B$.

To explain the second term, write

$$B = B_0 + B_1 I + B_2 J + B_3 K.$$

for some Hermitian $k \times k$ matrices $B_m, 0 \leqslant m \leqslant 3$. Then

$$(B_m)_{ij} = \int_{\mathbb{R}^4} \langle \phi_i, x_m \phi_j \rangle$$

for $1 \leqslant i, j \leqslant k$ and $0 \leqslant m \leqslant 3$. Since $\langle \phi_i, x_m \phi_j \rangle$ decays as $1/r^5$, it is integrable over $\mathbb{R}^4$.

Remark. Tom improvised a little bit at this point. This inverting process apparently has a $U(k)$-symmetry: we can choose any unitary basis of $\ker D_A$. In addition, Λ is a vector in $\mathbb{C}^{4k}$, not in $\mathbb{H}^k$. This approach is not the one described in Atiyah's book. For interested readers, we recommend §3.3 of

- Donaldson, S. K.; Kronheimer, P. B.
  The geometry of four-manifolds. Oxford Mathematical Monographs. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1990. x+440 pp.

Exercise 10.2.1. *Compute the index of D_A^+.*

Proof. For an oriented 4-manifold X, the $\hat{A}$-genus is defined by

$$\hat{A}(X) = 1 - \frac{1}{24}p_1(X).$$

Since $c_1(E) = 0$, the Chern-character of E is

$$\mathrm{ch}(E) = 2 - c_2(E) + \cdots .$$

By the Atiyah-Singer index theorem, we have

$$\begin{aligned}\mathrm{Ind}_{\mathbb{C}}(D_A^+) &= \int_X \hat{A}(X)\,\mathrm{ch}(E) = \int_X (1 - \frac{1}{24}p_1(X))(2 - c_2(E)). \\ &= -c_2(E)[X] - \frac{1}{12}p_1(X)[X] = k - \frac{1}{12}p_1(X)[X].\end{aligned}$$

It remains to verify that for $X = S^4$, $p_1(TS^4) = 0$. One way to see this is to make use of the twistor space $\mathbb{CP}^3$:

$$\begin{tikzcd} S^2 \arrow[r] \arrow[d] & \mathbb{CP}^3 \arrow[d, "\pi"] \\ * \arrow[r] & S^4 \end{tikzcd}$$

Let $F = \pi^* TS^4$, then

$$T\mathbb{CP}^3 = F \oplus V$$

where V denotes the bundle of vertical spaces. Since V is a line bundle and its restriction to each fiber S^2 is the tangent bundle, its total Chern class is

$$c(V) = 1 + 2a.$$

where $a \in H^2(\mathbb{CP}^3;\mathbb{Z})$ is the canonical generator. On the other hand, by the Euler exact sequence,

$$c(T\mathbb{CP}^3) = (1 + a)^4 = 1 + 4a + 6a^2 + 4a^3.$$

For a complex vector bundle E, $p_1(E_{\mathbb{R}}) = c_1^2(E) - 2c_2(E)$. Therefore,

$$\begin{aligned}\pi^* p_1(TS^4) = p_1(F_{\mathbb{R}}) &= p_1(T\mathbb{CP}^3) - p_1(V) \\ &= (4a)^2 - 12a^2 - (2a)^2 = 0.\end{aligned}$$

It is interesting to note that F is a complex bundle even though TS^4 is not. $\square$

Lecture 11. Gauge Transformations

11.1. Stabilizers

We start with the gauge equivalence of connections. Let G be a compact Lie group and $P \to X$ be a principal $G-$bundle:

$$\begin{tikzcd} G \arrow[r] & P \arrow[d] \\ & X \end{tikzcd}$$

We adopt the following notations:

- $\operatorname{Ad} P = P \times_{\operatorname{Ad}} G$ is a bundle of Lie groups
- $\operatorname{ad} P = P \times_{\operatorname{ad}} \mathfrak{g}$ is a bundle of Lie algebras
- $\mathscr{A}_P$ is the space of smooth connections on P. It is an affine space over $\Omega^1(\operatorname{ad} P) = \Gamma(X, T^*X \otimes \operatorname{ad} P)$.
- $\mathscr{G}_P = \Gamma(X, \operatorname{Ad} P)$ is the space of smooth sections of $\operatorname{Ad} P$. The point-wise multiplication defines a group structure. It is the so-called gauge group.

A gauge transformation $g \in \mathscr{G}_P$ is also a bundle map

$$\begin{tikzcd} \tilde{g} : P \arrow[rr] \arrow[dr] & & P \arrow[dl] \\ & X & \end{tikzcd}$$

that commutes with right multiplications. For any $p \in P$, $\tilde{g}(p) = p \cdot g(p)$ for some $g(p) \in G$ and

$$g(ph) = h^{-1}g(p)h = \operatorname{Ad}(h^{-1})g(p).$$

Hence, g is the corresponding section of $\operatorname{Ad} P \to X$ determined by $\tilde{g}$. We will use these two points of view interchangeably.

Let $V \to X$ be a bundle induced by a faithful linear representation of G. Then both $\operatorname{Ad} P$ and $\operatorname{ad} P$ are sub-bundles of

$$\operatorname{End}(V).$$

The gauge group $\mathscr{G}_P$ acts on $\mathscr{A}_P$ by the formula

$$\begin{aligned} \mathscr{A}_P \times \mathscr{G}_P &\to \mathscr{A}_P \\ (A, g) &\mapsto g^*A = A + g d_A g^{-1}. \end{aligned}$$

where $d_A = \nabla_A$ is the covariant derivative induced on $\operatorname{End}(V)$.

Exercise 11.1.1. *Check that g^*A is the pull-back connection.*

Proof. The induced connection ∇_A on $\operatorname{End}(V)$ is defined by the property:

$$(\nabla_A \phi)v + \phi(\nabla_A v) = \nabla_A(\phi(v)).$$

Let us compute the pull-back connection $g^*(\nabla_A) = g \circ \nabla_A \circ g^{-1}$:

$$\begin{aligned} g \circ \nabla_A(g^{-1}v) &= g \circ (\nabla_A g^{-1})v + g \circ g^{-1}\nabla_A v \\ &= (\nabla_A + g\nabla_A g^{-1})v. \end{aligned}$$

Using a trivialization of the bundle, we usually have a formula:

$$g^*A = gAg^{-1} + gdg^{-1}.$$

where A is an 1-form valued matrix and g is a matrix. This formula agrees with the previous description because

$$\begin{aligned} gAg^{-1} + gdg^{-1} &= A + g(dg^{-1} + Ag^{-1} - g^{-1}A) \\ &= A + g\underbrace{\left(dg^{-1} + [A, g^{-1}]\right)}_{=d_A g^{-1}}. \end{aligned}$$ □

The quotient space $\mathscr{B} = \mathscr{B}_P = \mathscr{A}_P/\mathscr{G}_P$ is the moduli space of connections on P.

How bad is this action? Orbits with non-trivial stabilizers give rise to "non-smooth" points in $\mathscr{B}$. Given a central element $u \in Z(G)$, the bundle map

$$\tilde{u} : P \to P,\ p \mapsto p \cdot u$$

fixes all connections on P, because, by definition, a connection is invariant under the right multiplication. This shows for any $A \in \mathscr{A}_P$,

$$u^*A = A,$$

so $u \in \mathrm{Stab}_A$.Working with the quotient group $\mathscr{G}_P/Z(G)$ can make this action faithful, so $Z(G)$ does not cause an issue.

If $g^*A = A$, then $g\nabla_A g^{-1} = 0$. This means both g^{-1} and g are parallel sections of $\mathrm{End}(V)$, so g is determined by its value $g(p)$ at a base point $p \in P$. Hence,

$$\mathrm{Stab}_A \subset G.$$

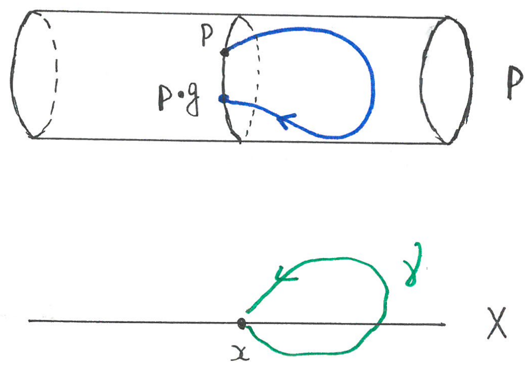
To understand it more concretely, take a loop $\gamma : [0, 1] \to X$ around $x = \pi(p) \in X$. Let $\tilde{\gamma}$ be the horizontal lift of γ with $\tilde{\gamma}(0) = p$. Then $\tilde{\gamma}(1) = p \cdot u^{-1}$ for some $u \in G$. This element

$$u = \rho_{A,p}(\gamma) \in G$$

is the holonomy of A around γ, and $\rho_{A,p}$ defines a representation:

$$\rho_{A,p} : \Omega(X, x) \to G.$$

The inverse in the definition of u is to make $\rho_{A,p}$ a group homomorphism. If $g \in \mathrm{Stab}_A$, then $\tilde{g} \circ \tilde{\gamma}$ is a horizontal lift of γ with $\tilde{g} \circ \tilde{\gamma}(0) = p \cdot g(p)$, so it must agree



with the path $\tilde{\gamma}(t) \cdot g(p)$ for any $t \in [0,1]$. In particular, comparing the end point gives:

$$\begin{aligned} p \cdot g(p) \cdot u^{-1} &= \tilde{g}(p) \cdot u^{-1} = \tilde{g}(p \cdot u^{-1}) = \tilde{g}(\tilde{\gamma}(1)) \\ &= \tilde{\gamma}(1) \cdot g(p) = p \cdot u^{-1} \cdot g(p). \end{aligned}$$

This shows $g(p)$ lies in the commutant of $\operatorname{Im} \rho_{A,p}$. This is also a sufficient condition. Hence,

$$\mathrm{Stab}_A = C(\operatorname{Im} \rho_{A,p}) \subset G.$$

Any subgroup of G is the image of $\rho_{A,p}$ for some A and $p \in P$. We have to decide which subgroups of G can be realized as commutants of other subgroups.

For a subgroup $H \subset G$, let

$$C(H) := \text{the commutant of } H \subset G.$$

We list some properties and examples:

(1) If $H = C(K)$ for some $K \subset G$, then
$$C(C(H)) = H,$$
but this is not true in general if the assumption $H = C(K)$ is removed.

(2) All commutants form a directed graph $\mathcal{G}$ where arrows are given by inclusions of subgroups. The center $Z(G)$ is the initial object, while G is the final object. By (1), taking commutants identifies $\mathcal{G}$ with its dual graph (where arrows are reversed).

(3) Let $G = U(n)$. Let H consist of elements of the block matrices
$$\begin{pmatrix} x_1 I_{n_1} & & & & & \\ & \cdots & & & & \\ & & x_s I_{n_s} & & & \\ & & & U(k_1) & & \\ & & & & \cdots & \\ & & & & & U(k_r) \end{pmatrix}$$

where $x_i \in S^1$. Then $C(H)$ is given by

$$\begin{pmatrix} U(n_1) & & & & & \\ & \cdots & & & & \\ & & U(n_s) & & & \\ & & & x_1 I_{k_1} & & \\ & & & & \cdots & \\ & & & & & x_r I_{k_r} \end{pmatrix}.$$

Up to conjugacy, all commutants in $U(n)$ are given in this form.

(4) For a simply connected structure group G, the graph of commutants has a relatively simpler shape. But in general there will be more commutants if $\pi_1(G) \neq \{e\}$. Take $SO(3)$ for an example. The graph of commutants is

$$\begin{tikzcd} & & K_4 = \mathbb{Z}_2 \times \mathbb{Z}_2 \arrow[dr] & & \\ 1 \arrow[r] & \mathbb{Z}_2 \arrow[ur] \arrow[dr] & & O_2 \arrow[r] & SO(3). \\ & & SO(2) \arrow[ur] & & \end{tikzcd}$$

- $C(SO(3)) = 1$.
- $\mathbb{Z}_2 = \{e, \begin{pmatrix} -1 & & \\ & -1 & \\ & & 1 \end{pmatrix}\}$. $C(\mathbb{Z}_2) = O(2)$.
- $K_4 = \{A = \begin{pmatrix} \pm 1 & & \\ & \pm 1 & \\ & & \pm 1 \end{pmatrix} : \det A = 1\}$. $C(K_4) = K_4$.
- $SO(2) = \{\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} : A \in SO(2)\}$. $C(SO(2)) = SO(2)$.
- $O(2) = \{\begin{pmatrix} A & 0 \\ 0 & \det A \end{pmatrix} : A \in O(2)\}$. $C(O(2)) = \mathbb{Z}_2$.

Arrows denote inclusions of subgroups. It is easily seen that this graph is dual to itself.

Definition 11.1.2. A connection $A \in \mathscr{A}_P$ is called reducible if $\mathrm{Stab}_A \neq Z(G)$.

A reducible connection will make the quotient space singular.

11.2. Sobolev Completions

To analyze $\mathscr{A}_P/\mathscr{G}_P$, we need to complete this space in some Sobolev topology. For an oriented Riemannian manifold X and $E \to X$ a vector bundle, pick a smooth connection A. For a section $s : X \to E$, define its Sobolev norm to be

$$\|s\|^p_{L^p_{k,A}} = \int_X \sum_{l=0}^{k} |\nabla_A^{(l)} s|^p dvol_X$$

By completing $\mathcal{C}^\infty(X, E)$ with respect to $\| \cdot \|_{L^p_{k,A}}$, we obtain the Sobolev space

$$L^p_k(X, E).$$

This space is independent of A: norms $\| \cdot \|_{L^p_{k,A}}$ induced by different connections are equivalent as long as connections are smooth.

Let $E = T^*X \otimes \operatorname{ad} P$ or $E = \operatorname{End}(V)$. We obtain the Sobolev completion of $\mathscr{A}_P$ and $\mathscr{G}_P$ with respect to $\| \cdot \|_{L^p_k}$:

$$\mathscr{A}^p_k, \mathscr{G}^p_k.$$

We have regarded $\mathscr{G}_P$ as a subset of $\mathcal{C}^\infty(X, \operatorname{End}(V))$. If k is not large enough, then $L^2_k \not\hookrightarrow \mathcal{C}^0$. For a section s in the completion $\mathscr{G}^P_k$, s_x might not lie in

$$(\operatorname{Ad} P)_x \ (\subset \operatorname{End}(V_x))$$

for some $x \in X$. We have to be careful about the regularity of sections.

Theorem 11.2.1 (The Sobolev Embedding Theorem). *In dimension n, we have embeddings:*

(1) $L^p_k \hookrightarrow L^q_l$ if $k - \frac{n}{p} \geqslant l - \frac{n}{q}$ and $k > l$.
(2) $L^p_k \hookrightarrow \mathcal{C}^{l,\alpha}$ if $k - \frac{n}{p} \geqslant l + \alpha$, $k > l$ and $0 < \alpha < 1$.

In particular, $k - \frac{p}{n} > 0$ implies $L^p_k(X, E) \hookrightarrow \mathcal{C}^0(X, E)$. In addition to the embedding theorem, we also have the multiplication theorem which says in this case $L^p_k(X, E)$ is a Banach algebra. We will show

Proposition 11.2.2. *If $k - \frac{p}{n} > 0$, then $\mathscr{G}^p_k$ is a Banach Lie group.*

We will work out the case when $G = U(n)$. Let V be the bundle induced by the fundamental representation. Then

$$(\operatorname{Ad} P)_x \subset \operatorname{End}(V_x)$$

is the space of unitary transformations.

Proof. Consider the map

$$\phi : L^p_k(X, \operatorname{End}(V)) \to L^p_k(X, \operatorname{Herm}(V))$$
$$f \mapsto f^* f.$$

where $\operatorname{Herm}(V)$ denotes the bundle of Hermitian endomorphisms of V. Then $\mathscr{G}^p_k = \phi^{-1}(\operatorname{Id}_V)$. For any $g \in \mathscr{G}^p_k$, the tangent map

$$d_g\phi : L^p_k(X, \operatorname{End}(V)) \to L^p_k(X, \operatorname{End}(V)).$$

is surjective. By the implicit function theorem, $\ker d_g\phi$ gives a local coordinate chart of $\mathscr{G}_k^k$. It suffices to check that the multiplication and the inverse map:

$$\mathscr{G}_k^p \times \mathscr{G}_k^p \xrightarrow{\times} \mathscr{G}_k^p,\ \mathscr{G}_k^p \xrightarrow{inv} \mathscr{G}_k^p$$

are smooth maps between Banach manifolds. This follows from the fact that L_k^p is an algebra. $\square$

Compare with the case in symplectic geometry. We need to deal with the diffeomorphism group $\mathrm{Diff}(X)$, which becomes much trickier. Instead of Sobolev completions, it is more natural to use the $\mathcal{C}^k$-norm. The composition

$$\mathrm{Diff}^k(X) \times \mathrm{Diff}^k(X) \to \mathrm{Diff}^k(X)$$

is still valid, but it is not smooth any more. Some derivatives will lose if we study the tangent map. For interested readers, Tom recommended

- Ebin, David G. On the space of Riemannian metrics. Bull. Amer. Math. Soc. 74 1968 1001-1003.

This was part of Ebin's thesis at MIT under the supervision of Professor Singer. The idea is to include carefully higher and higher $\mathcal{C}^k$-norms. Reader can also see the prototype of polyfolds in his thesis.

The action of $\mathscr{G}_{k+1}^p$ on $\mathscr{A}_k^p$ is given by

$$\begin{aligned} \mathscr{G}_{k+1}^p \times \mathscr{A}_k^p &\to \mathscr{A}_k^p \\ (g, A) &\mapsto A + g d_A g^{-1}. \end{aligned}$$

This action is smooth when

$$k + 1 - \frac{n}{p} > 0.$$

When $n = 4$ and $p = 2$, we need $k > 1$. In the borderline case when $k = 1$, $L_2^2 \not\hookrightarrow \mathcal{C}^0$. The completion $\mathscr{G}_2^2$ is defined to be

$$\{g \in L_2^2(X^4, \mathrm{End}(V)) : g(x) \in AdP\ a.e.\ x \in X\}.$$

It is still useful to consider this group. For $A \in \mathscr{A}_1^2$, $A = A_0 + a$ for some $a \in L_1^2(X, \mathrm{ad}\, P)$ (A_0 is a smooth background connection). We have

$$F_A = F_{A_0} + d_{A_0} a + \frac{1}{2}[a \wedge a] \in L^2.$$

The L^2-norm of F_A is the one controlled by the energy identity, so the borderline case is still important.

Lecture 12. The configuration space is Hausdorff

12.1. The configuration space is Hausdorff

We study the property of the configuration space in more details. Recall that G is compact Lie group and $P \to X$ is a principal $G-$bundle:

$$\begin{array}{ccc} G & \longrightarrow & P \\ & & \downarrow \\ & & X \end{array}$$

Let $E \to X$ be a bundle induced by a faithful linear representation of G. Then both $\operatorname{Ad} P$ and $\operatorname{ad} P$ are sub-bundles of

$$\operatorname{End}(E) = E \otimes E^*.$$

We adopt the following notations:

- $\mathscr{A}_P$ is the space of smooth connections on P.
- $\mathscr{A}_k^p$ is the L_k^p-completion of $\mathscr{A}_P$.
- $\mathscr{G}_P = \Gamma(X, \operatorname{Ad} P)$ is the space of smooth sections of $\operatorname{Ad} P$. It is also the automorphism group of P.
- $\mathscr{G}_{k+1}^p$ is the L_{k+1}^p-completion of $\mathscr{G}_P$.

When the condition

$$k + 1 - \frac{n}{p} > 0 \tag{13}$$

is satisfied, $L_k^p \hookrightarrow \mathcal{C}^0$ and the action

$$\mathscr{G}_{k+1}^p \times \mathscr{A}_k^p \to \mathscr{A}_k^p$$

is smooth. For $A \in \mathscr{A}_k^p$, the stabilizer is given by

$$\operatorname{Stab}_A = \{g \in \mathscr{G}_{k+1}^p : d_A g = 0\}.$$

We form the quotient space

$$\mathscr{B}_k^p = \mathscr{A}_k^p / \mathscr{G}_{k+1}^p.$$

Assuming (**??**), we will show that

Theorem 12.1.1. *The configuration space $\mathscr{B}_k^p$ is a Banach manifold, i.e.$\mathscr{B}_k^p$ is locally the quotient of a Banach space by a compact Lie group.*

We will achieve Step 0 in this section:

Theorem 12.1.2. *$\mathscr{B}_k^p$ is Hausdorff.*

Recall that an equivalence condition for a topological space S being Hausdorff is that the diagonal

$$\Delta \subset S \times S$$

is a closed subset. For a Hausdorff space S with an action by a Hausdorff topological group G, for the quotient space S/G to be Hausdorff, we need the graph

$$\Gamma_G = \{(s, g\cdot s) : s \in S, g \in G\} \subset S \times S$$

to be closed. We give a proof of Theorem 12.1.2 in the borderline case.

Exercise 12.1.3. *When $k+1-\frac{n}{p}=0$, define*

$$\mathscr{G}^p_{k+1} = \{g \in L^p_{k+1}(\mathrm{End}(E)), g \in \mathrm{Ad}\, P \text{ a.e.}\}.$$

then it is a group.

Proof. We will focus on the case when $n = 4$, $p = 2$ and $k = 1$. For $g, g' \in L^2_2(X, \mathrm{End}(E))$ with $g, g' \in \mathrm{Ad}\, P$ a.e., $g \cdot g' \in \mathrm{Ad}\, P$ a.e.. It remains to check the regularity:

$$\nabla^2_{A_0}(g\cdot g') = \nabla^2_{A_0}(g)\cdot g' + g\cdot \nabla^2_{A_0}(g') + 2\nabla_{A_0}(g)\cdot\nabla_{A_0}(g').$$

Usually, we run into trouble for the first two terms because $L^2_2 \not\hookrightarrow \mathcal{C}^0$. But we use the fact that G is **compact**, so $g, g' \in L^\infty$ by default. The multiplication theorem

$$L^2 \times L^\infty \hookrightarrow L^2, L^2_1 \times L^2_1 \hookrightarrow L^2.$$

implies $\nabla^2_{A_0}(g\cdot g') \in L^2$. □

Remark. $\mathscr{G}^2_2$ is not a very nice group. We will not talk about the smoothness of multiplications.

Proposition 12.1.4. *$\mathscr{B}^2_1 = \mathscr{A}^2_1/\mathscr{G}^2_2$ is Hausdorff.*

Proof. We have to show the graph

$$\Gamma = \{(A, g^*A) : A \in \mathscr{A}^2_1, g \in \mathscr{G}^2_2\} \subset \mathscr{A}^2_1 \times \mathscr{A}^2_1$$

is a closed subset. Suppose we have a sequence (A_i, g_i) such that

$$(A_i, g_i^*A_i) \to (A, B) \in \mathscr{A}^2_1 \times \mathscr{A}^2_1$$

as $i \to \infty$. Suppose

$$\begin{aligned} A_i &= A_0 + a_i, \\ g^*A_i &= A_0 + b_i \\ a_i, b_i &\in L^2_1(X, T^*X \otimes \mathrm{ad}\, P). \end{aligned}$$

Then $a_i \xrightarrow{L^2_1} a := A - A_0$ and $b_i \xrightarrow{L^2_1} b := B - A_0$. We have

$$\begin{aligned} A_0 + b_i = g^*A_i &= A_0 + g_i\nabla_{A_0}g_i^{-1} + g_i a_i g_i^{-1} \\ \Rightarrow \qquad b_i &= -(\nabla_{A_0}g_i)g_i^{-1} + g_i a_i g_i^{-1} \\ \Rightarrow \qquad \nabla_{A_0}g_i &= g_i a_i - b_i g_i. \end{aligned}$$

We wish to show that $\{g_i\}$ is bounded in L^2_2, so $\{\nabla_{A_0}g_i\}$ has to be bounded in L^2_1. This is achieved in two steps:

(1) $\{\nabla_{A_0} g_i\}$ is bounded in L^4-norm. This is because $\{a_i\},\{b_i\}$ have uniformly bounded L_1^2-norm and $\{g_i\}$ has uniformly bounded L^∞ norm. We use the multiplication theorem (when $n = 4$)

$$L_1^2 \times L^\infty \hookrightarrow L^4 \times L^\infty \hookrightarrow L^4.$$

(2) $\{\nabla_{A_0} g_i\}$ is bounded L_1^2-norm. It suffices to control $\|\nabla_{A_0}^2 g_i\|_2$:

$$\nabla_{A_0}(a_i g_i) = \underbrace{\nabla_{A_0}(a_i)}_{\in L^2} \cdot \underbrace{g_i}_{\in L^\infty} + \underbrace{a_i}_{\in L_1^2} \cdot \underbrace{\nabla_{A_0}(g_i)}_{\in L^4}.$$

Now we use the multiplication structures $L^2 \times L^\infty \hookrightarrow L^2$ and $L_1^2 \times L^4 \hookrightarrow L^2$. The boundedness of the L^4-norm of $\nabla_{A_0} g_i$ follows from (1).

This shows that $\{g_i\}$ has a weakly convergent subsequence in L_2^2. Let the limit be g. Since $L_2^2 \hookrightarrow L^2$ is compact,

$$(g_i \xrightarrow{L^2} g) \Rightarrow (g_i \xrightarrow{a.e.} g) \Rightarrow (g \in \operatorname{Ad} P \ a.e.).$$

Hence, $g \in \mathscr{G}_2^2$ is a gauge transformation. Finally, we have to show $g^*A = B \in \mathscr{A}_1^2$. It is enough to prove that they are equal in a weaker Sobolev space. Take limit of the equation

$$\nabla_{A_0} g_i = g_i a_i - b_i g_i.$$

Since $a_i \xrightarrow{L_1^2} a$ and $g_i \xrightarrow{L_1^2} g$, the right hand side converges to

$$ga - bg$$

as $i \to \infty$ in $L^2(X)$. The left hand side converges to $\nabla_{A_0} g$ in L^2, so

$$\nabla_{A_0} g = ga - bg. \qquad \square$$

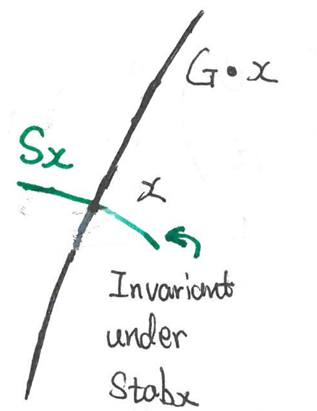
The proof relies heavily on the fact that $G \subset \operatorname{End}(V_x, V_x)$ is compact. If this assumption is removed, the quotient space might not be Hausdorff any more:

- Take $S = \mathbb{R}$ and $G = \mathbb{R}_*$. Then the quotient space consists of two points $\bar{0}, \bar{1}$. Open sets are
$$\varnothing, \{\bar{0}, \bar{1}\}, \{\bar{1}\},$$
so S/R is not Hausdorff.
- Take $S = \mathfrak{sl}(2,\mathbb{C})$ and $G = SL(2,\mathbb{C})$ with the adjoint action (taking conjugation). Then S/G consists of
  (1) $a_\lambda = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}$, $\lambda \in \mathbb{C}$. $a_\lambda \sim a_{-\lambda}$.
  (2) $b_\mu = \begin{pmatrix} 0 & \mu \\ 0 & 0 \end{pmatrix}$, $\mu \in \mathbb{C}^*$. $b_\mu \sim b_1$.

  As a set, $S/G = \mathbb{C}/\mathbb{Z}_2 \cup \{*\}$. The topology is non-Hausdorff.

12.2. Local Charts

The next step is to construct orbifold charts for the quotient space $\mathscr{B}_k^p$. Suppose we have a finite dimensional manifold M acted by a compact Lie group G. Pick $x \in M$. A transverse S_x is a sub-manifold of M that **intersects transversely with the orbit** $G \cdot x$. The idea is to find a transverse S_x **invariant under the action of** Stab_x.



A tidy way to organize the construction is the following:

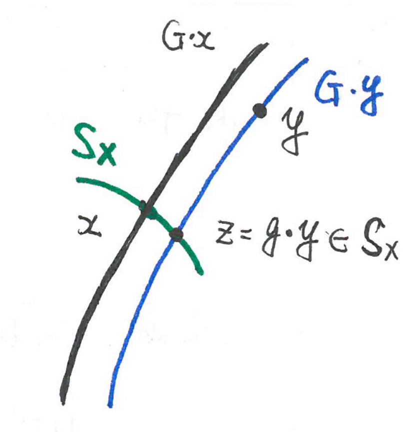
(1) If the action is free, consider the map
$$\begin{aligned} G \times S_x &\to M \\ (g, s) &\mapsto g \cdot s. \end{aligned}$$
and show it is a diffeomorphism onto its image when S_x is small enough. This reduces to the fact that $T_x S_x \oplus T_x G \cong T_x M$.

(2) Suppose the action is not free and S_x is Stab_x-invariant. Then Stab_x acts on $G \times S_x$
$$h(g, s) \mapsto (gh^{-1}, hs).$$
This action is free. By the first case, the quotient space
$$G \times S_x / \mathrm{Stab}_x$$
is a manifold. Consider the map
$$\begin{aligned} \phi : G \times S_x / \mathrm{Stab}_x &\to M \\ [(g, s)] &\mapsto g \cdot s. \end{aligned}$$
When S_x is small enough, ϕ is a diffeomorphism onto a neighborhood of the orbit $G \cdot x$. Then the orbifold chart comes from
$$S_x / \mathrm{Stab}_x \cong (G \times S_x / \mathrm{Stab}_x)/G \subset M/G.$$

If someone is ever confused with the definition of an orbifold, he/she can actually read off all necessary conditions to define an orbifold from this picture. If y is contained in the image of ϕ, then it is identified with

$$[(g, z)] \in G \times S_x / \operatorname{Stab}_x$$

for some $g \in G$ and $z \in \operatorname{Stab}_x$. Therefore, $\operatorname{Stab}_y \subset \operatorname{Stab}_x$. One can also realize the construction of the transition map ϕ_{xy} from an orbifold chart (S_y, ϕ_y) to (S_x, ϕ_x).



Return to the infinite dimensional case. We wish to define slices for the $\mathscr{G}$-action on $\mathscr{A}$:

$$(g, A) \mapsto A + g d_A g^{-1} = A - g^{-1} d_A g.$$

Consider the differential of this action:

$$T_e\mathscr{G} = \Omega^0(X, \operatorname{ad} P) \to \Omega^1(X, \operatorname{ad} P)$$
$$\xi \mapsto -d_A\xi.$$

Since the L^2-inner product is $\mathscr{G}$-invariant, consider the L^2-complement of $\operatorname{Im} T_e\mathscr{G}$. Define

$$S_A = \{a \in \Omega^1(X, \operatorname{ad} P) : \langle d_A\xi, a\rangle_{L^2(X)} = 0,\ \forall \xi \in \Omega^0(X, \operatorname{ad} P)\}.$$

When X is closed,

$$(\langle d_A\xi, a\rangle_{L^2(X)} = 0,\ \forall\xi) \Leftrightarrow (d_A^* a = 0).$$

When X is not closed,

$$(\langle d_A\xi, a\rangle_{L^2(X)} = 0,\ \forall\xi) \Leftrightarrow (d_A^* a = 0, *a|_{\partial X} = 0).$$

An interesting question is how large a local orbifold chart could be. In our case, it is necessary to fix a Sobolev completion. A stronger Sobolev topology we use, a smaller chart we get. But it is enough for applications, since there are bijections between moduli spaces $\{F_A^+ = 0\}/\mathscr{G}$ in different Sobolev completions.

It is convenient to work with a canonical (gauge invariant) smooth metric. Another example is the moduli space of Riemannian metrics on X. The space $\operatorname{Met}(X)$ of smooth metrics on X is acted on by the diffeomorphism group $\mathrm{Diff}(X)$. It is known that for a fixed metric g, the isometry group is a compact Lie group if X is compact. This question was addressed in Ebin's Thesis. For an announcement of results, see

- Ebin, David G. On the space of Riemannian metrics. Bull. Amer. Math. Soc. 74 1968 1001-1003.

In this case,

$$\begin{aligned} T_e \,\mathrm{Diff}(X) &= \Gamma(X, TX) \\ T\,\mathrm{Met} &= \mathrm{Sym}^2(T^*X), \end{aligned}$$

and the differential of the group action is

$$X \mapsto \frac{\partial}{\partial t}\phi^*_{X,t}g\Big|_{t=0} = \mathscr{L}_X g.$$

One can show this first-order operator has an injective symbol. Taking Sobolev completions, Ebin figured out a $\mathrm{Diff}^2_{k+1}(X)$-invariant metrics on Met^2_k and tried the same strategy as above.

Lecture 13. **The Construction of Slices**

This Monday happened to be the Open House for prospective (graduate) students at MIT. Tom expected some of them would come to his class, so he prepared a special lecture introducing applications of gauge theory on low dimensional topology and made a PPT. Unfortunately, we waited for 10 minutes and none of them showed up. Tom was very frustrated and decided to continue on his normal lectures.

This is the reason why this lecture was shorter than others.

Here is a digression. Taubes studied $SL(2,\mathbb{C})$-connections and the set up is as follows. Suppose we have an $SU(2)$-principal bundle $P \to X$. It induces an $SL(2,\mathbb{C})$ bundle:

$$P \times_\iota SL(2,\mathbb{C}) \to X.$$

by the fundamental representation $\iota : SU(2) \hookrightarrow SL(2,\mathbb{C})$. Since

$$\mathfrak{sl}(2,\mathbb{C}) = \mathfrak{su}(2) \oplus i\mathfrak{su}(2),$$

an $SL(2,\mathbb{C})$-connection can be written as

$$A + i\Phi$$

with $\Phi \in \Omega^1(Y, \operatorname{ad} P)$ and an $SU(2)$-connection $A \in \mathscr{A}_P$. We still use $SU(2)$-gauge transformations:

$$\mathscr{G}_P.$$

It is a good set-up for some questions. We mentioned in the last lecture that $\mathfrak{sl}(2,\mathbb{C})/SL(2,\mathbb{C})$ is not Hausdorff. Nevertheless, if we can get rid of nilpotent elements, then it is still fine.

13.1. The construction of Slices

We wish to construct orbifold charts for $\mathscr{A}^p_k/\mathscr{G}^p_{k+1}$. For $A \in \mathscr{A}^p_k$, let

$$S_{A,\epsilon} = \{A + a : d_A^* a = 0, *a|_{\partial X} = 0, \|a\|_{L^p_k} < \epsilon\}.$$

Then the Hodge theory tells us (we need to solve a Neumann boundary value problem for $d_A^* d_A$):

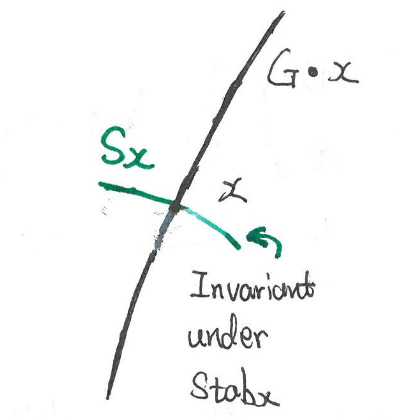
$$L^p_k(X, T^*X \otimes \operatorname{ad} P) = \operatorname{Im} d_A \oplus \{a \in L^p_k(X, T^*X \otimes \operatorname{ad} P) : d_A^* a, *a|_{\partial X} = 0\}$$

where

$$d_A : L^p_{k+1}(X, \operatorname{ad} P) \to L^p_k(X, T^*X \otimes \operatorname{ad} P).$$

Consider the map

$$\begin{aligned} m : \mathscr{G}^p_{k+1} \times S_{A,\epsilon} &\to \mathscr{A}^p_k. \\ (g, A + a) &\mapsto g^*(A + a) = A + a + g d_{A+a} g^{-1}. \end{aligned}$$



and its differential at (e, A):

$$\mathcal{D}_{(e,A)}m : L^p_{k+1}(X, \operatorname{ad} P) \times \{a \in L^p_k : d_A^* a = 0, *a|_{\partial X}\} \to L^p_k.$$
$$(\xi, a) \mapsto -d_A\xi + a.$$

Hence, by the Hodge theory, we have

Theorem 13.1.1. *The differential $\mathcal{D}m$ is surjective at (e, A) for all $g \in \mathscr{G}^p_{k+1}$.*

If (ξ, α) lies in the kernel of $\mathcal{D}_{(e,A)}m$, then

$$\begin{aligned} & d_A\xi + \alpha = 0 \\ \Rightarrow \quad & d_A\xi = 0, \alpha = 0 \\ \Rightarrow \quad & \ker \mathcal{D}_{(e,A)}m \cong \text{ the Lie algebra of } \operatorname{Stab}_A. \end{aligned}$$

The goal is to show that

$$\bar{m} : \mathscr{G}^p_{k+1} \times S_{A,\epsilon}/\operatorname{Stab}_A \to \mathscr{A}^p_k.$$

is a diffeomorphism if $\epsilon \ll 1$.

Theorem 13.1.2. *When $\epsilon \ll 1$, $\bar{m}$ is a local diffeomorphism onto its image.*

Proof. We have shown that $D_{(e,A)}\bar{m}$ is a bijection. By implicit function theorem, for a tiny Stab_A-invariant neighborhood U of $\operatorname{Stab}_A \subset \mathscr{G}^p_{k+1}$ and $\epsilon \ll 1$,

$$U \times S_{A,\epsilon}/\operatorname{Stab}_A$$

is a diffeomorphism onto its image, so the theorem is proved at the point

$$[(e, A)]$$

For other points, we use the fact that $\bar{m}$ is $\mathscr{G}^p_{k+1}$-equivariant:

$$\begin{tikzcd}
U \times S_{A,\epsilon}/\operatorname{Stab}_A \arrow[r, "m"] \arrow[d, "g"] & \mathscr{G}^p_k \arrow[d, "g"] \\
(g \cdot U) \times S_{A,\epsilon}/\operatorname{Stab}_A \arrow[r, "m"] & \mathscr{G}^p_k.
\end{tikzcd}$$

Since vertical maps are diffeomorphisms, m is a diffeomorphism on $(g \cdot U) \times S_{A,\epsilon}/\operatorname{Stab}_A$ which is a neighborhood of $[(g, A)]$. This proves the theorem. $\square$

To prove $\bar{m}$ is a diffeomorphism globally, it remains to show $\bar{m}$ is injective for $\epsilon \ll 1$.

Suppose we have

$$g^*(A + a) = A + b$$

for some $a, b \in S_{A,\epsilon}$ and $g \in \mathscr{G}^p_{k+1}$. We wish to show that $g \in \operatorname{Stab}_A$. We know from the last lecture that

$$d_A g = ga - bg$$

as a section of $\operatorname{End}(E)$. Apply d_A^*, then

$$\begin{aligned}
d_A^* d_A g &= d_A^*(ga - bg) \\
&= - * d_A\big(g(*a) - (*b)g\big) \\
(\text{since } d_A^* a = d_A^* b = 0) &= - * (d_A g \wedge *a - (-1)^{n-1} * b \wedge d_A g).
\end{aligned}$$

The above computation shows

$$d_A^* d_A g = \Phi(d_A g, a) - \Psi(b, d_A g).$$

for some bilinear bundle maps Φ, Ψ.

Lemma 13.1.3. *For some $C > 0$,*

$$\|d_A^* d_A g\|_{(L^2_1)^*} \geqslant C \|d_A g\|_{L^2}.$$

for all $g \in L^2_1(X, \operatorname{End}(E))$.

Assuming this lemma, we have

$$\begin{aligned}
C\|d_A g\|_{L^2} &\leqslant \|d_A^* d_A g\|_{(L^2_1)^*} \\
&\leqslant \|\Phi(d_A g, a)\|_{(L^2_1)^*} + \|\Psi(b, d_A g)\|_{(L^2_1)^*}. \\
&\leqslant C_1(\|d_A g\|_{L^2}\|a\|_{L^n} + \|d_A g\|_{L^2}\|b\|_{L^n}).
\end{aligned} \tag{14}$$

In the middle, we used the fact that in dimension $n > 2$, $L^2_1 \hookrightarrow L^{2n/(n-2)}$ and

$$L^{2n/(n-2)} \times L^2 \times L^n \to L^1 \xrightarrow{\int_X} \mathbb{R}$$

is bounded linear. Indeed,

$$\frac{n-2}{2n} + \frac{1}{2} + \frac{1}{n} = 1.$$

This shows $L^2 \times L^n \to (L^2_1)^*$ is bounded linear.

Using (14), if

$$\|a\|_{L^n} + \|b\|_{L^n} \leqslant \frac{C}{2C_1},$$

then $\|d_A g\|_2$ has to be zero and we are done. This requirement is independent of the choice of Sobolev completion of $\mathscr{G}$. The injectivity holds for a ball of a much weaker Sobolev norm (L^n). The L^n-norm is important here, because the action

$$\mathscr{G}_1^n \curvearrowright \mathscr{A}_0^n$$

is precisely at the borderline of the Sobolev embedding. But we can still obtain injectivity as above.

In the next lecture, we will discuss Uhlenbeck's Fundamental Lemma:

Theorem 13.1.4. *There exists $C, \epsilon > 0$ such that for any L_1^2-connection A on B^4 with*

$$\int_{B^4} |F_A|^2 \leqslant \epsilon,$$

we can find a gauge transformation $g \in \mathscr{G}_2^2$ such that

$$g^* A = \Gamma + a$$

where Γ is the trivial connection and

(1) $d_\Gamma^* a = 0$*,*
(2) $*a|_{\partial B^4} = 0$*,*
(3) $\int_{B^4} |\nabla_\Gamma a|^2 + |a|^2 \leqslant C \int_{B^4} |F_A|^2$.

There are generalizations of this result in higher dimensions. The energy

$$\int_{B^4} |F_A|^2$$

is conformally invariant in dimension 4, which is no longer true in higher dimensions. If we stick to the L^2-norm, we must keep track of the size of n-ball $B^n(0, R)$. In this case, the constant in (3)

$$C = C(R)$$

depends on the radius. Instead, we can work with

$$\int_{B^n} |F_A|^{\frac{n}{2}}$$

which is still invariant under the rescaling of the metric. A modification of Uhlenbeck's theorem will continue to hold then.

Exercise 13.1.5. *Prove Lemma* 13.1.3*.*

Proof. We assume the regularity $g \in L_1^2$ and

$$a, b \in L_1^{n/2}.$$

We think of g as a section of $\mathrm{End}(E) \to X$, not as a a gauge transformation. The equation

$$d_A g = ag - gb$$

still holds in L^2. Because of the multiplication structure,

$$L_1^{n/2} \times L_1^2 \to L_1^{\frac{2n}{n+2}}$$

$d_A g \in L_1^{\frac{2n}{n+2}}$ and it makes sense to talk about its boundary value. Since $*a, *b = 0$ on ∂X,

$$*d_A g = 0$$

on ∂X. Therefore, we are allowed to use integration by parts without worrying about the boundary term:

$$\begin{aligned} \|d_A^* d_A g\|_{L_1^2(X)^*} &= \sup_{\|v\|_{L_1^2}=1} \langle d_A^* d_A g, v\rangle_{L^2} \\ &= \sup_{\|v\|_{L_1^2}=1} \langle d_A g, d_A v\rangle_{L^2}. \end{aligned}$$

Write $g = g_1 + g_2$ with

$$g_1 \in \ker d_A, \langle g_2, g_1\rangle_{L^2} = 0.$$

Then $\|d_A g\|_2 = \|d_A g_2\|_2 \geqslant C\|g_2\|_{L_1^2}$ for some $C > 0$. Let

$$v = \frac{g_2}{\|g\|_{L_1^2}}$$

be the test function. Hence,

$$\begin{aligned} \|d_A^* d_A g\|_{L_1^2(X)^*} &\geqslant \frac{1}{\|g_2\|_{L_1^2}} \langle d_A g, d_A g\rangle_{L^2} \\ &\geqslant \frac{C}{\|d_A g\|_2} \|d_A g\|_2^2 \\ &\geqslant C\|d_A g\|_2. \end{aligned}$$

The heuristic for this estimate is that d_A^* is injective on $\mathrm{Im}\, d_A$:

$$\begin{tikzcd} \Omega^0 \arrow[r, "d_A"'] & \Omega^1, \arrow[l, bend right, "d_A^*"'] \end{tikzcd}$$

so we expect that $\|d_A^* d_A g\|_{L_{k+1}^2} \geqslant \|d_A g\|_{L_k^2}$ for any k. When X has a non-empty boundary, it is not true that

$$L_{-k}^2 = (L_k^2)^*.$$

Instead,

$$L_{-k}^2 = (L_{k,0}^2)^*.$$

Here, $L^2_{k,0}$ denotes the space of L^2_k functions whose restriction at the boundary is zero ($k > \frac{1}{2}$). This space allows us to do integration by parts without worrying about the boundary term. For this exercise, we have to be careful about the boundary.

$\square$

Lecture 14. The Big-Slice theorem

14.1. The Big-Slice theorem

Consider the space $\mathscr{A}_0^p$ of L^p-connections and L_1^p gauge transformations $\mathscr{G}_1^p$. We assume $p > n$ where n is the dimension of the base manifold X, so

$$L_1^p \hookrightarrow L^\infty$$

and L_1^p is an algebra. Let

$$S_{A,\epsilon}^{\text{big}} = \{a \in L^p : d_A^* a = 0, *a|_{\partial X}, \|a\|_{L^n} < \epsilon\}.$$

be the big slice around $A \in \mathscr{A}^p$ of radius ϵ. It is called big because the L^n-norm is used in the definition, which is weaker than the L^p-norm.

Remark. For an L^p section a, the Sobolev restriction theorem fails and $*a|_{\partial X}$ is not well-defined. Nevertheless, if $d_A^* a = 0$, the boundary value $\langle a, \vec{n}\rangle dvol_{\partial X} = *a|_{\partial X}$ still exists in the sense of distribution:

$$\langle x, \langle a, \vec{n}\rangle\rangle_{\partial X} := \langle d_A \tilde{x}, a\rangle_{L^2(X)} - \underbrace{\langle \tilde{x}, d_A^* a\rangle}_{=0}.$$

Here, x is a test function on ∂X and $\tilde{x}$ is a reasonable extension of x on X. For this lecture, it is recommended to ignore the boundary and assume X is closed. The analysis will become easier then. $\square$

In the previous lecture, we studied the $\mathscr{G}_1^p$ equivariant map

$$\bar{m} : (\mathscr{G}_1^p \times S_{A,\epsilon}^{\text{big}})/\operatorname{Stab}_A \to \mathscr{A}^p.$$

and proved that

Theorem 14.1.1. *If $\epsilon \ll 1$, then $\bar{m}$ is injective.*

We wish to extend this result and show that

(1) $\bar{m}$ is a diffeomorphism onto its image.
(2) $\operatorname{Im}(\bar{m})$ contains an L^n-neighborhood of A.

The first statement only implies that $\operatorname{Im}(\bar{m})$ contains an L^p-neighborhood of A. The second statement is stronger, since it implies

- There exists $\delta(\epsilon) > 0$ such that for any $A' \in \mathscr{A}_0^p$ with $\|A' - A\|_{L^n} < \delta$, we can find a gauge transformation $g \in \mathscr{G}_1^p$ with $g^* A' \in S_{A,\epsilon}^{\text{big}}$.

This is the content of **the Big-Slice theorem**.

For (1), it suffices to check that for any $a \in S_{A,\epsilon}^{\text{big}}$, the linearization

$$\mathcal{D}_{(e,A+a)}\bar{m}$$

is invertible.

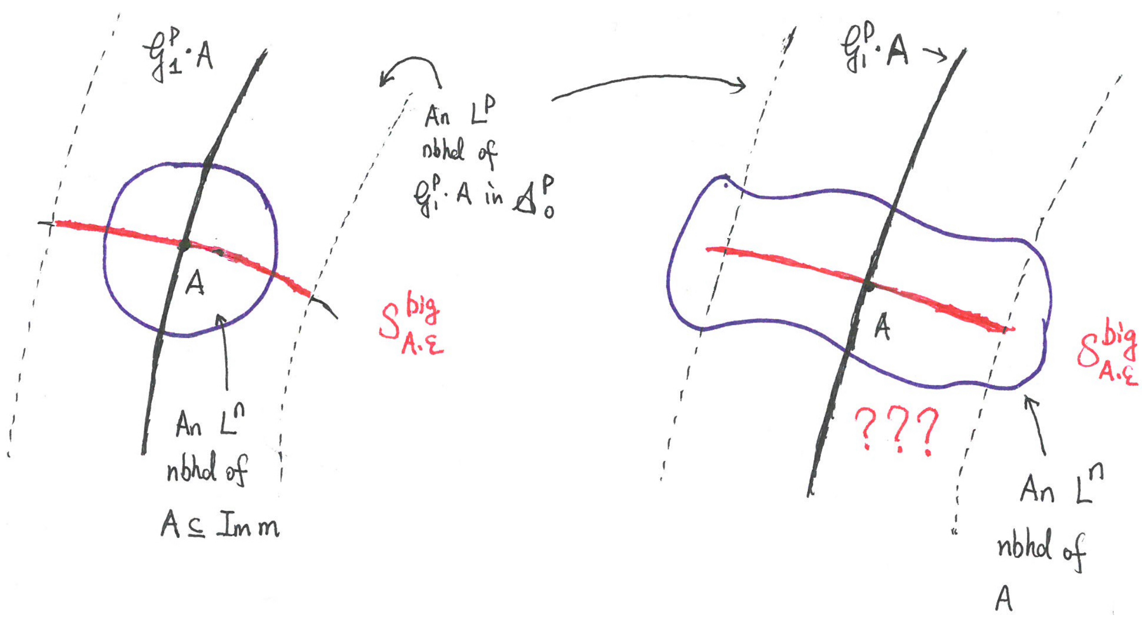


FIGURE 1. The left picture is accurate.

Before we pass to the quotient space, the linearized operator of

$$m : \mathscr{G}_1^p \times S^{\text{big}}_{A,\epsilon} \to \mathscr{A}^p$$
$$(g, A + a) \mapsto (A + gag^{-1} + gd_A g^{-1}).$$

at $(e, A + a)$ is

$$\mathcal{D}_{(e,A+a)} m(\xi, \alpha) = \alpha - (d_A \xi + [a, \xi]).$$

When $a = 0$, it is a surjection between spaces:

$$L_1^p(X, \operatorname{ad} P) \oplus \{\alpha \in L^p : d_A^* \alpha = 0, \alpha|_{\partial X}\} \to L^p(X, T^*X \otimes \operatorname{ad} P).$$

When $a \in S^{\text{big}}_{A,\epsilon}$ is allowed to change, applying the multiplication theorem (*Remark*: **this estimate is incorrect**)

$$L_1^p \times L^n \to L^p$$

to the term $[a, \xi]$, we see that when $\|a\|_n \ll \epsilon$ for some ϵ small enough,

$$\mathcal{D}_{(e,A+a)} m$$

remains surjective. Hence, (1) follows from the inverse function theorem.

Remark. This proof is incorrect because the multiplication theorem here was wrong when $p > n$. Tom corrected this mistake in the next lecture.

The second part is harder. The first step is an a priori estimate:

Lemma 14.1.2. *Let $A \in \mathscr{A}^p$, we can find $\delta, K > 0$ such that the following property holds*

- *Suppose $A + b \in \mathscr{A}^p$ and for some $g \in \mathscr{G}_1^p$,*
$$g \cdot (A + b) = A + a$$
with $A + a \in S^{big}_{A,\epsilon}$. If $\|a\|_n + \|b\|_n \leqslant \delta$, then
$$\|d_A g\|_n + \|a\|_n \leqslant K\|b\|_n$$
$$\|d_A g\|_p + \|a\|_p \leqslant K\|b\|_p.$$

This lemma is saying that if $A + b$ is gauge equivalent to a connection $A + a$ in the big slice $S^{\text{big}}_{A,\epsilon}$, the L^n-norm of a and L^n_1-norm of g is controlled by $\|b\|_n$ given that $\|b\|_n$ and $\|a\|_n$ are small. It is an a priori estimate because we have assumed the existence of a gauge transformation g.

Proof. Recall from previous lectures that
$$g \cdot (A + b) = A + a$$
implies that
$$d_A g = gb - ag.$$
Apply d_A^* to this equation and use the fact that $d_A^* a = 0$:
$$d_A^* d_A g = \Phi_1(d_A g, b) + \Phi_2(g, d_A^* b) + \Psi(a, d_A g).$$
where Φ_1, Φ_2 and Ψ are bilinear bundle maps. Their expressions are not important (see Lecture 13). What we need to know from analysis is that

- If $\partial X = \varnothing$, $d_A^* d_A : L^n_1 \to (L^n_1)^*$ is bounded linear and Fredholm of index 0. $\ker d_A^* d_A = \ker d_A$ consists of parallel sections.
- $d_A^* d_A : L^n_1 \to (L^n_1)^*$ various continuously for $A \in \mathscr{A}^p$ (but not for $A \in \mathscr{A}^n$).
- For some $K(A) > 0$,
$$\|d_A^* d_A \xi\|_{(L^n_1)^*} \geqslant K(A)\|d_A \xi\|_n.$$
The constant $K(A)$ depends continuously on $A \in \mathscr{A}^p$, (but not in $\mathscr{A}^n$).

Hence, we have
$$\begin{aligned} K(A)\|d_A g\|_n &\leqslant \|d_A^* d_A g\|_{(L^n_1)^*} \\ &\leqslant \|\Phi_1(d_A g, b)\|_{(L^n_1)^*} + \|\Phi_2(g, d_A^* b)\|_{(L^n_1)^*} + \|\Psi(a, d_A g)\|_{(L^n_1)^*}. \end{aligned}$$
We wish to control the last three terms in terms of $\|d_A g\|_n$. We will use the multiplication theorem:

- For Φ_1 and Ψ, use $L^n \times L^n \times L^n_1 \to L^1 \xrightarrow{\int_X} \mathbb{R}$. For this to hold, since $L^n_1 \hookrightarrow L^q$ for any $q < \infty$, we need
$$\frac{1}{n} + \frac{1}{n} + \frac{1}{q} < 1,$$
which is $n > 2$. This implies that
$$L^n \times L^n \to (L^n_1)^*$$

and

$$\|\Phi_1(d_A g, b)\|_{(L_1^n)^*} \lesssim \|d_A g\|_n \|b\|_n,$$
$$\|\Psi(a, d_A g)\|_{(L_1^n)^*} \lesssim \|d_A g\|_n \|a\|_n.$$

- For Φ_2, use $L_1^n \times L_{-1}^n \times L^\infty \to L^1$, so
$$(L_1^n)^* \times L^\infty \to (L_1^n)^* = L_{-1}^n \ (\partial X = \varnothing).$$
Therefore, we have
$$\|\Phi_2(g, d_A^* b)\|_{(L_1^n)^*} \lesssim \|g\|_\infty \|d_A^* b\|_{(L_1^n)^*} \lesssim \|g\|_\infty \|b\|_n.$$

Putting all these together, we obtain

$$\begin{aligned} K(A)\|d_A g\|_n &\leqslant C_1 \|d_A g\|_n (\|a\|_n + \|b\|_n) + C_2 \|b\|_n \\ &\leqslant C_1 \delta \|d_A g\|_n + C_2 \|b\|_n. \end{aligned}$$

if $\|a\|_n + \|b\|_n \leqslant \delta$. Suppose $2C_1\delta \leqslant K(A)$, then

$$\|d_A g\|_n \leqslant \frac{C_2}{K(A) - C_1\delta} \|b\|_n \leqslant \frac{2C_2}{K(A)} \|b\|_n.$$

As for $\|a\|_n$, we exploit the equation:

$$a = gbg^{-1} - d_A g \cdot g^{-1}$$

and the fact that $g \in L^\infty$. Hence,

$$\|a\|_n \leqslant \|b\|_n + \|d_A g\|_n \leqslant C_3 \|b\|_n. \qquad \square$$

Exercise 14.1.3. *Prove Lemma* 14.1.2 *for the* L^p*-norm.*

Proof. We will use the following facts from analysis:

- If $\partial X = \varnothing$, $d_A^* d_A : L_1^p \to (L_1^p)^*$ is bounded linear and Fredholm of index 0. $\ker d_A^* d_A = \ker d_A$ consists of parallel sections.
- $d_A^* d_A : L_1^p \to (L_1^p)^*$ various continuously for $A \in \mathscr{A}^p$.
- For some $K(A) > 0$,
$$\|d_A^* d_A \xi\|_{(L_1^p)^*} \geqslant K(A) \|d_A \xi\|_p.$$
The constant $K(A)$ depends continuously on $A \in \mathscr{A}^p$.

Hence, we have

$$\begin{aligned} K(A)\|d_A g\|_p &\leqslant \|d_A^* d_A g\|_{(L_1^p)^*} \\ &\leqslant \|\Phi_1(d_A g, b)\|_{(L_1^p)^*} + \|\Phi_2(g, d_A^* b)\|_{(L_1^p)^*} + \|\Psi(a, d_A g)\|_{(L_1^p)^*}. \end{aligned}$$

We wish to control the last three terms in terms of $\|d_A g\|_p$. We will use the multiplication theorem:

- For Φ_1 and Ψ, use $L^p \times L^n \times L^p_1 \to L^1 \xrightarrow{\int_X} \mathbb{R}$. For this to hold, since $L^p_1 \hookrightarrow L^\infty$, we need
$$\frac{1}{p} + \frac{1}{n} \leqslant 1$$
which is true if $n \geqslant 2$. This implies
$$L^p \times L^n \to (L^p_1)^*$$
and
$$\|\Phi_1(d_A g, b)\|_{(L^p_1)^*} \lesssim \|d_A g\|_p \|b\|_n,$$
$$\|\Psi(a, d_A g)\|_{(L^p_1)^*} \lesssim \|d_A g\|_p \|a\|_n.$$
- For Φ_2, use $L^p_1 \times L^p_{-1} \times L^\infty \to L^1$, so
$$(L^p_1)^* \times L^\infty \to (L^p_1)^* = L^p_{-1} \ (\partial X = \varnothing).$$
Therefore, we have
$$\|\Phi_2(g, d_A^* b)\|_{(L^p_1)^*} \lesssim \|g\|_\infty \|d_A^* b\|_{(L^p_1)^*} \lesssim \|g\|_\infty \|b\|_p.$$

Putting all these together, we obtain
$$\begin{aligned} K(A)\|d_A g\|_p &\leqslant C_1 \|d_A g\|_p (\|a\|_n + \|b\|_n) + C_2 \|b\|_p \\ &\leqslant C_1 \delta \|d_A g\|_p + C_2 \|b\|_p. \end{aligned}$$
if $\|a\|_n + \|b\|_n \leqslant \delta$. Suppose $2C_1\delta \leqslant K(A)$, then
$$\|d_A g\|_p \leqslant \frac{C_2}{K(A) - C_1\delta} \|b\|_p \leqslant \frac{2C_2}{K(A)} \|b\|_p.$$

As for $\|a\|_p$, we exploit the equation:
$$a = gbg^{-1} - d_A g \cdot g^{-1}$$
and the fact that $g \in L^\infty$. Hence,
$$\|a\|_p \leqslant \|b\|_p + \|d_A g\|_p \leqslant C_3 \|b\|_p.$$
□

Now we are ready to prove the Big-Slice theorem:

Proof of the Big-Slice theorem. It remains to prove (2). We will use the method of continuity. Let
$$U_\delta = \{A + b : b \in L^p, \|b\|_n < \delta\}.$$
The goal is to show if δ is small enough,
$$U_\delta \subset m(\mathscr{G}^p_1 \times S^{\mathrm{big}}_{A,\epsilon}).$$
The proof is divided into three steps:

- *Step* 1. U_δ is connected. This is obvious.
- *Step* 2. $U_\delta \cap m(\mathscr{G}^p_1 \times S^{\mathrm{big}}_{A,\epsilon})$ is open. Both of them are open, so is their intersection. The L^n-ball U_δ is open, because $L^p \hookrightarrow L^n$.

- *Step* 3. $U_\delta \cap m(\mathscr{G}_1^p \times S_{A,\epsilon}^{\text{big}})$ is closed inside U_δ.

Only the last step is at issue. Suppose we have a convergent sequence of $b_i \in U_\delta \cap m(\mathscr{G}_1^p \times S_{A,\epsilon}^{\text{big}})$ with $\lim b_i = b \in U_\delta$, so there are pairs $(g_i, a_i) \in \mathscr{G}_1^p \times S_{A,\epsilon}^{\text{big}}$ such that

$$g_i \cdot (A + b_i) = A + a_i \in S_{A,\epsilon}^{\text{big}}.$$

This implies that

$$\|a_i\|_n + \|b_i\|_n \leqslant \epsilon + \delta.$$

for each pair (a_i, b_i). By making $\epsilon + \delta$ small, (a_i, b_i) satisfies the condition of Lemma 14.1.2. Therefore, for some $K > 0$,

$$\|d_A g_i\|_n + \|a_i\|_n \leqslant K\|b_i\|_n \tag{15}$$

$$\|d_A g_i\|_p + \|a_i\|_p \leqslant K\|b_i\|_p. \tag{16}$$

Since $b_i \to b$ in L^p, $\|b_i\|_p$'s are uniformly bounded. The second estimate then implies

$$\|d_A g_i\|_p, \|a_i\|_p$$

are uniformly bounded and we can find weakly convergent subsequence:

$$g_i \xrightarrow{w-L_p^1} g, a_i \xrightarrow{w-L^p} a.$$

Since $L_1^p \hookrightarrow L^\infty$ is compact, $g \in \mathscr{G}_1^p$ is a gauge transformation. Taking the weak limit of the equation in L^p,

$$d_A g_i = g_i b_i - a_i g_i.$$

we obtain

$$d_A g = gb - ag.$$

It remains to verify that a lies in the big slice $S_{A,\epsilon}^{\text{big}}$:

- $d_A^* a = 0$ because it is the weak limit of $\{d_A^* a_i\}$.
- Since $a_i \to a$ weakly in L^n, $\|a\|_n \leqslant K\|b\|_n \leqslant K\delta < \epsilon$, where we use the estimate (15). The last inequality $K\delta < \epsilon$ is achieved by making δ small.

This completes the proof. $\square$

We are on the way towards Uhlenbeck's Fundamental Lemma. We focus on dimension 4 and $X = B^4$ with the Euclidean metric:

$$\begin{array}{c} P = B^4 \times G \\ \downarrow \\ B^4. \end{array}$$

Theorem 14.1.4. *There exists $C, \epsilon > 0$ such that for any L_1^2-connection $A = \Gamma + a$ on B^4 with*

$$\int_{B^4} |F_A|^2 \leqslant \epsilon,$$

we can find a gauge transformation $g \in \mathscr{G}_2^2$ *such that*

$$g^*A = \Gamma + b$$

where Γ *is the trivial connection and*

(1) $d_\Gamma^* b = 0$,
(2) $*b|_{\partial B^4} = 0$.
(3) $\int_{B^4} |\nabla_\Gamma b|^2 + |b|^2 \leqslant C \int_{B^4} |F_A|^2$.

Informally, $[\Gamma] \in \mathscr{A}_1^2/\mathscr{G}_2^2$ is an isolated non-degenerate critical point (in fact, a minimizer) of the energy functional:

$$E(A) := \int_{B^4} |F_A|^2.$$

Remark. If f is a Morse function on a finite dimensional manifold M with a unique minimizer p, then for x close to p:

$$0 \leqslant f(x) - f(p) \sim |x-p|^2,$$

since the Hessian at p is non-degenerate. If we view $E(A)$ as a functional on L_1^2, then (3) is saying

$$E(A) \sim \inf_{g \in \mathscr{G}_2^2} \|g \cdot A - \Gamma\|_{L_1^2}.$$

Therefore, the minimizer $[\Gamma] \in \mathscr{A}_1^2/\mathscr{G}_2^2$ of $E(A)$ has to be non-degenerate (informally), and so isolated. □

The proof is again by the method of continuity. Let

$$V_\epsilon = \{\Gamma + a : a \in L_1^2, E(A) < \epsilon\},$$
$$W_\epsilon = \{A \in V_\epsilon : \exists g \in \mathscr{G}_2^2, s.t. \text{ conclusions hold}\}.$$

The proof is again subdivided into three steps. We have to show that when $\epsilon \ll 1$ and $C \gg 0$,

- *Step* 1. V_ϵ is connected.
- *Step* 2. W_ϵ is closed in V_ϵ.
- *Step* 3. W_ϵ is open in V_ϵ.

These steps will imply $V_\epsilon = W_\epsilon$. The Big-Slice theorem will imply *Step* 3. The question is how to get the estimate (3) out of (1)(2). We will present a proof in the next lecture.

Lecture 15. Uhlenbeck's Fundamental Lemma

15.1. A correction to the proof of the Big-Slice Theorem

Consider the space $\mathscr{A}_0^p$ of L^p-connections and L_1^p gauge transformations $\mathscr{G}_1^p$. We assume $p > n$ where n is the dimension of the base manifold X, so

$$L_1^p \hookrightarrow L^\infty$$

and L_1^p is an algebra. Let

$$S_{A,\epsilon}^{\mathrm{big}} = \{a \in L^p : d_A^* a = 0, *a|_{\partial X}, \|a\|_{L^n} < \epsilon\}.$$

be the big slice around $A \in \mathscr{A}^p$ of radius ϵ.

Previously, we studied the $\mathscr{G}_1^p$-equivariant map

$$\bar{m} : (\mathscr{G}_1^p \times S_{A,\epsilon}^{\mathrm{big}})/\operatorname{Stab}_A \to \mathscr{A}^p.$$

and proved the Big-Slice theorem:

Theorem 15.1.1 (The Big-Slice theorem)**.** *If $\epsilon \ll 1$, then*

(1) $\bar{m}$ is a diffeomorphism onto its image.
(2) $\operatorname{Im}(\bar{m})$ contains an L^n-neighborhood of A.

The proof of (1) is divided in two steps:

(a) When $\epsilon \ll 1$, $\bar{m}$ is injective.
(b) When $\epsilon \ll 1$, the linearized operator $\mathcal{D}_{(e,A+a)}\bar{m}$ is invertible.

Step (a) is proved in Lecture 13. The proof of (b) in Lecture 14 was incorrect, and Tom did a correct proof this time. Occasionally, Tom did not want to look at his notes in hopes that he may come up with a simpler proof on his own. But sometimes the proof is too simple to be correct.

Before we pass to the quotient space, the linearized operator of

$$\begin{aligned} m : \mathscr{G}_1^p \times S_{A,\epsilon}^{\mathrm{big}} &\to \mathscr{A}^p \\ (g, A + a) &\mapsto (A + gag^{-1} + g d_A g^{-1}). \end{aligned}$$

at $(e, A + a)$ is

$$\mathcal{D}_{(e,A+a)} m(\xi, \alpha) = \alpha - (d_A \xi + [a, \xi]).$$

When $a = 0$, it is a surjection between spaces:

$$L_1^p(X, \operatorname{ad} P) \oplus \{\alpha \in L^p : d_A^* \alpha = 0, \alpha|_{\partial X}\} \to L^p(X, T^*X \otimes \operatorname{ad} P).$$

Since $L_1^p \hookrightarrow L^\infty$ is compact (X is compact), the composition

$$L_1^p \times L^p \to L^\infty \times L^p \to L^p$$

is also compact. When $a \neq 0$, by applying this multiplication theorem to the term $[a, \xi]$, we see that the linearized operator

$$\mathcal{D}_{(e,A+a)} m = \mathcal{D}_{(e,A)} m - [a, \cdot]$$

is the sum of a Fredholm operator and a compact operator, so it is Fredholm. Moreover,

$$\operatorname{Ind} \mathcal{D}_{(e,A)} m = \operatorname{Ind} \mathcal{D}_{(e,A+a)} m.$$

To prove (b), it suffices to show that

- If $\|a\|_n \ll 1$ and $(\xi, \alpha) \in \ker \mathcal{D}_{(e,A+a)} m$, then
$$d_A \xi = 0.$$

This implies $\dim \ker \mathcal{D}_{(e,A+a)} m = \dim \ker \mathcal{D}_{(e,A)} m = \dim \operatorname{Stab}_A$, so these linearized operators are surjective. The linearization of $\bar{m}$ becomes invertible then.

We only need a linearized proof of *Step* (a). Suppose

$$d_A \xi = \alpha - [a, \xi].$$

By applying d_A^*, we obtain:

$$\begin{aligned} d_A^* d_A \xi &= d_A^*(\alpha - [a, \xi]). \\ (\text{since } d_A^* \alpha = 0) &= -d_A^*[a, \xi] \\ &= -[d_A^* a, \xi] + *[*a, d_A \xi] \\ (\text{since } d_A^* a = 0) &= *[a, *d_A \xi] = \Phi(a, d_A \xi) \end{aligned}$$

for a bilinear bundle map Φ. Using estimates from Lecture 13:

$$c\|d_A \xi\|_2 \leqslant \|d_A^* d_A \xi\|_{(L_1^2)^*}$$

$$L^n \times L^2 \times L_1^2 \to L^1 \xrightarrow{\int_X} \mathbb{R},$$

we obtain that

$$c\|d_A \xi\|_2 \leqslant \|d_A^* d_A \xi\|_{(L_1^2)^*} \leqslant C\|a\|_n \|d_A \xi\|_2$$

which forces $\|d_A \xi\|_2 \equiv 0$ when $\|a\|_n \ll 1$. This completes the proof of the Big-Slice theorem 15.1.1.

Part (2) of the Big-Slice theorem 15.1.1 is concerned with an L^n-neighborhood of A. By using $g \in \mathscr{G}_1^p$ to translate this neighborhood, it actually implies

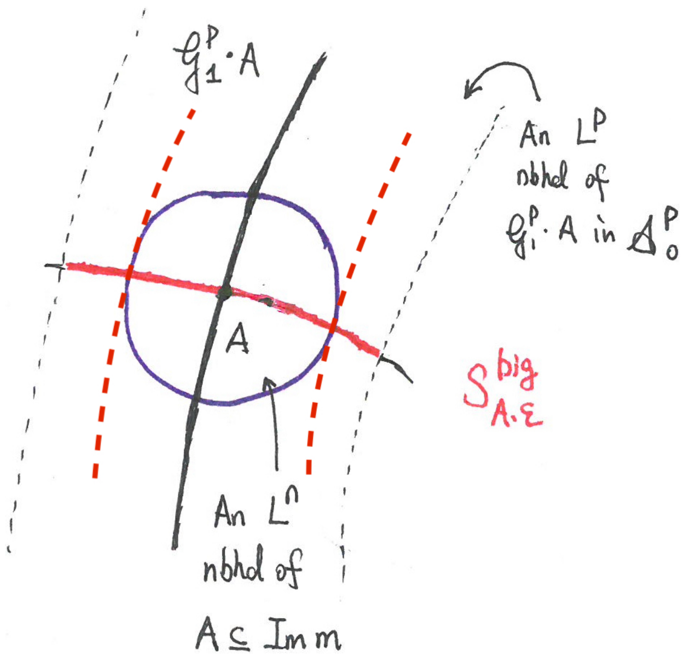
(2') For $A \in \mathscr{A}^p$ and a big slice $S_{A,\epsilon}^{\mathrm{big}}$, $\mathscr{G}_1^p \cdot S_{A,\epsilon}^{\mathrm{big}}$ contains an L^n-neighborhood of the orbit $\mathscr{G}_1^p \cdot A$.

Let us state an immediate generalization of (2') that will be used in the proof of Uhlenbeck's Fundamental Lemma:

Corollary 15.1.2. *For $A \in \mathscr{A}^p \subset \mathscr{A}^n$ and a big slice $S_{A,\epsilon}^{big}$, $\mathscr{G}_1^n \cdot S_{A,\epsilon}^{big}$ contains an L^n-neighborhood of the orbit $\mathscr{G}_1^n \cdot A$ inside $\mathscr{A}^n$.*

Remark. The connection A has to be in L^p $(p > n)$ for this corollary to hold.

Exercise 15.1.3. *Prove Corollary* 15.1.2.



Proof. The proof is by approximation. Let δ be the size of the L^n-neighborhood of A in the Big-Slice theorem. It suffices to show if $A + b \in \mathscr{A}^n$ and $\|b\|_n < \delta/2$, then there exists $g \in \mathscr{G}_1^n$ such that

$$g \cdot (A + b) \in S^{\text{big}}_{A,\epsilon}.$$

Let $b_i \in L^p$ such that $b_i \to b$ in L^n. By the Big-Slice theorem 15.1.1, we have find $g_i \in \mathscr{G}_1^p$ such that

$$g_i \cdot (A + b_i) = A + a_i \in S^{\text{big}}_{A,\epsilon}.$$

Therefore, $\|a_i\|_n \leqslant \epsilon$. By Lemma 1.2 from Lecture 14, for some $K > 0$,

$$\|a_i\|_n + \|g_i\|_{L_1^n} \leqslant K\|b_i\|_n.$$

Hence, we can find a weakly convergent subsequence $a_i \to a \in L^n, g_i \to g \in L_1^n$. By passing to the limit, we know that

$$\|a\|_n \leqslant \epsilon, d_A^* a = 0, *a|_{\partial X} = 0$$

and

$$g \cdot (A + b) = (A + a) \in S^{\text{big}}_{A,\epsilon}.$$

Here, the big slice is actually the closure

$$\overline{S^{\text{big}}_{A,\epsilon}} \subset \mathscr{A}^n.$$

□

15.2. Uhlenbeck's Fundamental Lemma

We start with a lemma:

Lemma 15.2.1. *There exists $C_1, \epsilon_1 > 0$ such that for all $F_A = \Gamma + b$ with*

- $\|b\|_4 \leqslant \epsilon_1$,
- $d^*b = 0$,
- $*b|_{\partial B^4} = 0$,

we have

$$\int_{B^4} |\nabla_\Gamma b|^2 + |b|^2 \leqslant C_1 \int_{B^4} |F_A|^2.$$

Proof. By the Weitzenböck formula, we have

$$\int_{B^4} |(d + d^*)b|^2 = \int_{B^4} |\nabla_\Gamma b|^2 + \int_{\partial B^4} *b \wedge d^*b + \int_{\partial B} |b|^2. \tag{17}$$

for any $b \in \Omega^1(B^4)$. Since the Euclidean metric on B^4 is flat, the Ricci curvature term does not show up. Moreover, $S^3 = \partial B^4 \subset B^4$ has constant mean curvature, so the last term is positive.

Exercise 15.2.2. *Prove the Weitzenböck formula.*

We postpone the solution of this exercise to the end of the note.

By assumption $d^*b = 0$ and $*b|_{\partial B^4} = 0$. The equation (17) then implies

$$\int_{B^4} |db|^2 \geqslant \int_{B^4} |\nabla_\Gamma b|^2. \tag{18}$$

Since $L^2_1 \hookrightarrow L^4$, for some $C_{Sob} > 0$,

$$\|b\|_4 \leqslant C_{Sob} \|b\|_{L^2_1} := (\int_{B^4} |\nabla_\Gamma b|^2 + |b|^2)^{\frac{1}{2}}.$$

Putting all these together, we have

$$\begin{aligned}
\int_{B^4} |F_A|^2 = \int_{B^4} |db + b \wedge b|^2 &\gtrsim \int_{B^4} |db|^2 - |b|^4 \\
\text{(by (18))} &\geqslant (\int_{B^4} |\nabla_\Gamma b|^2) - \|b\|_4^4 \\
\text{(since } \|b_1\|_4 < \epsilon) &\geqslant (\int_{B^4} |\nabla_\Gamma b|^2) - \epsilon_1^2 C_{Sob}^2 \|b\|^2_{L^2_1} \\
&\geqslant \frac{\lambda_1^2}{\lambda_1^2 + 1} \|b\|^2_{L^2_1} - \epsilon_1^2 C_{Sob}^2 \|b\|^2_{L^2_1} \\
&\geqslant (\frac{\lambda_1^2}{\lambda_1^2 + 1} - \epsilon_1^2 C_{Sob}^2) \|b\|^2_{L^2_1} \geqslant C_1^{-1} \|b\|^2_{L^2_1},
\end{aligned}$$

if $\epsilon_1 \ll 1$. Here, we used that fact that

$$\int_{B^4} |\nabla_\Gamma b|^2 \geqslant \lambda_1^2 \int_{B^4} |b|^2. \qquad \square.$$

for some $\lambda_1 > 0$. To see this, one can use proof by contradiction because a parallel 1-form b on B^4 with the boundary condition $*b|_{\partial X}$ has to be zero.

We are ready to state and prove Uhlenbeck's Fundamental Lemma. Let $P = B^4 \times G$ be the principal bundle:

Theorem 15.2.3. *There exists $C_1, \epsilon > 0$ such that for any L_1^2-connection $A = \Gamma + a$ on B^4 with*

$$E(A) := \int_{B^4} |F_A|^2 < \epsilon,$$

we can find a gauge transformation $g \in \mathscr{G}_2^2$ such that

$$g^* A = \Gamma + b$$

where Γ is the trivial connection and

(1) $d_\Gamma^* b = 0$,
(2) $*b|_{\partial B^4} = 0$.
(3) $\int_{B^4} |\nabla_\Gamma b|^2 + |b|^2 \leqslant C_1 E(A)$.

Proof. The proof is by the method of continuity. Let

$$V_\epsilon = \{\Gamma + a : a \in L_1^2, E(A) < \epsilon\},$$
$$W_\epsilon = \{A \in V_\epsilon : \exists g \in \mathscr{G}_2^2, s.t.\ (1)(2)(3) \text{ hold for } g \cdot A\}.$$

The proof is divided into three steps. Let ϵ_1, C_1 be the constants in Lemma 18 and choose $\epsilon \ll 1$ such that $2C_1 C_{sob}\epsilon \leqslant \epsilon_1$. We have to show

- *Step* 1. V_ϵ is connected.
- *Step* 2. W_ϵ is closed in V_ϵ.
- *Step* 3. W_ϵ is open in V_ϵ.

Three steps together will imply $V_\epsilon = W_\epsilon$.

Step 1. Connectedness. For each $t \in [0, 1]$, Define

$$\tau_t : \mathscr{A}_1^2 \to \mathscr{A}_1^2$$
$$A(x) \mapsto A(tx)|_{B^4}.$$

For each $A \in \mathscr{A}_1^2$, the path

$$t \in [0, 1] \mapsto \tau_t(A)$$

is continuous, but the "global" map

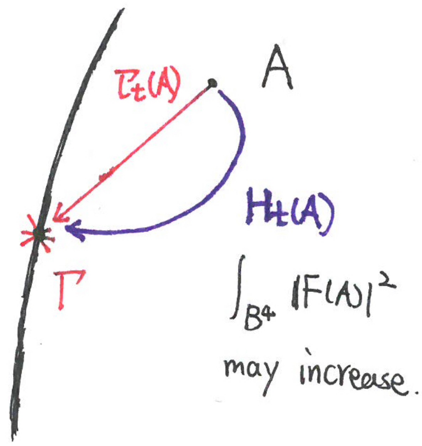
$$\tau : [0, 1] \times \mathscr{A}_1^2 \to \mathscr{A}_1^2$$
$$(t, A) \mapsto \tau_t(A)$$

is not continuous. In this case, we say that τ is point-wise continuous in $\mathscr{A}_1^2$.

Remark. An analogue of τ is the translation operator T on $L^p(\mathbb{R}^n)$:

$$\begin{aligned} T : \mathbb{R}^n \times L^p(\mathbb{R}^n) &\to L^p(\mathbb{R}^n) \\ (h, f(x)) &\mapsto f(x-h). \end{aligned}$$

For each $f \in L^p(\mathbb{R}^n)$, $T(\cdot, f)$ is continuous as a map $\mathbb{R}^n \to L^p(\mathbb{R}^n)$, but T is not continuous itself.



The advantage of the path τ_t is that $\tau_t(A)$ has uniformly small energy (using the fact that $E(A)$ is **scale invariant** or even **conformal invariant**):

$$E(\tau_t(A)) = \int_{B^4} |F_{\tau_t(A)}|^2 = \int_{|x|\leqslant t} |F_A|^2 \leqslant \int_{|x|\leqslant 1} |F_A|^2 < \epsilon.$$

and $\tau_0(A) = \Gamma$. This shows if $A \in V_\epsilon$, then $t \mapsto \tau_t(A)$ is a path that lies in V_ϵ and connects Γ and A. V_ϵ is path connected. *Step* 1 is done.

Remark. There is an obvious deformation retract of $\mathscr{A}_1^2$ onto Γ:

$$H_t(A) = tA + (1-t)\Gamma.$$

which is continuous as a map $[0,1] \times \mathscr{A}_1^2 \to \mathscr{A}_1^2$. The problem with the linear homotopy is that the path $H_t(A)$ does not necessarily lie in V_ϵ given that $A \in V_\epsilon$. The energy $E(H_t(A))$ may not decrease monotonically and might exceed ϵ for some $t \in [0,1]$.

Step 2. Closedness. Suppose

$$A_i = \Gamma + a_i \in W_\epsilon$$

and $a_i \to a \in L_1^2$ as $i \to \infty$. We can find $g_i \in \mathscr{G}_2^2$ such that

$$g_i \cdot (\Gamma + a_i) = \Gamma + b_i \tag{19}$$

satisfies (1)(2)(3). We need to prove some a priori estimates:

$$\|b_i\|_4 \leqslant C_{sob}\|b\|_{L_1^2} \leqslant C_1 C_{sob} \int_{B^4} |F_{A_i}|^2 \leqslant C_1 C_{sob}\epsilon \leqslant \frac{\epsilon_1}{2}.$$

and

$$\|b_i\|_{L^2_1} \leqslant C_1 \int_{B^4} |F_{A_i}|^2 \leqslant C_1\epsilon.$$

By passing to a weakly convergent subsequent, we have

$$b_i \xrightarrow{w-L^2_1} b \in L^2_1 \Rightarrow b_i \xrightarrow{s-L^4} b.$$

Moreover, $\|b\|_4 = \lim \|b_i\|_4 < \epsilon_1$. By (19),

$$dg_i = g_i a_i - b_i g_i. \tag{20}$$

Our goal is to show $\|g_i\|_{L^2_2}$ is uniformly bounded:

- $\|dg_i\|_4 \leqslant \|g_i\|_\infty \|a_i\|_4 + \|b_i\|_4 \|g_i\|_\infty < C_2$ is uniformly bounded.
- The next step is the second derivative of g_i:

$$\begin{aligned}\|\nabla(dg_i)\|_2 &\leqslant \|b_i \otimes \nabla g_i\|_2 + \|(\nabla b_i) g_i\|_2 + \|g_i \nabla a_i\|_2 + \|\nabla g_i \otimes a_i\|_2 \\ &\leqslant \|b_i\|_4 \|dg_i\|_4 + \|b_i\|_{L^2_1} \|g\|_\infty + \|a_i\|_{L^2_1} \|g\|_\infty + \|a_i\|_4 \|dg_i\|_4 \\ &\leqslant C_3.\end{aligned}$$

Hence, we can find a weakly convergent subsequence such that

$$g_i \xrightarrow{w-L^2_2} g \in L^2_2.$$

Since the embedding $L^2_2 \hookrightarrow L^q$ ($\forall 1 < q < \infty$) is compact, $g_i \xrightarrow{s-L^q} g$ and $g_i \xrightarrow{a.e.} g$. This shows $g \in \mathscr{G}^2_2$ is a gauge transformation.

Now we have the convergence:

$$a_i \xrightarrow{s-L^2_1} a,\ b_i \xrightarrow{w-L^2_1} b,\ g_i \xrightarrow{w-L^2_2} g.$$

Consider the limit of the equation (20) as $i \to \infty$:

$$dg = ga - bg$$

which holds in L^{4-c} for any $0 < c$ (< 1). This shows $g \cdot (A + a) = A + b$. It remains to verify that

$$(1)(2)(3) \text{ hold for } A + b \in \mathscr{A}^2_1.$$

Both (1) and (2) follow from $b_i \to b$ in $w - L^2_1$. The last follows from Lemma 15.2.1. Indeed, b satisfies (1)(2) and the estimate

$$\|b\|_4 < \epsilon_1.$$

Step 3. Openness. This follows from the Big-Slice theorem. □

Exercise 15.2.4. *Prove Step 3.*

Proof. For the proof, we need a variant of Corollary 15.1.2:

Corollary 15.2.5. *For $A \in \mathscr{A}^p_1 \subset \mathscr{A}^2_1$ ($p > 2$) and a big slice $S^{big}_{A,\epsilon}$, $\mathscr{G}^2_2 \cdot S^{big}_{A,\epsilon}$ contains an L^2_1-neighborhood of the orbit $\mathscr{G}^2_2 \cdot A$ inside $\mathscr{A}^2_1$.*

We choose $A = \Gamma$ to be the trivial connection in Corollary 15.2.5. Let ϵ' ($< \epsilon_1$) be the size of the big slice S' around Γ and δ be the size of the L_1^2-neighborhood around A.

Suppose
$$A + a \in W_\epsilon,$$
so there exists $g \in \mathscr{G}_2^2$ such that $g \cdot A = A + b$ satisfies (1)(2)(3). Then we know
$$\left(C_{sob}^{-1}\|b\|_4 \leqslant \right) \|b\|_{L_1^2} \leqslant C_1 E(A) < C_1\epsilon < \frac{\delta}{2}$$
if ϵ is small enough. This implies together with (1)(2) that
$$g \cdot A = A + b \in S'.$$
In order to show $W_\epsilon \subset V_\epsilon$ is an open subset, we need to find $\eta > 0$ and for any $a' \in L_1^2$ with $\|a'\|_{L_1^2} < \eta$, a gauge transformation $g' \in \mathscr{G}_2^2$ such that
$$g' \cdot (A + a + a')$$
satisfies (1)(2)(3). Note that
$$g \cdot (A + a + a') = g \cdot (A + a) + ga'g^{-1} = A + b + ga'g^{-1}.$$
Since $g \in L^\infty$, $\|ga'g^{-1}\|_4 \lesssim \|a'\|_4 \leqslant C_{sob}\|a'\|_{L_1^2} < \eta C_{Sob}$. Furthermore,
$$\|\nabla(ga'g^{-1})\|_2 \lesssim \|g\|_\infty \|a'\|_{L_1^2} + \|a'\|_4 \|dg\|_4 < C_2\eta.$$
By making η small, we can assure that
$$\|b + ga'g^{-1}\|_{L_1^2} < \delta.$$
By Corollary 15.2.5, we can find $g_1 \in \mathscr{G}_2^2$ such that
$$g_1 \cdot g \cdot (A + a + a') = A + b' \in S'$$
satisfies (1)(2). To prove (3), by Lemma 15.2.1, it remains to verify
$$\|b'\|_4 \leqslant \epsilon_1.$$
This is guaranteed, because the size ϵ' of S' is made small so that $\epsilon' < \epsilon_1$. □

Finally, we give a solution to Exercise 15.2.2. The transcriber believed that formula (17) had missed a boundary term.

Exercise 15.2.6. *Let b be a smooth 1-form on a compact manifold X with boundary ∂X. Let $V \in \Gamma(X, TX)$ be the dual tangent vector of b and $\vec{n}$ be the outward normal at ∂X. Then*
$$\begin{aligned}\int_X |db|^2 + |d^*b|^2 = \int_X |\nabla b|^2 + \int_X \mathrm{Ric}(V, V) \\ - \int_{\partial X} (\langle \nabla_V V, \vec{n}\rangle dvol_{\partial X} + {*b} \wedge d^*b)\end{aligned}$$

Proof. The local version of the Weitzenböck formula states that

$$\Delta b = \nabla^*\nabla b + \mathrm{Ric}(V, V)$$

where $\Delta = dd^* + d^*d$ is the Hodge Laplacian. For a proof, see [2, P. 156, Corollary 8.3]. To derived the global version, we use integration by parts:

$$\begin{aligned}
\int_X \langle db, db\rangle - \int_X \langle b, d^*db\rangle &= \int_{\partial X} b \wedge *db \\
\int_X \langle d^*b, d^*b\rangle - \int_X \langle b, dd^*b\rangle &= -\int_{\partial X} *b \wedge d^*b \\
\int_X \langle \nabla b, \nabla b\rangle - \int_X \langle b, \nabla^*\nabla b\rangle &= \int_{\partial X} \langle b, \nabla_{\vec{n}} b\rangle dvol_{\partial X}.
\end{aligned}$$

Note that for any m-form ω:

$$b \wedge *\omega = (-1)^{m-1} * (\iota(V)\omega).$$

Apply this formula for $\omega = db$:

$$\begin{aligned}
-\int_{\partial X} b \wedge *db = \int_{\partial X} *(\iota(V)db) &= \int_{\partial X} \langle \iota(V)db, \vec{n}\rangle dvol_{\partial X} \\
&= \int_{\partial X} db(V, \vec{n}) dvol_{\partial X}.
\end{aligned}$$

Extend $\vec{n}$ to a vector field in a neighborhood of ∂X. For any vector fields (W_1, W_2), we have:

$$\begin{aligned}
db(W_1, W_2) &= W_1(b(W_2)) - W_2(b(W_1)) - b([W_1, W_2]). \\
&= W_1\langle V, W_2\rangle - W_2\langle V, W_1\rangle - \langle V, \nabla_{W_1} W_2 - \nabla_{W_2} W_1\rangle \\
&= \langle \nabla_{W_1} V, W_2\rangle - \langle \nabla_{W_2} V, W_1\rangle.
\end{aligned}$$

Let $W_1 = V$ and $W_2 = \vec{n}$, then

$$db(V, \vec{n}) = \langle \nabla_V V, \vec{n}\rangle - \langle \nabla_{\vec{n}} V, V\rangle.$$

Therefore,

$$\begin{aligned}
\int_X |db|^2 + |d^*b|^2 &= \int_X |\nabla b|^2 + \int_X \mathrm{Ric}(V, V) - \int_{\partial X} \big(\langle b, \nabla_{\vec{n}} b\rangle + db(V, \vec{n})\big) \\
&\quad - \int_{\partial X} *b \wedge d^*b \\
&= \int_X |\nabla b|^2 + \int_X \mathrm{Ric}(V, V) \\
&\quad - \int_{\partial X} (\langle \nabla_V V, \vec{n}\rangle dvol_{\partial X} + *b \wedge d^*b)
\end{aligned}$$

Now we use the fact that $d^*b \equiv 0$ and $*b|_{\partial X} \equiv 0$. This means $V \perp \vec{n}$ and $b(\vec{n}) \equiv 0$ at ∂X. Therefore,

$$\int_X |db|^2 - \int_X |\nabla b|^2 = \int_X \mathrm{Ric}(V,V) + \int_{\partial X} \langle V, \nabla_V \vec{n} \rangle$$

The last term is related to the shape operator on ∂X. When $X = D^n$ and $\partial X = S^n$, we have

$$Ric(V,V) \equiv 0, \langle V, \nabla_V \vec{n} \rangle = |V|^2.$$

When $X \subset S^n$ is a hemisphere with the round metric and ∂X is the equator of S^n,

$$Ric(V,V) = (n-1)|V|^2, \langle V, \nabla_V \vec{n} \rangle \equiv 0.$$

In either case, the right hand side is positive. The best way to convince yourself that all signs in the computation are correct is to work out the case $X = D^n$ directly (using coordinates), as is done in Uhlenbeck's original paper [3]. $\square$

Lecture 16. Uhlenbeck's Compactness Theorem I

16.1. The curvature map is proper

Last time, we proved Uhlenbeck's Fundamental Lemma. Let $P = B^4 \times G$ be the principal bundle:

Theorem 16.1.1. *There exists $C_1, \epsilon > 0$ such that for any L^2_1-connection $A = \Gamma + a$ on B^4 with*

$$E(A) := \int_{B^4} |F_A|^2 < \epsilon,$$

we can find a gauge transformation $g \in \mathscr{G}^2_2$ such that

$$g^* A = \Gamma + b$$

where Γ is the trivial connection and

(1) $d^*_\Gamma b = 0$,
(2) $*b|_{\partial B^4} = 0$.
(3) $\int_{B^4} |\nabla_\Gamma b|^2 + |b|^2 \leqslant C_1 E(A)$.

Let the curvature slice be

$$SC_\epsilon = \{A = \Gamma + b : d^* b = 0, *b|_{\partial B^4} = 0, \int_{B^4} |F_A|^2 < \frac{\epsilon}{2}\}$$

with the same ϵ as in Theorem 16.1.1.

Theorem 16.1.2. *The curvature map*

$$A \in SC_\epsilon \mapsto F_A \in L^2$$

is proper on the slice SC_ϵ.

Proof. Let $A_i = \Gamma + b_i \in SC_\epsilon$. Suppose their curvatures $\{F_{A_i}\}$ are Cauchy in L^2, then

$$\begin{aligned}
\int_{B^4} |F_{A_i} - F_{A_j}|^2 &\gtrsim \int_{B^4} |d(a_i - a_j)|^2 - \int_{B^4} |(a_i - a_j) \wedge a_i + a_j \wedge (a_i - a_j)|^2 \\
&\geqslant \int_{B^4} |(d + d^*)(a_i - a_j)|^2 - C_2 \|a_i - a_j\|^2_{L^2_1} (\|a_i\|^2_{L^2_1} + \|a_j\|^2_{L^2_1}) \\
&= C_3 \|a_i - a_j\|^2_{L^2_1} - \epsilon C_2 C_1 \|a_i - a_j\|^2_{L^2_1} \\
&= (C_3 - \epsilon C_2 C_1) \|a_i - a_j\|^2_{L^2_1}.
\end{aligned}$$

This implies $\{a_i\}$ is Cauchy when $\epsilon \ll 1$. $\square$

16.2. The Removable Singularities Theorem

Now let us consider the global picture. Let

$$P \to X^4$$

be a principal G-bundle over a closed 4-manifold X and $\{A_i\}$ be a sequence of anti-self dual connections. Then

$$\begin{aligned} E(A) := \int_X |F_A|^2 = -\int_X \operatorname{tr}(F_A \wedge *F_A) = \int_X \operatorname{tr}(F_A \wedge F_A) \\ = 8\pi^2 \langle c_2(P), [X^4] \rangle. \end{aligned}$$

is a constant independent of A. We can cover

$$X = \bigcup_{i \in \Lambda} B(x_i, \delta)$$

by balls of a given radius δ such that

- For any $x \in X$, at most N balls contain x.

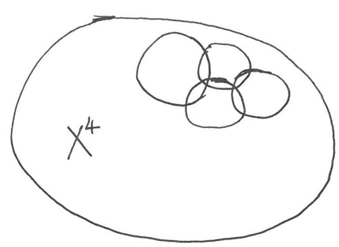
Then we can find a subsequence of connections $\{A_j\}$ that except for

$$\frac{E(A)N}{\epsilon}$$

balls the curvature constraint

$$\int_{B(x_i,\delta)} |F_{A_j}|^2 < \epsilon \tag{21}$$

holds for all connections A_j in this subsequence.



Remark. One way to produce such a cover is as follows. Find a maximal collection of disjoint balls of radius $\frac{\delta}{3}$ inside X. Let them be

$$B(x_i, \frac{\delta}{3}), 1 \leqslant i \leqslant m.$$

Then the collection $\{B(x_i, \delta)\}$ necessarily covers X. For any $x \in X$, if $x \in B(x_i, \delta)$, then

$$B(x_i, \frac{\delta}{3}) \subset B(x, 2\delta).$$

Therefore,

$$N \leqslant N_0 := \frac{\max_{x\in X} Vol(B(x, 2\delta))}{\min_{1\leqslant i\leqslant m} Vol(B(x_i, \delta/3))}.$$

At this step, we can apply Gromov-Bishop comparison theorem to estimate N_0 in terms of a curvature bound of X. Alternatively, we shrink the size of δ. Locally, any metric is close to the standard Euclidean metric. Then N is bounded by the corresponding number N_0 in $\mathbb{R}^n$. □

We can iterate this process and get a cover of

$$X^c := X - \{x_1, x_2, \cdots, x_k\}$$

such that for any balls in this cover, the constraint (21) holds and we can apply Uhlenbeck's Fundamental Lemma. We wish to show that after proper gauge fixing,

$$[A_i]|_{X^c} \xrightarrow{\mathcal{C}^\infty_{loc}} [A] \text{ on } X^c$$

for a smooth connection A in $P|_{X^c}$. However, if

$$\int_{X^c} |F_A|^2 < \int_X |F_{A_i}|^2 = 8\pi^2 \langle c_2(P), [X^4]\rangle,$$

then we can only have weak convergence $|F_{A_i}| \xrightarrow{w-L^2} |F_A|$.

We also need the Removable Singularities Theorem:

Theorem 16.2.1. *Suppose $A = \Gamma + a$ is a smooth connection in $(B^4 - \{0\}) \times G$ and $F_A^+ = 0$. If $E(A) < \infty$, then we can find a smooth gauge transformation $g : B^4 - \{0\} \to G$ such that*

$$g \cdot A = \Gamma + a'$$

and a' extends to a smooth connection on $B^4 \times G$.

The point is that the topology of the bundle might change if $g \neq 0 \in \pi_3(G)$.

This theorem allows us to extend the limit A to a connection A' in a possibly different bundle $P' \to X$. If $\pi_1(G) \neq \{0\}$, then P might have characteristic classes in dimension 2, but they are unaffected under the transformation g in Theorem 16.2.1. Only a 4-dimensional class could be changed.

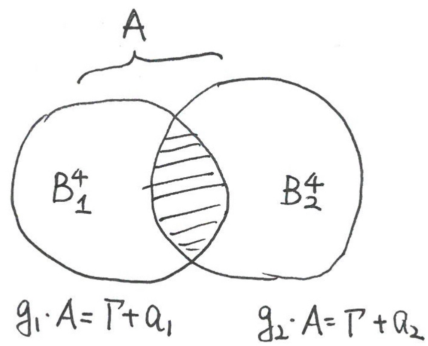
16.3. The Patching Argument: A Sketch

The 4-ball B^4 allows us to use the rescaling argument. If we can prove the connectedness of V_ϵ for other 4-manifolds, then Uhlenbeck's Fundamental Lemma still holds for them. In particular, to prove the Removable Singularities theorem, we wish to work on an annulus, but the argument has to be refined.

We need **a patching argument**.

Suppose we have

$$X = B_1^4 \cup B_2^4$$



and a smooth connection A in $P \to X$ with small energy $E(A)$. By Uhlenbeck's Fundamental Lemma, on B_i^4, we find a gauge transformation g_i such that

$$g_i \cdot A = \Gamma + a_i.$$

and $\|a_i\|_{L_1^2} < C_1 E(A)$. On $B_1^4 \cap B_2^4$, let $g_{12} = g_2 \cdot g_1^{-1}$, then

$$g_{12} \cdot (\Gamma + a_1) = \Gamma + a_2 \Rightarrow dg_{12} = g_{12} a_1 - a_2 g_{12}. \tag{22}$$

If a_i has small L^4-norm, then dg_{12} has small L^4-norm and $g_{12} - c$ has small L_1^4-norm for some $c \in G$. Compute the covariant derivative of (22), then

$$\nabla(dg_{12}) = \nabla g_{12} \otimes a_1 + g_{12} \cdot \nabla a_1 - \nabla a_2 \cdot g_{12} - a_2 \otimes \nabla g_{12}.$$

Using the multiplicative theorem

$$L^4 \times L^4 \hookrightarrow L^2, L^\infty \times L^2 \hookrightarrow L^2,$$

we see that $g_{12} - c$ is small in L_2^2. Our goal is to factorize

$$g_{12} = \eta_2^{-1} \eta_1$$

with η_2, η_1 small in L_2^2. We have to estimate the $\mathcal{C}^0$-norm of g_{12}. Apply d^* to (22), then

$$\begin{aligned} d^* dg_{12} &= - * (dg_{12} \wedge *a_1) \pm *(*a_2 \wedge dg_{12}) \\ &= \Phi_1(dg_{12}, a_1) + \Phi_2(a_2, dg_{12}) \in \operatorname{Im}(L_1^2 \times L_1^2). \end{aligned}$$

It is an observation by Taubes that

$$L_1^2 \times L_1^2 \hookrightarrow L^2 \cap \Delta(\mathcal{C}^0).$$

The proof uses Hardy's Inequality.

Lemma 16.3.1 (Hardy's Inequality)**.** *If $f \in L_1^2(B(0,1))$ is supported on $B(0,1/2)$, then*

$$\left| \int_{B^4} \frac{|f(x)|^2}{|x-y|^2} dx \right| \leqslant C \|f\|_{L_1^2}^2$$

for any $y \in B(0,1/2)$.

Proof. WLOG, assume y is the origin. Using polar coordinates:

$$\int_{B^4} \frac{|f(x)|^2}{|x-y|^2} dx = \int_0^1 \int_{S^3} |f(r,\omega)|^2 r dr d\omega,$$

and

$$\begin{aligned}\int_0^1 |f(r)|^2 r dr &= -\int_0^1 f f_r r^2 dr \\ &\leqslant (\int_0^1 f^2 r dr)^{\frac{1}{2}} (\int_0^1 (\frac{\partial f}{\partial r})^2 r^3 dr)^{\frac{1}{2}}\end{aligned}$$

which implies

$$\int_0^1 |f(r)|^2 r dr \leqslant \int_0^1 (\frac{\partial f}{\partial r})^2 r^3 dr.$$

Putting all these together, we obtain

$$|\int_{B^4} \frac{|f(x)|^2}{|x-y|^2} dx| \leqslant \int_0^1 \int_{S^3} (\frac{\partial f}{\partial r})^2 r^3 dr d\omega = \int_{B^4} |\frac{\partial f}{\partial r}|^2.$$

□

This inequality[1] will imply g_{12} has small $\mathcal{C}^0$-norm since

$$\frac{1}{|x-y|^2}$$

is proportional to the Green's function of the Laplacian Δ. By possibly shrinking the domain,

$$\|g_{12} - c\|_{\mathcal{C}^0(B')}$$

for some $B' \subset B_1^4 \cap B_2^4$. The goal[2] is to find smooth gauge transformations

$$\eta_i \text{ on } B_i^4$$

such that $g_{12} = \eta_1^{-1} \cdot \eta_2$ with η_i small in L_2^2. Then

$$\eta_1 \cdot (\Gamma + a_1) = \eta_2 \cdot (\Gamma + a_2)$$

equal on $B_1^4 \cap B_2^4$ and we have estimates on the L_1^2-norm of the resulting connection.

[1]Tom didn't explain the details

[2]Tom didn't finish the construction. He intended to give a sketch only.

Lecture 17. Uhlenbeck's Compactness Theorem II

We record a short conversation happened in the class:

Tom: " The timing is extremely good. We talked about the Chern-Simons functional and Simons came to MIT. We are studying Uhlenbeck's theorems and she won the Abel prize."
A student:" So you should talk about one of your theorems. [Audience laughs] "
Tom paused for a second and made a face.
" Well, only if a prize is coming."

Remark. James Simons received the Donald Sussman Fellowship Award from the Sloan School, MIT and gave three Fireside Chats in early March. The first talk, moderated by Tom, was on his mathematical life.
Karen Uhlenbeck was awarded the Abel Prize in March 19, 2019.

17.1. A local compactness result

Let $B_r = B_r^4$ be the ball of radius r in $\mathbb{R}^4$. Let

$$\mathcal{M}(B_r) = \{[A] : F_A = - * F_A\} \subset \mathscr{A}_1^2/\mathscr{G}_2^2.$$

be the moduli space of anti-self dual connections on B_r. When $r = 1$, we studied the space

$$\mathcal{M}^\epsilon(B_1) = \{[A] \in \mathcal{M}(B_1) : \int_{B_1} |F_A| < \epsilon\}.$$

By restricting connections to a smaller ball, we get a restriction map:

$$r : \mathcal{M}^\epsilon(B_1) \hookrightarrow \mathcal{M}(B_r).$$

Theorem 17.1.1 (Uhlenbeck)**.** *When $\epsilon \ll 1$, The restriction map r is compact.*

This theorem is analogous to a classical theorem in complex analysis. Consider the space of holomorphic functions on the unit disk $B(0,1) \subset \mathbb{C}$ with uniformly bounded L^2-norms, then the restriction map to a smaller disk $B(0,r)$ is compact in $\mathcal{C}^k$-topology for any $k > 0$:

$$\{f : B(0,1) \to \mathbb{C} : \bar{\partial} f = 0, \|f\|_2 < C\} \xrightarrow{r} \mathcal{C}^k(B(0,r)).$$

For a sequence of connections $\{A_i\} \subset \mathcal{M}^\epsilon(B_1)$, Uhlenbeck's fundamental lemma allows us to show they are weakly convergent in L_1^2. We wish to prove strong L_k^2 convergence in the interior.

Here is a rough idea of the proof[3]. For any connection A, there is an inequality

$$\int_{B_r} |F_A|^2 \geqslant -\mathcal{CS}(A|_{B_r}). \tag{23}$$

[3]See the next section for more details

The equality holds if and only if A is anti-self dual. Write

$$A + a.$$

By Fubini's theorem, over the sphere S^3_r, the quantity

$$\int_{S^3_r} |\nabla_\Gamma a|^2 + |a|^2 \tag{24}$$

is finite for almost all $r \in [0,1]$. On the other hand, the Chern-Simons functional $\mathcal{CS}$ is continuous in $L^2_{1/2}$-topology:

$$\mathcal{CS}(\Gamma + a) = -\int_{S^3_r} \operatorname{tr}(\frac{1}{2} a \wedge da + \frac{1}{3} a \wedge a \wedge a).$$

Indeed, $da \in L^2_{-1/2}$ and $L^2_{1/2} \hookrightarrow L^3$ in dimension 3:

$$\frac{1}{2} - \frac{3}{2} = 0 - \frac{3}{3}.$$

Apply multiplication theorems:

$$L^2_{1/2} \times L^2_{-1/2} \hookrightarrow L^1, L^3 \times L^3 \times L^3 \hookrightarrow L^1.$$

Suppose (24) is uniformly bounded for all a_i and for a sequence of numbers $r_j \to 1$, by (23), we conclude

$$\lim_{i \to \infty} \int_{B_{r_j}} |F_{A_i}|^2 = \int_{B_{r_j}} |F_A|^2.$$

for each r_j. This will imply the strong convergence on B_{r_j}:

$$F_{A_i} \xrightarrow{s-L^2(B_{r_j})} F_A.$$

Remark. If $f_i \to f$ weakly in L^2, then $\|f\|_2 \leqslant \liminf \|f_i\|_2$. If $\|f\|_2 = \liminf \|f_i\|_2$, then the convergence is in fact strong. □

We wish to use the properness of the curvature map (see Lecture 15). By Uhlenbeck's fundamental lemme, we can achieve

(1) $d^*_\Gamma a_i = 0$,
(2) $*a_i|_{\partial B_1} = 0$.
(3) $\int_{B_1} |\nabla_\Gamma a_i|^2 + |a_i|^2 \leqslant C_1 E(A_i)$.

But instead of (2), what we really want is

(2') $*a_i|_{\partial B_{r_j}} = 0$ for $r_j < 1$.

Nevertheless, we can still prove properness if

(1) $a_i|_{\partial B_r}$ is small in $L^2_{1/2}$
(2) a_i is small in $L^2_1(B_1)$.

Let us go through some details in this sketch.

17.2. A more organized sketch

Our goal is to prove Theorem 17.1.1.

Step 1. Put A_i in good gauge. Using Uhlenbeck's fundamental lemma, we write

$$A = \Gamma + a_i$$

with $a_i \in L^2_1(B^4_1, T^*B^4_1 \otimes \operatorname{ad} P)$ and

(1) $d^*_\Gamma a_i = 0$,
(2) $*a_i|_{\partial B_1} = 0$.
(3) $\int_{B_1} |\nabla_\Gamma a_i|^2 + |a_i|^2 \leqslant C_1 E(A_i) \leqslant C_1 \epsilon$.

By passing to a subsequence, we assume $a_i \to a$ weakly in L^2_1 for some connection matrix a. Let $A = \Gamma + a$, then

$$F_{A_i} \xrightarrow{w-L^2(B_1)} F_A.$$

Step 2. For each i, define

$$f_i(r) = \int_{B_r} |F_{A_i}|^2.$$

They are monotonically increasing functions with a uniform bound on the supremum norm. Using polar coordinates, we have

$$f_i(r) = \int_0^r \int_{S^3} |F_{A_i}|^2 r^3 dr d\omega = 2\int_0^r \int_{S^3} |F_{A_i}|_{S^3}|^2 r^3 dr d\omega.$$

Here, we used the fact that F_{A_i} is anti-self dual and

$$|dt \wedge e_j + e_{j+1} \wedge e_{j+2}|^2 = 2|e_{j+1} \wedge e_{j+2}|^2.$$

Hence,

$$f_i'(r) = 2\int_{S^3} |F_{A_i}|_{S^3_r}|^2 r^3 d\omega = 2r^3 \int_{S^3} |F_{A_i}|_{S^3_r}|^2 d\omega.$$

Since $f_i' \geqslant 0$ and $\int_0^1 f_i(r)dr \leqslant M$, by a theorem in measure theory, we can find $M' > 0$, a subsequence of $\{f_i\}$ and $0 < r_j \to 1$, such that

$$f_i(r_j)' \leqslant M'$$

for all i, j. Hence, when $j \gg 1$,

$$\int_{S^3} |F_i|_{S^3_{r_j}}|^2 \leqslant 2M',$$

for any i.

Step 3. For each fixed r_j, $\{F_i|_{S^3_{r_j}}\}$ is a sequence of connections on S^3 with bounded energy. By **another compactness theorem**[4], for each A_i, we can find a gauge transformation g_i on S^3 such that

$$g_i \cdot (A_i|_{S^3_{r_j}}) - \Gamma$$

[4]Tom didn't explain this.

has bounded L^2_1-norm and g_i is in the identity component of $\mathscr{G}$. In particular, we can find a weakly convergent subsequence in L^2_1, so they converge strongly in $L^2_{1/2}$ (the inclusion $L^2_1 \hookrightarrow L^2_{1/2}$ is compact). Since $\mathcal{CS}$ is continuous in $L^2_{1/2}$-topology,

$$\mathcal{CS}(A_i|_{S^3_{r_j}}) = \mathcal{CS}(g_i \cdot (A_i|_{S^3_{r_j}})) \to \mathcal{CS}(A'|_{S^3_{r_j}}).$$

if $g_i \cdot A_i \to A'$ in w-$L^2_1(S^3)$.

Step 4. Since each A_i is anti-self dual, the equality in (23) is achieved. *Step* 3 then implies that

$$\lim_{i\to\infty} \int_{B_{r_j}} |F_{A_i}|^2 = \int_{B_{r_j}} |F_A|^2$$

for each fixed r_j. Here, A is the limit of $\{A_i\}$ found in *Step* 1. Note that $A|_{S^3_{r_j}}$ is gauge equivalent to A' on $S^3_{r_j}$. By a theorem from functional analysis, the weak convergence (on a smaller ball B_{r_j}):

$$F_{A_i} \xrightarrow{w-L^2(B_{r_j})} F_A.$$

is in fact strong:

$$F_{A_i} \xrightarrow{s-L^2(B_{r_j})} F_A.$$

Step 5. We can almost apply the properness result from Lecture 14, except that the boundary condition is not fulfilled:

$$*a_i|_{\partial B_r} \neq 0.$$

Nevertheless, we know that

$$\| * a_i|_{\partial B_r}\|_{L^2_{1/2}}$$

is uniformly small. **By the proof of the Big-Slice theorem**[5], we can find gauge transformations g_i on B_{r_j} with controlled L^2_2-norm putting A_i into the slice. Since $g_i|_{S^3_{r_j}}$ is in the identity component of the gauge group,

$$\mathcal{CS}(g_i \cdot A_i) = \mathcal{CS}(A_i)$$

is convergent. By passing to subsequence, we assume

$$g_i \cdot A_i \xrightarrow{w-L^2_1(B_{r_j})} A'', F_{g_i\cdot A_i} \xrightarrow{w-L^2(B_{r_j})} F_{A''}$$

for some L^2_1-connection A'' on B_{r_j}. By the argument in *Step* 3,

$$\lim_{i\to\infty} \int_{B_{r_j}} |F_{g_i\cdot A_i}|^2 = \int_{B_{r_j}} |F_{A''}|^2.$$

which implies

$$F_{g_i\cdot A_i} \xrightarrow{s-L^2} F_A.$$

Finally, we use the properness result to finish the argument.

[5]Tom didn't explain this.

17.3. The global picture

Suppose X is a closed 4-manifold and $[A_i] \in \mathcal{M}(X)$ is a sequence of anti-self dual connections in $P \to X$. By the covering argument from Lecture 16, we can find a finite collection of points $\{x_1, \cdots, x_k\} \subset X$ and a cover of $X^c = X - \{x_1, \cdots, x_k\}$:

$$\{B(x_\alpha, r_\alpha)\}_{\alpha\in\Lambda}$$

such that

(1) $\{B(x_\alpha, r_\alpha/2)\}_{\alpha\in\Lambda}$ is still a cover of X^c.
(2) For any $x \in X^c$, there are at most N balls that contains x.
(3) For each A_i,
$$\int_{B(x_\alpha,r_\alpha)} |F_{A_i}|^2 \leqslant \epsilon,$$
so we can apply Uhlenbeck's Fundamental Lemma.

Using the local compactness theorem, the restriction map

$$\mathcal{M}^\epsilon(B(x_\alpha, r_\alpha)) \to \mathcal{M}(B(x_\alpha, r_\alpha/2))$$

is compact. By passing to a subsequence, connections

$$[A_i]|_{B(x_\alpha,r_\alpha/2)}$$

converge for all $\alpha \in \Lambda$. Use the patching argument to get g_i on $P|_{X^c}$ such that

$$g_i \cdot A_i \xrightarrow{s-L^2_{1,loc}} A$$

on $P|_{X^c}$. To extend A to a connection on X, we need Uhlenbeck's Removable Singularities theorem:

Theorem 17.3.1. *Suppose $A = \Gamma + a$ is a smooth connection in $(B^4 - \{0\}) \times G$ and $F^+_A = 0$. If $E(A) < \infty$, then we can find a smooth gauge transformation g : $B^4 - \{0\} \to G$ such that*

$$g \cdot A = \Gamma + a'$$

and a' extents to a smooth anti-self dual connection on $B^4 \times G$.

Therefore, by applying a gauge transformation on $P|_{X^c}$, A becomes a smooth anti-self dual connection on $P' \to X$, but P' is not necessarily the same bundle as P.

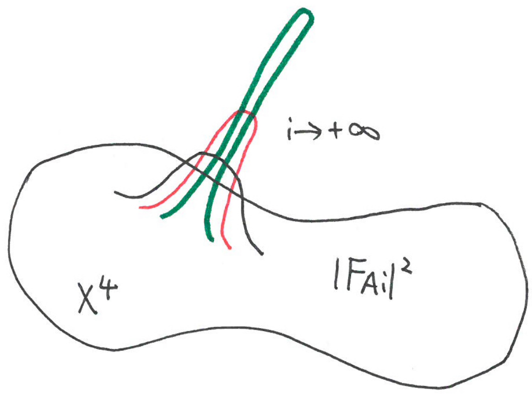
17.4. Bubbling Phenomena

What can we say about P'? Since

$$\int_X |F_A|^2 < \lim \int_X |F_{A_i}|^2 = 8\pi^2 c_2(P),$$

we must have

$$c_2(P') < c_2(P).$$



For each distinguished point x_j, there exist $\alpha > 0$ and a sequence of numbers $\{r_i\}$ with $r_i \to 0$ such that

$$0 < \alpha < \int_{B(x_j, r_i)} |F_{A_i}|^2 (\leqslant 8\pi^2 c_2(P)).$$

We rescale $B(x_j, r_j)$ so that its radius is normalized to be 1:

$$\tau_{1/r_i} : B(x_j, r_i) \to B(x_j, 1) \subset T_{x_j} X.$$

The metric

$$g_i = (\tau_{1/r_i}^{-1})^* g$$

on $B(x_j, 1)$ is changing and converges to the Euclidean metric g_∞. The connection

$$(\tau_{1/r_i})_* A_i$$

on $B(x_j, 1)$ is anti-self dual with respect to g_i. We can still prove a similar compactness theorem, even through they are ASD with respect to different metrics.

Fix $R > 0$, then under above rescaling,

$$\tau_{1/r_i} : B(x_j, R) \to B(x_j, \frac{R}{r_j}) \subset T_{x_j} X.$$

and at the limit,

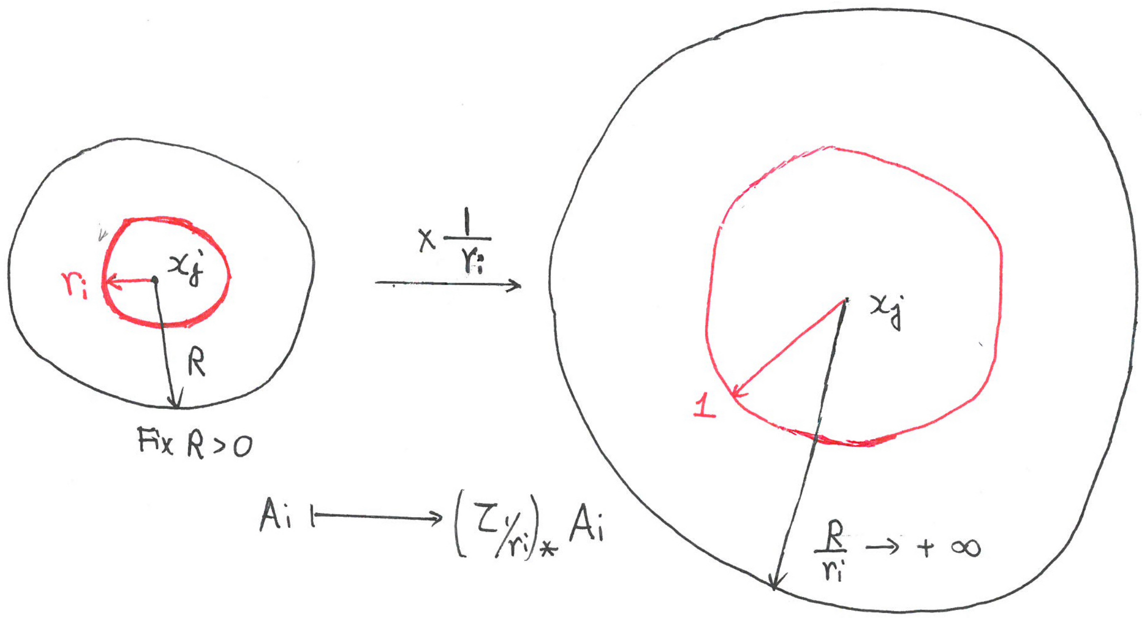
$$\lim \tau_{1/r_i}(B(x_j, R)) = T_{x_j} X \cong \mathbb{R}^4.$$

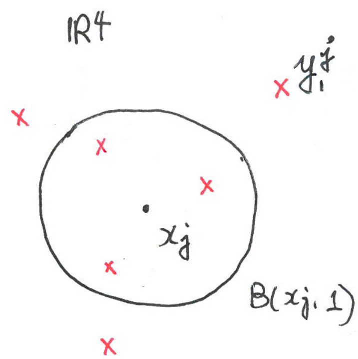
Therefore, away from a finite collection of points $\{y_1^j, y_2^j, \cdots, y_m^j\} \subset \mathbb{R}^4$,

$$(\tau_{1/r_i})_* A_i \xrightarrow{L^2_{1,loc}} A^j$$

for some smooth anti-self dual connection A^j on $\mathbb{R}^4$ with finite energy:

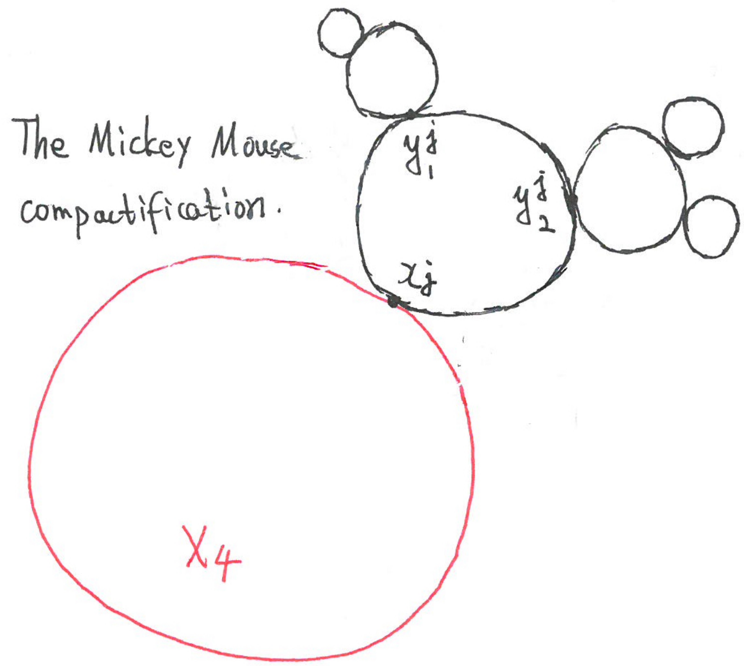
$$\int_{\mathbb{R}^4} |F_{A^j}|^2 = \lim_i \int_{B(x_j, \frac{R}{r_i})} |F_{A^j}|^2 \leqslant \lim_i \int_{B(x_j, R)} |F_{A_i}|^2 \leqslant 8\pi^2 c_2(P).$$





Here, $\mathbb{R}^4$ is regarded as $S^4 - \{S\}$, S^4 with the South pole removed. By Uhlenbeck's Removable Singularities theorem, A^j extends to a smooth connection on S^4 for some bundle $P^j \to S^4$.

Finally, we can iterated this construction and obtain a bubble tree compactification of the moduli space, which is finer than Uhlenbeck's compactification to be introduced in the next lecture. Uhlenbeck realized this bubble tree compactification first in her study of harmonic maps, but she didn't formalize the construction. Nowadays, it is called Kontsevich's compactification, which is improper from a historical point of view.


The Mickey Mouse compactification.
y_1^j
y_2^j
x_j
X_4

Lecture 18. **The structure of the moduli space**

18.1. The local structure

Let X^4 be a closed 4-manifold and $\mathcal{M}_k(X)$ be the moduli space of ASD[6] connections on $P_k \to X$. Here, $P_k \to X$ is a principle $SU(2)$-bundle and

$$c_2(P_k)[X] = k \geqslant 0.$$

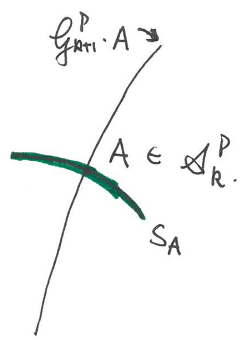
Let $\mathrm{Sym}^l(X) = X^l/\Sigma_l$ be the symmetric product of X. Then Uhlenbeck's compactness theorem states that any sequence of connections $\{[A_i]\} \subset \mathcal{M}_k(X)$ has a subsequence whose limit is

$$([A], (x_1, \cdots, x_l))$$

with $[A] \in \mathcal{M}_{k-l}$ and $x_i \in X$. The l-tuple of points is unordered. Thus, this limit is contained in

$$\mathcal{M}_{k-l}(M) \times \mathrm{Sym}^l(X)$$

for some $0 \leqslant l \leqslant k$.



We examine the local structure of $\mathcal{M}_k(X)$ more carefully. If $[A] \in \mathcal{M}(X)$, then locally, $\mathcal{M}(X)$ is described by the zero locus of

$$G_A : \{A + a \in S_A\} \to \Omega^2_+(\mathrm{ad}\, P)$$
$$A + a \mapsto F^+_{A+a}.$$

whose linearization at $a = 0$ is d^+_A. With appropriate Sobolev completions, G_A becomes a Fredholm map. Note that the chain of maps:

$$\Omega^0(\mathrm{ad}\, P) \xrightarrow{d_A} \Omega^1(\mathrm{ad}\, P) \xrightarrow{d^+_A} \Omega^2_+(\mathrm{ad}\, P) \tag{25}$$

is a complex if A is ASD. Indeed,

$$d^+_A \circ d_A = F^+_A = 0.$$

[6]"ASD" stands for "anti-self dual"

In fact, (25) is an elliptic complex, that is, the symbol sequence is exact. This implies that the sequence

$$L^p_{k+1}(X, \operatorname{ad} P) \xrightarrow{d_A} L^p_k(X, T^*X \otimes \operatorname{ad} P) \xrightarrow{d_A^+} L^p_{k-1}(X, \Lambda^2_+ \otimes \operatorname{ad} P)$$

is a Fredholm complex, which means

(1) It has finite dimensional cohomology groups.
(2) All maps have closed range.

There are two ways to compute the **index** of $d_A^+ \oplus d_A^*$:

(1) Use Atiyah-Singer Index theorem.
(2) Compute some model cases, then apply the excision theorem.

It will save us some time if we adopt the second one.

18.2. Index Computation: When P is trivial

If P is trivial and A is the trivial connection, then the complex (**??**) becomes

$$\Omega^0 \otimes \mathfrak{g} \xrightarrow{d} \Omega^1 \otimes \mathfrak{g} \xrightarrow{d^+} \Omega^2_+ \otimes \mathfrak{g}. \tag{26}$$

Its cohomology groups are:

$$H^0 \cong \mathfrak{g}, H^1 = H^1(X, \mathbb{R}) \otimes \mathfrak{g}, H^2 = \mathcal{H}_2^+ \otimes \mathfrak{g}.$$

We elaborate on $\mathcal{H}_2^+$. If $\omega \in \operatorname{Coker} d^+$, then $\omega \in \Omega^2_+$ and for any $a \in \Omega^1$,

$$0 = \int_X \langle d^+ a, \omega \rangle = \int_X da \wedge *\omega = \int_X da \wedge \omega.$$

By Stokes' theorem,

$$\int_X a \wedge d\omega = 0,$$

so $d\omega \equiv 0$. Since $\omega = *\omega$, $d^*\omega = 0$ as well. Hence, ω is a harmonic 2-form.

For each 4-manifold X, we associate a quadratic form $Q_X : H_2(X) \to \mathbb{R}$. If $\alpha \in H_2(X)$ is represented by a immersed surface Σ, we perturb Σ into Σ' such that they intersect transversely. Then, Q_X is the signed count of intersections of Σ and Σ':

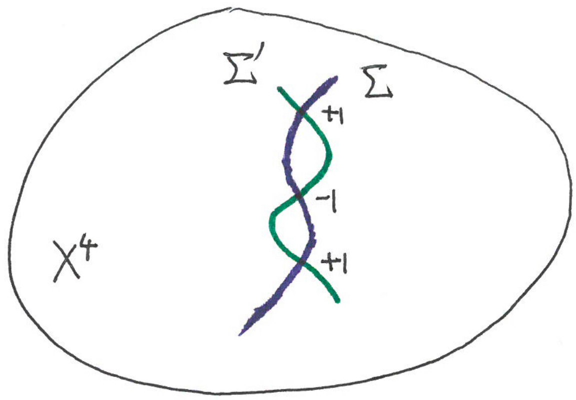
$$Q_X(\alpha) = \sum_{x \in \Sigma \cap \Sigma'} \epsilon(x).$$

In terms of cohomology, if $[\alpha] \in H^2(X, \mathbb{R})$,

$$Q_X([\alpha]) = \int_X \alpha \wedge \alpha.$$

if α is a closed 2-form representing $[\alpha]$. The induced bilinear form Q_X is the intersection form on $H^2(X, \mathbb{R})$, which is symmetric. By Poincaré duality, Q_X is non-degenerate. We write

$$H^2(X) \cong W^+ \oplus W^-.$$



where W^+ (W^-) is a maximal positive (negative) definite subspace of $H^2(X)$. Define

$$b^{\pm}(X) = \dim W^{\pm}.$$

By Hodge theory,

$$H^2(X, \mathbb{R}) \cong \{\omega \in \Omega^2(X) : d\omega = 0, d^*\omega = 0\} = \mathcal{H}^2(X).$$

Since the hodge star operator acts on $\mathcal{H}^2(X)$, $\mathcal{H}^2(X)$ decomposes into eigenspaces:

$$\mathcal{H}^2(X) = \mathcal{H}^+ \oplus \mathcal{H}^- = (+1) \oplus (-1).$$

Then, $\mathcal{H}^{\pm}$ is also the maximal positive (negative) definite subspaces of Q_X:

$$\omega \in \mathcal{H}^{\pm} \Rightarrow \int_X \omega \wedge \omega = \pm \int_X \omega \wedge *\omega = \pm \int_X |\omega|^2 \geqslant (\leqslant)\ 0.$$

Thus, the second cohomology of (18) is

$$H^2 = \mathcal{H}^2_+ \otimes \mathfrak{g}.$$

When A is trivial,

$$\text{the index of (25)} = \dim \mathfrak{g} \cdot (1 - b_1 + b^+).$$

It is convenient to write it in a different way. Let

$$\sigma(X) = b^+(X) - b^-(X)$$

be the signature of X. Let $\chi(X)$ be the Euler characteristic:

$$\chi(X) = b_0 - b_1 + b_2 - b_3 + b_4 = 2b_0 - 2b_1 + b_2 = 2b_0 - 2b_1 + (b^+ + b^-).$$

Then,

$$1 - b_1 + b^+ = \frac{\chi + \sigma}{2} \Rightarrow \text{the index of (25)} = \frac{\dim \mathfrak{g}}{2} \cdot (\chi(X) + \sigma(X)).$$

What if A is not trivial? In fact, the index of

$$L^p_{k+1}(X, \operatorname{ad} P) \xrightarrow{d_A} L^p_k(X, T^*X \otimes \operatorname{ad} P) \xrightarrow{d_A^+} L^p_{k-1}(X, \Lambda^2_+ \otimes \operatorname{ad} P)$$

is independent of the choice of A. As A varies, the complex is changed by 0-order terms (hence, by compact operators).

18.3. Index Computation: When P is not trivial

We constructed $SU(2)$-instantons on S^4 using the round metric and

$$\dim \mathcal{M}_k(S^4) = 8k - 3.$$

when $k \geqslant 1$. These moduli spaces are regular. By the Weitzenböck formula,

$$d_A^+(d_A^+)^* = \nabla_A^* \nabla_A + \Phi_1(F_A^+) \cdot + \Phi_2(W^+) \cdot + \Phi_3(s) \cdot$$

where Φ_1, Φ_2, Φ_3 are bundle maps that depend on F_A^+, the Weyl curvature tensor and the scalar curvature s respectively. Since A is ASD and the round metric is conformally flat, $F_A^+ \equiv 0$ and $W^+ \equiv 0$. Since s is point-wise positive, $d_A^+(d_A^+)^*$ has no kernel, so

$$d_A^+ \text{ is surjective.}$$

Therefore, the expected dimension of $\mathcal{M}_k(S^4)$ is at least $8k-3$. But they are actually equal.

To apply the excision principle of elliptic operators, we need the following picture:

(1) $X_1 = U_1 \cup V_1$ and $X_2 = U_2 \cup V_2$.
(2) $W_1 = U_1 \cap V_1$ is diffeomorphic to $W_2 = U_2 \cap V_2$.
(3) We have elliptic differential operators:

$$D_1 : \Gamma(E_1) \to \Gamma(F_1) \text{ on } X_1$$
$$D_2 : \Gamma(E_2) \to \Gamma(F_2) \text{ on } X_2.$$

(4) There exist bundle isomorphisms

$$\phi : E_1|_{W_1} \to E_2|_{W_2}, \psi : F_1|_{W_1} \to F_2|_{W_2}$$

that cover the diffeomorphism in (2) and intertwine D_1 and D_2:

$$D_2 = \psi \circ D_1 \circ \phi^{-1} \text{ on } W_2.$$

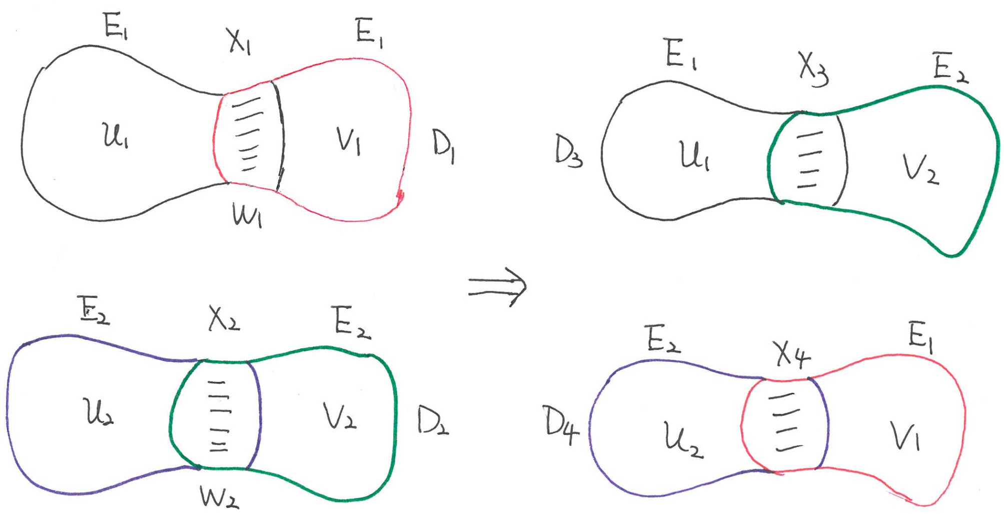
Under these assumptions, we can

(1) Define $X_3 = U_1 \cup V_2$ and $X_4 = U_2 \cup V_1$.
(2) Obtain $E_3 \to X_3$ by gluing $E_1|_{U_1}$ and $E_2|_{V_2}$.
(3) Obtain $E_4 \to X_4$ by gluing $E_2|_{U_2}$ and $E_1|_{V_1}$. Similarly, define $F_3 \to X_3$ and $F_4 \to X_4$.
(4) Define elliptic differential operators:

$$D_3 : \Gamma(E_3) \to \Gamma(F_3), D_4 : \Gamma(E_4) \to \Gamma(F_4).$$

The conclusion is that

$$\operatorname{Ind} D_1 + \operatorname{Ind} D_2 = \operatorname{Ind} D_3 + \operatorname{Ind} D_4.$$



To apply the excision theorem, let

$$X_1 = X, c_2(P_1) = k,\ X_2 = S^4, c_2(P_2) = 0,$$
$$X_3 = X, c_2(P_3) = 0,\ X_4 = S^4, c_2(P_4) = k.$$

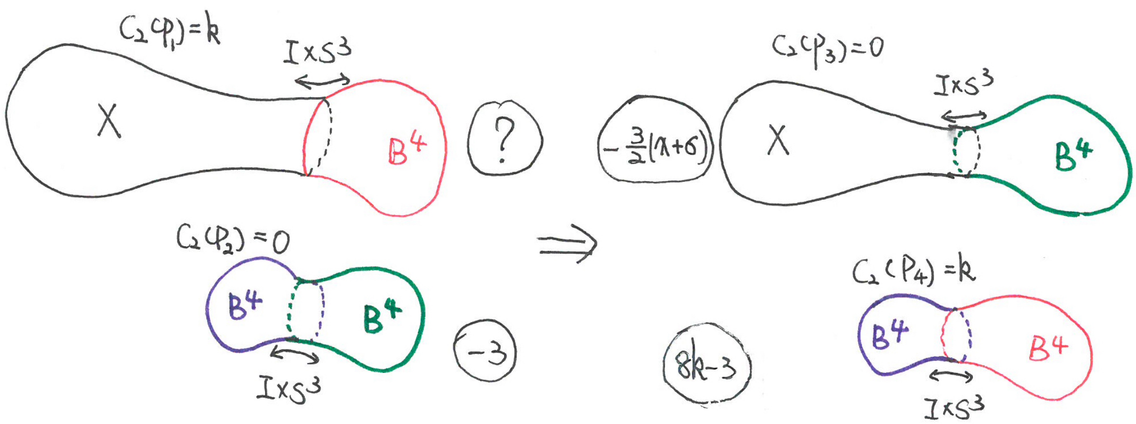
and $D_i = d^+_{A_i} \oplus d^*_{A_i} : \Omega^1(\operatorname{ad} P_i) \to \Omega^0(\operatorname{ad} P_i) \oplus \Omega^2_+(\operatorname{ad} P_i)$. Since $c_2(P_2) = c_2(P_3) = 0$,

$$\operatorname{Ind} D_2 = -\frac{3}{2}(\chi(S^4)) + \sigma(S^4)) = -3, \operatorname{Ind} D_3 = -\frac{3}{2}(\chi(X)) + \sigma(X)).$$

We have already known $\operatorname{Ind} D_4 = 8k - 3$, so

$$\operatorname{Ind} D_1 = \operatorname{Ind} D_3 + (\operatorname{Ind} D_4 - \operatorname{Ind} D_2) = 8k - \frac{3}{2}(\chi(X)) + \sigma(X)).$$

In the same manner, we can compute the index in general, without assuming the



structure group is $SU(2)$.

Theorem 18.3.1 (Uhlenbeck). *For a generic metric on X, the moduli space $\mathcal{M}_k(X)$ is smooth, i.e., d_A^+ is surjective for all $A \in \mathcal{M}_k(X)$ which is not* ***flat*** *or* ***reducible*** *and*

$$\dim \mathcal{M}_k(X) = 8k - \frac{3}{2}(\chi(X) + \sigma(X)).$$

The Uhlenbeck's compactification sits inside a larger space:

$$\begin{aligned}
\overline{\mathcal{M}_k(X)}^{Uhl} \subset \mathcal{M}_k(X) & \qquad \dim_k = 8k - \frac{3}{2}(\chi(X) + \sigma(X)) \\
\bigcup \mathcal{M}_{k-1}(X) \times X & \qquad \dim = \dim_k - 4 \\
\bigcup \mathcal{M}_{k-2}(X) \times \operatorname{Sym}^2 X & \qquad \dim = \dim_k - 4 \times 2 \\
\bigcup \cdots & \\
\bigcup \mathcal{M}_0(X) \times \operatorname{Sym}^k X & \qquad \dim = 4k.
\end{aligned}$$

The dimension of the l-th stratum is $\dim_k - 4l$ except when $l = k$. Suppose $b^+(X) = 0, \pi_1(X) = \{0\}$ and $k = 1$. Then $b^1 = 0$, and

$$\dim_1 = 8 \cdot 1 - 3 \cdot (1 - 0 + 0) = 5.$$

Then the compactification looks like

$$\begin{aligned}
\overline{\mathcal{M}_1(X)}^{Uhl} \subset \mathcal{M}_1(X) & \qquad \dim_1 = 5 \\
\bigcup \mathcal{M}_0(X) \times X & \qquad \dim = 4.
\end{aligned}$$

Note that $\mathcal{M}_0(X)$ consists of flat connections. Since $\pi_1(X) = \{0\}$, the flat connection is unique with stabilizer $SU(2)$. Because of the $SU(2)$-action, the expected dimension of $\mathcal{M}_0(X)$ is

$$\dim \mathcal{M}_0(X) - \dim SU(2) = 0 - 3 = -3.$$

where $\dim \mathcal{M}_0(X)$ is the actual dimension.

If $X = S^4$, $M_1(X) = \mathring{B}^5 = \mathbb{H}^5 = SO(5,1)/SO(5)$ and its compactification is the closed 5-ball:

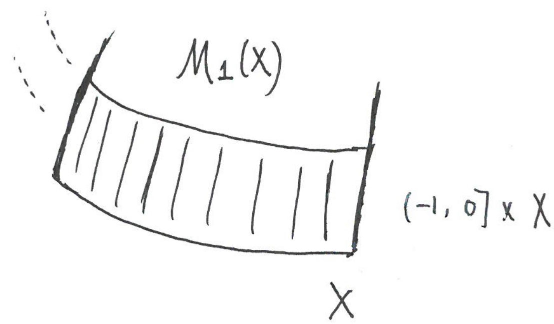
$$\overline{M_1(S^4)} = \mathring{B}^5 \cup S^4 = B^5.$$

Theorem 18.3.2 (Donaldson). *For any closed smooth 4-manifold X with $\pi_1(X) = \{0\}$ and $b^+(X) = 0$,*

$$\overline{M_1(X)} = M_1(X) \bigcup X.$$

and X has a product neighborhood $(-1, 0] \times X \subset \overline{M_1(X)}$.

Therefore, if $\mathcal{M}_1$ is a smooth manifold, then X is null-bordant. Theorem 18.3.1 has almost guaranteed that $\mathcal{M}_1$ is smooth, but we have to rule out reducible solutions. For instance, $\overline{\mathbb{CP}^2}$ has trivial π_1 and $b^+(\overline{\mathbb{CP}^2}) = 0$, but it is not null-bordant. Reducible solutions will occur in $\mathcal{M}_1(\overline{\mathbb{CP}^2})$.



$\overline{\mathbb{CP}^2}$ is $\mathbb{CP}^2$ with the reversed orientation. If you use the conjugate complex structure on $\mathbb{CP}^2$, you will end up with the same oriented manifold.

Tom:" Taubes were thinking about the moduli space on a connected sum of 4-manifolds and started with the case when one piece was $\overline{\mathbb{CP}^2}$. He started to think what was a $\overline{\mathbb{CP}^2}$. Is that the manifold with reversed complex structure? I believe that he didn't spend more than 15 minutes on this problem, but be careful, $\overline{\mathbb{CP}^2}$ is not this manifold."

18.4. Reducible Solutions

Let P be a $SU(2)$-principal bundle over X and A is a reducible ASD connection. The commutants graph of $SU(2)$ is simply:

$$\mathbb{Z}_2 \subset U(1) \subset SU(2).$$

If $\mathrm{Stab}_A = S^1$, then P is induced from a $U(1)$-bundle Q:

$$P = Q \times_{U(1)} SU(2).$$

This means that the canonical 2-plane bundle E splits as

$$E = L \oplus L^*,$$

and the connection A splits as

$$\begin{pmatrix} B & 0 \\ 0 & B^* \end{pmatrix}.$$

for a unitary connection B on Q. By the Chern-Weil theory,

$$[\frac{i}{2\pi} F_B] = c_1(Q) \in H^2(X, \mathbb{Z}).$$

Since A is ASD, so is B. When does Q admit an ASD connection? For a cohomology class $[\alpha] \in H^2(X)$, it admits an ASD representative if and only if

$$\int_X \alpha \wedge \theta = 0, \forall \theta \in \mathcal{H}^2_+(X).$$

If $b^+ = 0$, then any line bundle Q admits a connection with an ASD curvature.

We have to decide which principal bundles P arises in this way. We will address this problem in the next lecture and prove Donaldson's diagonalization theorem.

Lecture 19. Donaldson's Diagonalization Theorem

We will sketch a proof of Donaldson's diagonalization theorem.

Theorem 19.0.1 (Donaldson 1982). *If X^4 is a closed smooth simply connected 4-manifold with a definite intersection form q_X, then q_X is diagonalizable.*

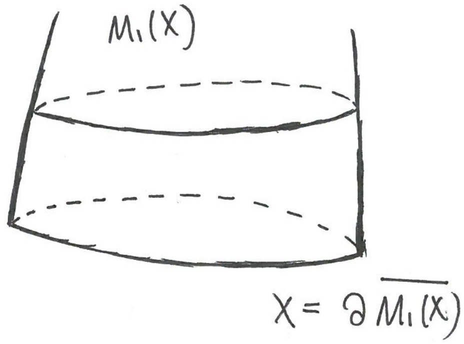
Fix an orientation of X such that the intersection form q_X is negative definite. Let $\mathcal{M}_1(X)$ be the moduli space of ASD connections on X of charge 1. Then

$$\dim \mathcal{M}_1(X) = 8 - 3(1 - b_1 + b_+) = 8 - 3(1 - 0 + 0) = 5.$$

In this case, Uhlenbeck's compactification of $\mathcal{M}_1(X)$ is

$$\overline{\mathcal{M}}_1^{Uhl} = \mathcal{M}_1(X) \cup X,$$

and X has a collar neighborhood in $\overline{\mathcal{M}}_1^{Uhl}$. The analysis used in the collar neighborhood theorem will be analogous to some techniques in Floer homology. We need the



generic metric theorem:

Theorem 19.0.2. *When the structure group $G = SU(2)$ or $SO(3)$, the linearized ASD equation:*

$$d_A^+ : \ker(d_A^*) \to \Omega_+^2(\operatorname{ad} P)$$

is surjective at every $A \in \mathcal{M}_k(X)$ which is not reducible or flat.

This result is unknown when G is a higher rank compact Lie group, for instance, when $G = SU(3)$. For any $x \in X$,

$$\begin{aligned}
\dim \Lambda_+^2 \otimes \operatorname{ad} P|_x &= 9 \text{ if } G = SU(2) \text{ or } SO(3),\\
\dim \operatorname{Sym}^2(T^*X)|_x &= 10,\\
\dim \operatorname{Sym}_0^2(T^*X)_x &= 9.
\end{aligned}$$

Note that $\operatorname{Sym}^2(T^*X)$ is the tangent space of metrics on X, while traceless symmetric 2-tensors $\operatorname{Sym}_0^2(T^*X)_x$ form the tangent space of conformal structures. Proof of the generic metric theorem holds because

$$\dim \operatorname{Sym}_0^2(T^*X)_x \geqslant \dim \Lambda_+^2 \otimes \operatorname{ad} P|_x,$$

which is no longer true when $\mathrm{rank}(G) \geqslant 3$.

19.1. The local structure near reducible solutions

The moduli space $\mathcal{M}_1(X)$ contains no flat connections, since the charge $k \neq 0$. What about reducible connections?

$$\begin{tikzcd} SU(2) \arrow[r] & P \arrow[d] \arrow[r, squiggly, "\exists ?"] & U(1) \arrow[r] & Q \arrow[d] \\ & X & & X \end{tikzcd}$$

The embedding $\rho : U(1) \hookrightarrow SU(2)$ is given by

$$\rho(e^{i\theta}) = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}.$$

Suppose P is induced from a $U(1)$-bundle: $P = Q \times_\rho SU(2)$. Fix a reference unitary connection A_0 of Q. If A is another unitary connection of Q, then

$$A = A_0 + a$$

for some $a \in i\Omega^1(X)$, and $F_A = F_{A_0} + da$. If F_A is ASD, then

$$d^+ a = -F_{A_0}^+. \tag{27}$$

The condition that $b^+ = 0$ implies that the operator

$$d^+ : \Omega^1 \to \Omega^2_+$$

is surjective, so the equation (27) has a solution. As for the uniqueness, $d^+ a = 0$ implies $da = 0$, since

$$0 = \int_X da \wedge da = -\int_X |d^+ a|^2 + \int_X |d^- a|^2.$$

Thus, $a = db$ for some $b \in i\Omega^0(X)$ if $b_1(X) = 0$. The solution of (27) is unique up to gauge transformations.

Note that P is determined by $c_2(P) \in H^4(X, \mathbb{Z})$ up to isomorphism, while Q is determined by $c_1(Q) = c_1(L) \in H^2(X, \mathbb{Z})$, where

$$L = Q \times_{U(1)} \mathbb{C}.$$

is the line bundle induced from the fundamental representation. Let $E = P \times_{SU(2)} \mathbb{C}^2$, then $E = L \oplus L^*$. By the Cartan formula,

$$c_2(E) = -c_1^2(L).$$

Define $\alpha(X) = \#\{c \in H^2(X, \mathbb{Z}) : c^2 = -1\}/\pm 1$. Replacing L by L^*, we will get the same SU(2)-bundle P, but the first Chern class

$$c_1(L^*) = -c_1(L)$$

is replaced by its opposite. Hence, we need to quotient out ± 1 in the definition of $\alpha(X)$. Furthermore,

$$\alpha(X) = \#\{\text{reducible ASD connections in } \mathcal{M}_1(X)\}.$$

To study the local structure near reducible solutions, consider the deformation complex at a reducible connection A:

$$0 \to \Omega^0(\operatorname{ad} P) \xrightarrow{d_A} \Omega^1(\operatorname{ad} P) \xrightarrow{d_A^+} \Omega^2_+(\operatorname{ad} P) \to 0. \tag{28}$$

If P is induced from Q, then

$$\begin{aligned}\operatorname{ad} P &= P \times_{\mathrm{ad}} \mathfrak{su}(2) = (Q \times_{U(1)} SU(2)) \times_{\mathrm{ad}} \mathfrak{su}(2)\\ &= Q \times_{\mathrm{ad}\circ\rho} \mathfrak{su}(2) = i\mathbb{R} \oplus (L^{\otimes 2})_{\mathbb{R}}.\end{aligned}$$

Indeed, the adjoint action of $\rho(e^{i\theta}) = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta}\end{pmatrix}$ is given by

$$\begin{pmatrix} it & z \\ -\bar{z} & -it\end{pmatrix} \mapsto \begin{pmatrix} it & ze^{2i\theta} \\ -\bar{z}^{-2i\theta} & -it\end{pmatrix} \in \mathfrak{su}(2).$$

The deformation complex decomposes into a direct sum of complexes:

$$0 \to \Omega^0(X, i\mathbb{R}) \xrightarrow{d} \Omega^1(X, i\mathbb{R}) \xrightarrow{d^+} \Omega^2_+(X, i\mathbb{R}) \to 0$$

$$0 \to \Omega^0(X, (L^{\otimes 2})_{\mathbb{R}}) \xrightarrow{d_B} \Omega^1(X, (L^{\otimes 2})_{\mathbb{R}}) \xrightarrow{d_B^+} \Omega^2_+(X, (L^{\otimes 2})_{\mathbb{R}}) \to 0.$$

Take the cohomology groups:

$$H^0(X, i\mathbb{R}) = i\mathbb{R}, H^1(X, i\mathbb{R}) = 0, H^2_+(X, i\mathbb{R}) = 0$$
$$H^0_B(X, (L^{\otimes 2})_{\mathbb{R}}) = 0, H^1_B(X, (L^{\otimes 2})_{\mathbb{R}}) = \mathbb{C}^{n+3}, H^2_{+,B}(X, (L^{\otimes 2})_{\mathbb{R}}) = \mathbb{C}^n.$$

The Euler characteristic number of the deformation complex is

$$-\dim \mathcal{M}_1(X) = -5.$$

This forces $\dim_{\mathbb{C}} H^1_B(X, (L^{\otimes 2})_{\mathbb{R}}) - \dim_{\mathbb{C}} H^2_{+,B}(X, (L^{\otimes 2})_{\mathbb{R}}) = 3$. The complex structure on $H^1_B(X, (L^{\otimes 2})_{\mathbb{R}})$ is not canonical. It depends on which complex structure is chosen on $(L^{\otimes 2})_{\mathbb{R}}$.

We could assume $n = 0$, by the generic metric theorem. $H^0(X, i\mathbb{R}) \cong i\mathbb{R}$ corresponds to the tangent space of the stabilizer. Thus,

$$T_{[A]}\mathcal{M}_1(X) = \mathbb{C}^3,$$

but S^1 acts on this space. The action is given by matrices:

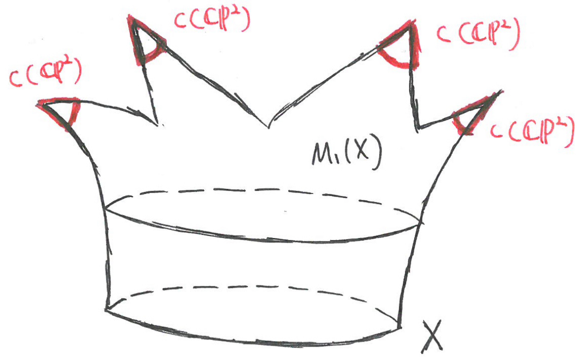
$$\begin{pmatrix} e^{2\theta i} & & \\ & e^{2\theta i} & \\ & & e^{2\theta i}\end{pmatrix},$$

so $\{\pm 1\} = Z(SU(2))$ acts trivially. Just like the slice theorem, $\mathcal{M}_1(X)$ locally looks like

$$\text{the slice/ stabilizers} = \mathbb{C}^3/S^1 \cong c(\mathbb{CP}^2)$$

which is a cone over $\mathbb{CP}^2$. Globally, we think of $\mathcal{M}_1(X)$ as a cobordism between

$$X \to \coprod_{\alpha(X)} \mathbb{CP}^2.$$



This is not an interesting result unless $\mathcal{M}_1(X)$ is an oriented cobordism.

Remark. The oriented cobordism group $\Omega_4^{SO} \cong \mathbb{Z}$ is generated by $[\mathbb{CP}^2]$ and is detected by the signature. The un-oriented cobordism group $\Omega_4 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ is generated by $[\mathbb{RP}^4]$ and $[\mathbb{RP}^2 \times \mathbb{RP}^2]$. $\square$

19.2. Orientation

We need to address the orientability of $\mathcal{M}_1(X)$. Consider the orientation line bundle of $\mathcal{M}_1(X)$:

$$\bigwedge^{max}(T\mathcal{M}_1(X))$$

where $\bigwedge^{max}$ denotes the highest exterior power. In fact, it is Quillen's observation that $\bigwedge^{max}(T\mathcal{M}_1(X))$ is the restriction of a universal determinant line bundle associated a family of Fredholm operators. Let $H_{\mathbb{R}}$ be a real infinite dimensional Hilbert space. Then the space of Fredholm operators is homotopy equivalent to

$$\mathrm{Fred}(H_{\mathbb{R}}) \cong \mathbb{Z} \times BO.$$

where the $\mathbb{Z}$-component corresponds to the index and BO is the classifying space of the infinite orthogonal group $O(\infty) = \lim_{n\to\infty} O(n)$. Note that

$$H^1(BO, \mathbb{Z}_2) \cong \mathbb{Z}_2.$$

The non-trivial element is the universal Stiefel-Whitney class w_1, and it is represented by a non-trivial real line bundle γ. The question is: can we construct γ concretely on $\mathrm{Fred}(H_{\mathbb{R}})$ without using this homotopy equivalence?

Let $T \hookrightarrow \mathrm{Fred}(H_{\mathbb{R}})$ be a compact family of Fredholm operators:

$$D_t : H_0 \to H_1, \ \forall t \in T.$$

Consider the real line:

$$\lambda_t = \bigwedge^{max} \ker(D_t) \otimes \bigwedge^{max} \mathrm{Coker}(D_t)^*.$$

This is a family of real lines over T. We have to show it is locally trivial, so it has a well-defined topology. The problem is that $\dim \ker(D_t)$ is not locally constant and $\ker(D_t)$ does not form a vector bundle over T.

Let $A : V_0 \to V_1$ be an invertible map between finite dimensional real vector spaces. Let $d = \dim V_0 = \dim V_1$. Then we can assign a canonical element in

$$\bigwedge^{max} V_0^* \otimes \bigwedge^{max} V_1 \ni \bigwedge^{d} A.$$

If $(u_1, \cdots, u_d)$ is a basis of V_0, let $(u_1^*, \cdots, u_d^*)$ be the dual basis in V_0^*. Define

$$\bigwedge^{d} A = (u_1^* \wedge \cdots \wedge u_d^*) \otimes Au_1 \wedge Au_2 \wedge \cdots \wedge Au_d.$$

This element is independent of the choice of $(u_1, \cdots, u_d)$. It is non-zero, because A is invertible. In particular, if there is a compact family of invertible maps A_t parametrized by $t \in T$, we obtain a section of

$$\bigwedge^{max} V_0^* \otimes \bigwedge^{max} V_1.$$

over T. This idea is generalized to an acyclic complex of vector spaces:

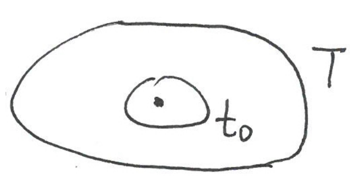
$$0 \to V_0 \xrightarrow{A_1} V_1 \xrightarrow{A_2} V_2 \xrightarrow{A_3} \cdots \xrightarrow{A_n} V_n \to 0.$$

An acyclic complex is a complex with trivial cohomology. We can assign to $A = (A_1, \cdots, A_n)$ a canonical element in

$$\bigwedge^{max} V_0^* \otimes \bigwedge^{max} V_1 \otimes \bigwedge^{max} V_2^* \otimes \bigwedge^{max} V_4 \cdots$$

Different literature may adopt different conventions. Though the complex is acyclic, this element is a meaningful invariant of A. This is the origin of Milnor torsion and Whitehead torsion. If we have a simplicial complex X with trivial rational cohomology, then applying this process to the simplicial chain complex of X will give us a canonical element in $\mathbb{Q}$. To make it an genuine invariant of X, we need to verify its invariance under the subdivision of simplicial structures of X.

Back to our problem. For fixed $t_0 \in T$, choose $J \subset H_1$ such that $\mathrm{Coker}\, D_{t_0} \subset J$. Then the composition $pr \circ D_t : H_0 \to H_1 \to H_1/J$ is surjective in a neighborhood of $t_0 \in T$.



Consider the acyclic complex:

$$0 \to \ker(D_t) \to D_t^{-1}(J) \xrightarrow{D_t} J \to \mathrm{Coker}(D_t) \to 0.$$

Then we obtain a canonical $\bigwedge D_t$ element in

$$\bigwedge^{max}(\ker D_t)^* \otimes \bigwedge^{max} D_t^{-1}(J) \otimes \bigwedge^{max} J^* \otimes \bigwedge^{max} \mathrm{Coker}\, D_t.$$

In particular, this is a canonical map between lines:

$$\bigwedge^{max} \ker D_t \otimes \bigwedge^{max}(\mathrm{Coker}\, D_t)^* \to \bigwedge^{max} D_t^{-1}(J) \otimes \bigwedge^{max} J^*. \tag{29}$$

Note that the right hand side is a genuine vector bundle near t_0, so it produces a trivialization of λ_t. When the auxiliary space J is changed, one can show different trivializations are compatible with each other.

The trivializing map defined in (29) is sensitive to the sign. If we start with $(-D_t)$, we would arrive at a different topology of λ_t.

Remark. There is a detailed construction of $\bigwedge D_t$ in Tom and Peter's book [4, Section 20]. Readers can also find some lecture notes on Milnor torsion on Google, so we did not fill in the definition here. □

We will finish the proof Theorem 19.0.1 in the next lecture.

Lecture 20. The Topology of Configuration Space I

20.1. The proof of Donaldson's Diagonalization Theorem

Recall from the last lecture that for a compact family $\{D_t : t \in T\}$ of real Fredholm operators, we constructed the determinant line bundle over T:

$$\begin{array}{ccc} \mathbb{R} & \longrightarrow & \lambda(D_t) \\ & & \downarrow \\ & & T. \end{array}$$

For the ASD moduli space $\mathcal{M}_1(X)$ of charge 1, we have Uhlenbeck's compactification $\overline{\mathcal{M}}_1^{Uhl} = \mathcal{M}_1(X) \cup M$. The number of reducible solutions is given by

$$\alpha(X) = \{c \in H^2(X, \mathbb{Z}) : c^2 = -1\}/\pm 1,$$

and each reducible solution has a neighborhood diffeomorphic to a cone over $\mathbb{CP}^2$:

$$c(\mathbb{CP}^2).$$

If $\mathcal{M}_1(X)$ is orientable, then X is oriented-cobordant to $\alpha(X)$ copies of $\mathbb{CP}^2$. Since the signature σ is a cobordism invariant, we have

$$|\sigma(X)| = |\sum_{\alpha(X)} \sigma(\mathbb{CP}^2) \text{ or } \sigma(\overline{\mathbb{CP}}^2)| \leqslant \alpha(X).$$

This is what we learn from the geometry. On the other hand, any two elements $c, c' \in H^2(X, \mathbb{Z})$ with $c^2 = c'^2 = -1$ and $c \neq \pm c'$ are orthogonal to each other, because q_X is negative definite.

Remark. Since $c \neq \pm c'$, $(c \pm c')^2 < 0$. Hence, $-2 < 2c \cdot c' < 2$. □

For each $c \in H^2(X, \mathbb{Z})$ with $c^2 = -1$, define

$$\Lambda_c = \{b \in H^2(X, \mathbb{Z}) : c \cdot b = 0\}.$$

Then $H^2(X, \mathbb{Z}) = \mathbb{Z}c \oplus \Lambda_c$. This decomposition can be continued. We conclude that

$$\alpha(X) \leqslant \operatorname{rank} H^2(X, \mathbb{Z}) = -\sigma(X).$$

Together with this algebraic input, we know $\alpha(X) = -\sigma(X)$ and $q_X = \oplus[-1]$ has to be diagonalizable.

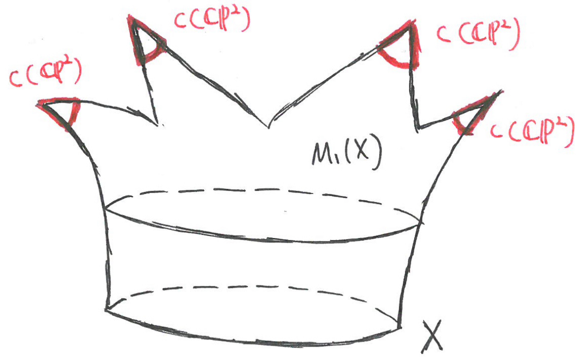
Remark. If $q_X = (-E_8)$, $\sigma(X) = -8$ and $\alpha(X) = 0$ since $(-E_8)$ is an even lattice.

Finally, let us explain why $\mathcal{M}_1(X) \subset \mathscr{B}_P = \mathscr{A}_P/\mathscr{G}_P$ is orientable. We will first sketch a strategy. In fact,

$$\bigwedge^{max} T\mathcal{M}_1(X) = \lambda(D_{[A]})|_{\mathcal{M}_1(X)}$$

where the family of operators $D_{[A]}$ is induced by

$$\Omega^1(X, \operatorname{ad} P) \xrightarrow{(d_A^*, d_A^+) = D_A} \Omega^0(X, \operatorname{ad} P) \oplus \Omega^2_+(X, \operatorname{ad} P).$$



Note that $\lambda(D_A) \to A$ is line bundle on $\mathscr{A}_P$. The gauge action of $\mathscr{G}$ lifts to $\lambda(D_A)$:

$$\begin{tikzcd} & \lambda(D_A) \arrow[d] \\ \mathscr{G} \arrow[ur, dashed, "action"] \arrow[r, "action"] & \mathscr{A}_P. \end{tikzcd}$$

However, $\lambda(D_A)$ does not push down directly to the quotient space $\mathscr{B}_P = \mathscr{A}_P/\mathscr{G}$ because of stabilizers. Let $\mathscr{A}^* \subset \mathscr{A}$ be the space of irreducible connections. Then the restriction $\lambda(D_A)|_{\mathscr{A}^*}$ descends to a determinant line bundle downstairs:

$$\begin{tikzcd} \lambda(D_A)|_{\mathscr{A}^*}/\mathscr{G} \arrow[d] \\ \mathscr{B}^* = \mathscr{A}^*/\mathscr{G}. \end{tikzcd}$$

Let $\mathcal{M}_1^*(X) = \mathcal{M}_1(X) \cap \mathscr{B}^*$. Then

$$\begin{tikzcd} \lambda(D_{[A]}) \arrow[r, "\subset"] \arrow[d] & \lambda(D_{[A]}) \arrow[d] \\ \mathcal{M}_1^*(X) \arrow[r, "\subset"] & \mathscr{B}^*. \end{tikzcd}$$

The trick is to show the determinant line bundle over $\mathscr{B}^*$ is orientable, and so is its restriction $\lambda(D_{[A]}) \to \mathcal{M}_1^*(X)$. In particular, even if $\mathcal{M}_1^*(X)$ might be disconnected, we get a consistent way to orient different components.

20.2. The algebraic topology of $\mathscr{B}^*$

Let us understand the algebraic topology of $\mathscr{B}^*$ in more generality. The materials below are based on [5]:

- Atiyah, M. F.; Bott, R.
  The Yang-Mills equations over Riemann surfaces. Philos. Trans. Roy. Soc. London Ser. A 308 (1983), no. 1505, 523-615.

Step 1 Let $\mathscr{G}^0$ be the based gauge group. Let $x_0 \in X$ be a base point, then

$$\mathscr{G}^o = \{g \in \mathscr{G} : g_{x_0} = \mathrm{Id}\}.$$

The action of $\mathscr{G}^o$ on $\mathscr{A}$ is free and $\mathscr{A}$ is contractible, so

$$\mathscr{A}/\mathscr{G}^o \xrightarrow[\cong]{\text{weak homotopy equivalent}} B\mathscr{G}^0.$$

Let $\mathscr{B}^{0,*} = \mathscr{A}^*/\mathscr{G}^o$, then it is a fiber bundle over $\mathscr{B}^*$ with fiber $G/Z(G)$:

$$\begin{tikzcd} G/Z(G) \arrow[r] & \mathscr{B}^{o,*} \arrow[d] \\ & \mathscr{B}^*. \end{tikzcd}$$

It is easier to start with $\mathscr{B}^{o,*}$ and $\mathscr{A}/\mathscr{G}^o$. To understand $B\mathscr{G}^o$, it suffices to find another model of $B\mathscr{G}^o$. Consider the classifying space of G:

$$\begin{tikzcd} G \arrow[r] & EG \arrow[d] \\ & BG. \end{tikzcd}$$

The total space EG of the universal bundle is contractible.

Lemma 20.2.1. *Let $[P, EG]^G$ be the mapping space of G-equivariant maps between P and EG, then $[P, EG]^G$ is contractible and is acted on freely by $\mathscr{G}^o$.*

Remark. Here, $[\cdot, \cdot]$ denotes the mapping space, not the homotopy classes of continuous maps.

Furthermore, the quotient space $[P, EG]^G/\mathscr{G}^o$ is $[X, BG]^*_{[P]}$. Here, $[\cdot, \cdot]^*$ denotes the mapping space that preserves base points, i.e. maps that send x_0 to a base point of BG. Since BG is the classifying space, the principal bundle P is pulled back from $EG \to BG$ via some map $f : X \to BG$ and the homotopy class of f is unique. Then $[\cdot, \cdot]^*_{[P]}$ is the connected component of $[\cdot, \cdot]^*$ that contains f.

Remark. Tom was a little sloppy here. Let us sketch a proof of Lemma 20.2.1. Suppose $P = X \times G$ is the trivial bundle, then

$$[P, EG]^G = [X \times G, EG]^G = [X, EG]$$

is contractible. In the general case, a map $f : S^n \to [P, EG]^G$ is equivalent to a G-equivariant map

$$S^n \times P \to EG$$

and it is a section of the bundle

$$\begin{tikzcd} EG \arrow[r] & (S^n \times P) \times_G EG \arrow[d, shift right] \\ & S^n \times X. \arrow[u, dashed, bend right, "s"'] \end{tikzcd}$$

If $\tilde{x} \in P$ is any lift of $x \in X$, then

$$s(y,x) = [(y,\tilde{x}), f(y,\tilde{x})] \in (S^n \times P) \times_G EG.$$

To show $\pi_n([P,EG]^G) = \{0\}$, it suffices to show any two sections are homotopy equivalent. Such a homotopy is given by the obstruction theory, using the fact that the fiber EG is contractible.

Then, it is easy to see $[P,EG]^G/\mathscr{G} = [X,BG]_{[P]}$, the non-based mapping space. To compare it with $[X,BG]^*_{[P]}$, we consider two fibrations:

$$\begin{tikzcd} \mathrm{Map}(X,BG)^*_{[P]} \arrow[r, "\iota"] & \boxed{\mathrm{Map}(X,BG)_{[P]}} \arrow[d, "ev"] \\ & BG \end{tikzcd}$$

$$\begin{tikzcd} G \arrow[r] & \mathrm{Map}(P,EG)^G/\mathscr{G}^o \arrow[d, "\pi"] & \\ & \boxed{\mathrm{Map}(P,EG)^G/\mathscr{G}} \arrow[r, dashed, "ev"] & BG. \end{tikzcd}$$

The vertical map in the first fibration is the evaluation map at the base point $x_0 \in X$:

$$ev : \mathrm{Map}(X,BG)_{[P]} \to BG, f \mapsto f(x_0) \in BG.$$

It becomes clear that the projection map π in the second diagram is the fibrant replacement of the inclusion map ι in the first map, because $\Omega BG \cong G$. It follows that $\mathrm{Map}(X,BG)^*_{[P]} \cong \mathrm{Map}(P,EG)^G/\mathscr{G}^o$. $\square$

The integral cohomology of $[X,BG]^*_{[P]}$ is delicate, while its rational cohomology is easier to understand. Take $G = SU(n)$. Then $H^*(BSU(n),\mathbb{Z})$ is a polynomial algebra generated by Chern classes

$$c_2, c_3, \cdots, c_n.$$

Note that $c_1 = 0$, because the structure group G is $SU(n)$, not $U(n)$. Each of them corresponds to a map to an Eilenberg-Maclane space:

$$f_m : BSU(n) \xrightarrow{c_m} K(\mathbb{Z},2m).$$

Putting them altogether, we obtain

$$f = \prod_{i=2}^n f_i : BSU(n) \to K(\mathbb{Z},4) \times K(\mathbb{Z},6) \times \cdots K(\mathbb{Z},2n).$$

This map f is not a homotopy equivalence, but it is a rational homotopy equivalence.

Remark. It suffices to prove that f induces isomorphisms on rational cohomology groups, so by Serre's theorem, f induces isomorphisms on $\mathbb{Q} \otimes \pi_*(\cdot)$. Using Serre's spectral sequence and induction, we know that

$$H^*(K(\mathbb{Z}, 2m), \mathbb{Q}) = \mathbb{Q}[u_{2m}]$$

is a polynomial algebra generated by the fundamental class $u_{2m} \in H^{2m}(K(\mathbb{Z}, 2m), \mathbb{Z})$, and

$$H^*(K(\mathbb{Z}, 2m-1), \mathbb{Q}) = \mathbb{Q}[u_{2m-1}]/(u_{2m-1}^2).$$

for any $m \geqslant 1$. □

Therefore, we know that

$$[X, BSU(n)]^* \xrightarrow[\cong]{\mathbb{Q}-w.h.e} [X, K(\mathbb{Z}, 4)]^* \times [X, K(\mathbb{Z}, 6)]^* \times \cdots \times [X, K(\mathbb{Z}, 2n)]^*.$$

Moreover, by a theorem of Thom,

$$[X, K(\mathbb{Z}, 2n)]^* \cong \prod_{i=0}^{\dim X} K(H^{2n-i}(X, \mathbb{Z}), i). \tag{30}$$

In particular When $n = 2$ and $\dim X = 4$, we have

$$[X, BSU(2)]^* \xrightarrow[\cong]{\mathbb{Q}-w.h.e} [X, K(\mathbb{Z}, 4)]^* \cong \prod_{i=0}^{4} K(H^{4-i}(X, \mathbb{Z}), i).$$

Take out the particular component correspondent to $[P]$:

$$[X, BSU(2)]^*_{[P]} \xrightarrow[\cong]{Q-w.h.e} \prod_{i=1}^{4} K(H^{4-i}(X, \mathbb{Z}), i).$$

Hence, the rational cohomology ring is given by

$$\begin{aligned} H^*([X, BSU(2)]^*_{[P]}, \mathbb{Q}) &\cong \mathbb{A}(H_*(X, \mathbb{Q})) \\ &:= \mathrm{Sym}(H_0(X)) \otimes \mathrm{Sym}(H_2(X)) \\ &\quad \otimes \bigwedge^*(H_1(X) \otimes H_3(X)). \end{aligned}$$

since $H^i(K(H^{4-i}(X, \mathbb{Z}), i), \mathbb{Q}) \cong H_{4-i}(X, \mathbb{Q})$.

20.3. Another description using the slant product

We have an evaluation map:

$$\begin{aligned} ev : [X, Y]^* \times X &\to Y \\ (f, x) &\mapsto f(x). \end{aligned}$$

If $\mu \in H^d(Y)$ and $a \in H_k(X)$, we form the slant product:

$$ev^*(\mu) \in H^d([X, Y]^* \times X), \ ev^*(\mu)/a \in H^{d-k}([X, Y]^*).$$

In the case of $G = SU(2)$, $\mathscr{B}_P^o = \mathscr{A}_P/\mathscr{G}_P^o = [X, BSU(2)]^*_{[P]}$ and we have an evaluation map:

$$\begin{tikzcd} SU(2) \arrow[r] & \mathscr{U} \arrow[r, equal] \arrow[d] & (\mathscr{A}_P \times P)/\mathscr{G}_P^o \\ & \mathscr{B}_P^0 \times X \arrow[r, "ev"] & BSU(2). \end{tikzcd}$$

Note that we have a universal $SU(2)$-bundle $\mathscr{U}$ over $\mathscr{B}^o \times X$ which is pulled back from the evaluation map. Therefore,

$$c_2(\mathscr{U}) = ev^* c_2.$$

For any $a \in H^*(X, \mathbb{Z})$, define

$$ev^*(c_2)/a = c_2(\mathscr{U})/a \in H^{4-\deg(a)}(\mathscr{B}_P^0).$$

Then these elements generate the rational cohomology ring of $\mathscr{B}_P^o$:

$$\begin{aligned} &\mathbb{A}(H_*(X, \mathbb{Q})) \cong H^*(\mathscr{B}_P^o, \mathbb{Q}) \\ &a \in H_i(X, \mathbb{Q}) \mapsto c_2(\mathscr{U})/a \in H^{4-i}(\mathscr{B}_P^o, \mathbb{Q}). \end{aligned}$$

where

$$\mathbb{A}(H_*(X, \mathbb{Q})) = \mathrm{Sym}(H_0(X)) \otimes \mathrm{Sym}(H_2(X)) \otimes \bigwedge^*(H_1(X) \otimes H_3(X)).$$

Each copy (either a polynomial algebra or a exterior algebra) corresponds to the rational cohomology ring of

$$K(H^{4-i}(X, \mathbb{Z}), i).$$

for $0 < i \leqslant 4$, where the only relation is

$$\alpha \smile \beta = (-1)^{\deg \alpha \deg \beta} \beta \smile \alpha.$$

When $G = SU(n)$ with $n \geqslant 3$, we also need to consider the slant product with respect to other characteristic classes of $\mathscr{U}$, and we obtain another copy of $\mathbb{A}(H_*(X, \mathbb{Q}))$ with a shifted degree:

deg	$\mathrm{Sym}(H_0(X))$	$\mathrm{Sym}(H_2(X))$	$\bigwedge^*(H_1(X))$	$\bigwedge^* H_3(X)$
$c_2(\mathscr{U})$	4	2	3	1
$c_3(\mathscr{U})$	6	4	5	3
...				

20.4. The topology of $\mathscr{B}_P^*$

Step 2. Finally, we return to $\mathscr{B}_P^*$ by studying the bundle:

$$\begin{tikzcd} G/Z(G) \arrow[r] & \mathscr{A}_P^* \times P/\mathscr{G}_P = \mathscr{U}^* \arrow[d] \\ & \mathscr{B}_P^* \times X. \end{tikzcd} \tag{31}$$

Claim 20.4.1. *$\mathscr{A}_P^*$ is weakly contractible.*

The point is that $\mathscr{A} - \mathscr{A}^* \subset \mathscr{A}$ has infinite co-dimension. Suppose $H \subset G$ is a subgroup, and P is induced from a principal H-bundle Q. Since $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{n}$ as a representation of H, we have

$$\operatorname{ad} P = Q \times_{ad_G \circ \iota} \mathfrak{g} = \operatorname{ad} Q \oplus V$$

for a vector bundle V. Then

$$\Omega^1(X, \operatorname{ad} P) = \Omega^1(X, \operatorname{ad} Q) \oplus \Omega^1(X, V).$$

A tangent vector in $\Omega^1(X, \operatorname{ad} Q)$ tends to preserve the reducibility, while a vector in $\Omega^1(X, V)$ tends to make the connection less reducible, i.e. the stabilizer becomes a smaller subgroup of G. When $G = SU(2)$, it just becomes irreducible. Hence,

$$\pi_n(\mathscr{A}^*) \cong \pi_n(\mathscr{A}) = 0.$$

This shows $\mathscr{A}^*$ is a classifying space of $\mathscr{G}/Z(G)$. We repeat the computation as above. Consider the classifying space $B(G/Z(G))$ and its rational cohomology:

$$H^*(B(G/Z(G)), \mathbb{Q}) \cong H^*(BG, \mathbb{Q}) = \operatorname{Sym}(\mathfrak{g}).$$

where $\operatorname{Sym}(\mathfrak{g})$ is the invariant sub-algebra of the polynomial algebra generated by $\mathfrak{t}$, under the action of the Weyl group. Here, $\mathfrak{t}$ is a Cartan sub-algebra of $\mathfrak{g}$. Note that this isomorphism is only valid over $\mathbb{Q}$. Consider

$$SU(2) \text{ v.s } SO(3).$$

While $H^4(BSU(2), \mathbb{Z})$ is generated by c_2, $H^4(BSO(3), \mathbb{Z})$ is generated by p_1. The two-fold cover $SU(2) \to SO(3)$ induces a map

$$\iota : BSU(2) \to BSO(3),$$

and $\iota^* p_1 = -4c_2$. Using the universal bundle (31), we define

$$\begin{aligned} H_*(X) &\to H^{4-*}(\mathscr{B}^*) \\ a &\mapsto -\frac{1}{4} p_1(\mathscr{U}^*)/a. \end{aligned}$$

We will continue the computation in the next lecture.

Exercise 20.4.2. *Explain the homotopy equivalence* (30).

Proof. In [5], there is no proof or reference provided for the homotopy equivalence (30), so we decide to sketch a proof. The transcriber would like to thank Greg Parker for helpful discussion.

To start, note that these spaces have the same homotopy groups:

$$\begin{aligned}\pi_i([X, K(\mathbb{Z}, 2n)]^*) &= [S^i, [X, K(\mathbb{Z}, 2n)]^*]_h^* \\ &= [S^i \wedge X, K(\mathbb{Z}, 2n)]_h^* \\ &= H^{2n}(S^i \wedge X, \mathbb{Z}) \\ &= H^{2n-i}(X, \mathbb{Z}).\end{aligned}$$

The issue is to find a map that induces isomorphisms on homotopy groups:

$$f = \prod_{i=0}^{\dim X} f_i : [X, K(\mathbb{Z}, 2n)]^* \to \prod_{i=0}^{\dim X} K(H^{2n-i}(X, \mathbb{Z}), i).$$

It is determined by each component:

$$f_i : [X, K(\mathbb{Z}, 2n)]^* \to K(H^{2n-i}(X, \mathbb{Z}), i).$$

Equivalently, it is an element in

$$H^i([X, K(\mathbb{Z}, 2n)]^*, H^{2n-i}(X, \mathbb{Z})).$$

This group fits into a short exact sequence (write $\mathcal{X} = [X, K(\mathbb{Z}, 2n)]^*$ for short):

$$\begin{aligned}0 \to \operatorname{Ext}(H_{i-1}(\mathcal{X}, \mathbb{Z}), H^{2n-i}(X, \mathbb{Z})) \to H^i(\mathcal{X}, H^{2n-i}(X, \mathbb{Z})) \\ \xrightarrow{\pi} \operatorname{Hom}(H_i(\mathcal{X}, \mathbb{Z}), H^{2n-i}(X, \mathbb{Z})) \to 0.\end{aligned}$$

It is not clear to us whether there is a canonical element in the group

$$H^i(\mathcal{X}, H^{2n-i}(X, \mathbb{Z})),$$

but we do have a canonical element in $\operatorname{Im} \pi$. Consider the evaluation map:

$$ev : [X, K(\mathbb{Z}, 2n)] \times X \to K(\mathbb{Z}, 2n).$$

Let u_{2n} be the canonical element in

$$H^{2n}(K(\mathbb{Z}, 2n), \mathbb{Z}).$$

Then the slant product defines map:

$$ev^*(u_{2n})/ : H_i(\mathcal{X}, \mathbb{Z}) \to H^{2n-i}(X, \mathbb{Z})$$

Denote this canonical element in $\operatorname{Hom}(H_i(\mathcal{X}, \mathbb{Z}), H^{2n-i}(X, \mathbb{Z}))$ by α. Then this exercise follows from a lemma. $\square$

Lemma 20.4.3. *If f_i is any lift of α in $H^i(\mathcal{X}, H^{2n-i}(X, \mathbb{Z}))$, then*

$$f_i : \mathcal{X} \to K(H^{2n-i}(X, \mathbb{Z}), i)$$

induces an isomorphism on π_i and the zero-map on π_k if $k \neq i$.

Proof of lemma. The statement is clear if $k \neq i$, since homotopy groups of

$$K(H^{2n-i}(X, \mathbb{Z}), i)$$

vanish in these dimensions. Take $\beta : (S^i, s_0) \to (\mathcal{X}, \bar{b})$ to be a based map. Let b be the base point of $K(\mathbb{Z}, 2n)$. Then the base point $\bar{b}$ of $\mathcal{X}$ is the constant map:

$$\bar{b}(x) = b, \forall x \in X.$$

Consider the following diagrams:

$$\begin{tikzcd}
S^i \times X \arrow[r, "\beta \times \mathrm{Id}"] \arrow[d, "\pi"] & {[X, K(\mathbb{Z}, 2n)]^* \times X} \arrow[r, "ev"] & K(\mathbb{Z}, 2n) \\
S^i \wedge X. \arrow[urr, "\tilde{\beta}"'] & &
\end{tikzcd} \tag{32}$$

The composition $ev \circ (\mathrm{Id} \times \beta)$ factorizes through $S^i \wedge X$, because $S^i \vee X$ is sent to the base point b. The map $\tilde{\beta}$ corresponds to β under the adjunction:

$$[S^i, \mathcal{X}]^* \cong [S^i \wedge X, K(\mathbb{Z}, 2n)]^*.$$

Hence, the homotopy class of $\tilde{\beta}$ is the image of $\phi[\beta]$ in the diagram:

$$\begin{tikzcd}
 & \phi[\beta] \in H^{2n}(S^i \wedge X, \mathbb{Z}) \arrow[dr, bend left, "\cong ?"] & \\
{[S^i] \in \pi_i(S^i)} \arrow[r, "\beta_*"] \arrow[d, "Hur"] & \pi_i(\mathcal{X}) \arrow[r, "(f_i)_*"] \arrow[u, "\cong"', "\phi"] \arrow[d, "Hur"] & \pi_i(K(H^{2n-i}(X, \mathbb{Z}), i)) \arrow[d, "\cong"', "Hur"] \\
{[S^i] \in H_i(S^i)} \arrow[r, "\beta_*"] & H_i(\mathcal{X}, \mathbb{Z}) \arrow[r, "(f_i)_*"] \arrow[dr, "\alpha"] & H_i(K(H^{2n-i}(X, \mathbb{Z}), i), \mathbb{Z}) \arrow[d, "\cong"', "\langle u_{2n}, \cdot \rangle"] \\
 & & H^{2n-i}(X, \mathbb{Z})
\end{tikzcd}$$

Here, Hur denotes the Hurewicz map. The goal is to show $(f_i)_* \circ \phi^{-1}$ is an isomorphism. Note that

$$Hur \circ (f_i)_*[\beta] = Hur \circ (f_i)_* \circ \beta_*[S^i] = (f_i)_* \circ \beta_*[S^i]$$

and $\langle u_{2n}, (f_i)_* \circ \beta_*[S^i] \rangle = \alpha \circ \beta_*[S^i]$. In terms of diagram (32),

$$\begin{aligned}
\alpha \circ \beta_*[S^i] = ev^*(u_{2n})/\beta_*[S^i] &= (\beta \times \mathrm{Id})^* \circ ev^*(u_{2n})/[S^i] \\
&= \pi^* \circ \tilde{\beta}^*(u_{2n})/[S^i] \\
&= \pi^*(\phi[\beta])/[S^i].
\end{aligned}$$

Finally, it suffices to show the map

$$H^{2n}(S^i \wedge X, \mathbb{Z}) \to H^{2n-i}(X, \mathbb{Z})$$
$$\gamma \mapsto \pi^*[\gamma]/[S^i]$$

is the desuspension map, where $\pi : S^i \times X \to S^i \wedge X$ is the projection map. Either we can take it as the definition, or it is an exercise of algebraic topology. $\square$

Lecture 21. The Topology of Configuration Space II

21.1. The algebraic topology of $\mathscr{B}^*$

Recall from the last lecture that the rational cohomology of $\mathscr{B}^*_P$ is given by

$$H^*(\mathscr{B}^*_P, \mathbb{Q}) \cong \underbrace{\mathbb{A} \otimes \cdots \otimes \mathbb{A}}_{n-1 \text{ copies}} \tag{33}$$

if P is a principle $SU(n)$-bundle with fixed Chern classes:

$$c_i \in H^{2i}(X, \mathbb{Z}),\ 2 \leqslant i \leqslant n.$$

Here, the algebra $\mathbb{A}(X)$ is defined as

$$\mathbb{A}(X) = \mathrm{Sym}(H_0(X)) \otimes \mathrm{Sym}(H_2(X)) \otimes \bigwedge\nolimits^*(H_1(X) \otimes H_3(X)).$$

with $H_i(X) = H_i(X, \mathbb{Q})$.

This isomorphism (33) is constructed by using the slant product:

$$\begin{aligned} \mu_i : H_*(X) &\to H^{2i-*}(\mathscr{B}^*_P),\ * \neq 4 \\ a &\mapsto \bar{c}_i(\mathscr{U}^*)/a. \end{aligned}$$

Let us explain the definition of $\bar{c}_i$. For the based configuration space $\mathscr{B}^o = \mathscr{A}/\mathscr{G}^o$, we have a fiber bundle:

$$\begin{array}{ccc} SU(n) & \longrightarrow & \mathscr{U} = \mathscr{A} \times P/\mathscr{G}^o \\ & & \downarrow \\ & & \mathscr{B}^o \times X, \end{array}$$

while for the irreducible configuration space $\mathscr{B}^* = \mathscr{A}^*/\mathscr{G}$, we have

$$\begin{array}{ccc} PSU(n) & \longrightarrow & \mathscr{U}^* = \mathscr{A}^* \times P/\mathscr{G} \\ & & \downarrow \\ & & \mathscr{B}^* \times X, \end{array}$$

Rationally, we have an isomorphism

$$\pi^* : H^*(BPSU(n), \mathbb{Q}) \to H^*(BSU(n), \mathbb{Q}) \cong \mathbb{Q}[c_2, \cdots, c_n]$$

induced by the projection map $\pi : SU(n) \to PSU(n)$. Then we define $\bar{c}_i$ as $(\pi^*)^{-1}(c_i)$ for $2 \leqslant i \leqslant n$, so

$$H^*(BPSU(n), \mathbb{Q}) \cong \mathbb{Q}[\bar{c}_2, \cdots, \bar{c}_n].$$

We address the orientability of the determinant line bundle $\lambda \to \mathscr{B}^*_P$ by studying $\pi_1(\mathscr{B}^*_P)$. The answer is already interesting when $G = SU(2)$. By the fibration,

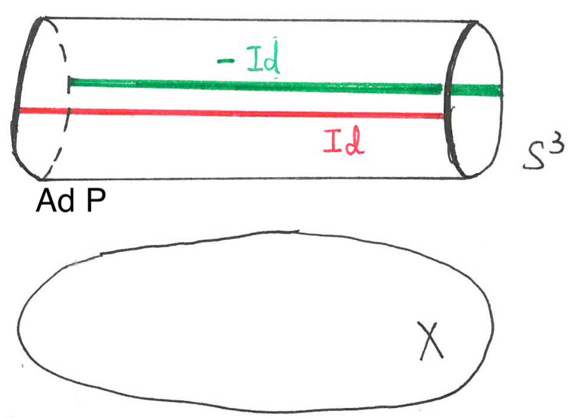
$$\begin{array}{ccc} \mathscr{G}/Z(G) & \longrightarrow & \mathscr{A}^* \\ & & \downarrow \\ & & \mathscr{B}^*, \end{array}$$

we have an isomorphism $\pi_1(\mathscr{B}^*_P) \cong \pi_0(\mathscr{G}/Z(G))$. We need to understand the component group of $\mathscr{G}/Z(G)$. Let us start with $\pi_0(\mathscr{G})$.

Remark. We need to compute $H^1(\mathscr{B}^*_P, \mathbb{Z}_2)$ which classifies real line bundles over $\mathscr{B}^*_P$. We computed $H^1(\mathscr{B}^*_P, \mathbb{Q}) \cong H_3(X, \mathbb{Q})$, but this is not enough.

21.2. The component group $\pi_0(\mathscr{G})$

The adjoint bundle $\operatorname{Ad} P = P \times_{Ad} SU(2)$ has a pair of distinguished sections $\{\pm \operatorname{Id}\}$.



Let $s : X \to \operatorname{Ad} P$ be a generic section, then $\gamma = s^{-1}(-\operatorname{Id})$ is a smooth 1-manifold inside X. It is clear that if $\gamma = \varnothing$, then s is homotopic to the constant section Id. The vertical tangent bundle at $-\operatorname{Id} \subset \operatorname{Ad} P$ is

$$N(-\operatorname{Id}) = VT \operatorname{Ad} P|_{-\operatorname{Id}} \cong \operatorname{ad} P.$$

Then the differential $ds|_\gamma$ identifies

$$N(\gamma)\ (\subset X) \cong \operatorname{ad} P|_\gamma.$$

By choosing an orientation of $\mathfrak{g}$, we obtain an orientation of ad P and a framing $\mathfrak{f}$ of γ, i.e. a trivialization of the normal bundle $N(\gamma)$ inside X. In particular, γ is oriented.

Suppose P is trivial, then we are trying to classify homotopy classes of maps:

$$[X, S^3]_h.$$

If we look for a homotopy between s and the constant section Id, the primary obstruction lies in $H^3(X, \pi_3(S^3))$ which is simply

$$s^*[u_3]$$

where $u_3 \in H^3(S^3, \mathbb{Z})$ is the fundamental class of S^3. Then γ is the Poincaré dual of $s^*[u_3]$:

$$\gamma = PD[s^*(u_3)] \in H_1(X, \mathbb{Z}).$$

If $[\gamma] = 0$, or equivalently, $s^*u_3 = 0 \in H^3(X, \pi_3(S^3))$, we have to deal with the secondary obstruction which lies in

$$H^4(X, \pi_4(S^3)) \cong H^4(X, \mathbb{Z}_2).$$

Let us understand this obstruction class more carefully. The discussion below is intended to mimic the Pontryagin-Thom construction.

Remark. The Pontryagin-Thom construction gives a bijection between

$$[X, S^3]_h \cong \Omega^1_{frame}(X)$$

where $\Omega^1_{frame}(X)$ is the framed cobordism group of framed 1-submanifolds in X. When $s = \mathrm{Id}$, the frame sub-manifold is an empty set.

The homology class $[\gamma] \in H_1(X, \mathbb{Z})$ is the primary obstruction class. If $[\gamma] = 0$, then γ is null-bordant inside X, i.e. $\gamma = \partial\Sigma$ for some oriented embedded surface $\Sigma \subset X \times I$. To exhibit a homotopy between s and Id, it is enough to see that

$$(\gamma, \mathfrak{f})$$

is null-bordant as a framed sub-manifold in X. In other words, the normal bundle $N\Sigma \to \Sigma$ is a real 3-plane bundle:

$$\begin{tikzcd} \mathbb{R}^3 \arrow[r] & N\Sigma \arrow[d] \\ & \Sigma. \end{tikzcd}$$

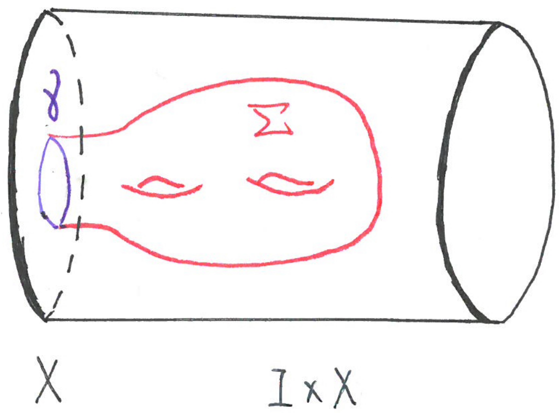
The question is:

- Can we extend the framing $\mathfrak{f}$ of $N\Sigma$ at the boundary $\gamma = \partial\Sigma$ to the whole surface Σ?

We need **an isomorphism of $N\Sigma \xrightarrow{\cong} \operatorname{ad} P|_\Sigma$ agreeing with a given one on the boundary** $\gamma = \partial\Sigma$. (Σ deformation retracts to an one-simplex, so both bundles are trivial over Σ).

A potential obstruction of this extension comes from the framing of $N(\gamma)$:

$$\begin{tikzcd} N(\gamma) \arrow[rr] \arrow[dr] & & \operatorname{ad} P|_\gamma \arrow[dl] \\ & \gamma. & \end{tikzcd}$$



which defines an element in

$$H^2(\Sigma, \partial\Sigma; \pi_1(SO(3))) \cong \mathbb{Z}_2. \tag{34}$$

Suppose $\Sigma^2 \subset X^4 \times I$ is closed and embedded. When is the normal bundle $N\Sigma$ trivial? Since

$$H^1(BSO(3), \mathbb{Z}) = \{0\}, H^2(BSO(3), \mathbb{Z}) = \mathbb{Z}_2,$$

the primary obstruction is the second Stiefel-Whitney class

$$w_2(N\Sigma) \in H^2(\Sigma, \pi_1(SO(3))).$$

Since $T\Sigma \oplus N\Sigma = TX \oplus \mathbb{R}|_\Sigma$ and $w_2(T\Sigma) = 0$, $w_2(N\Sigma) = w_2(TX|_\Sigma)$. If we further project Σ into X ($\Sigma \hookrightarrow X \times I \xrightarrow{pr_1} X$), then

$$\begin{aligned} w_2(TX|_\Sigma)[\Sigma] = w_2(N_X\Sigma)[\Sigma] &= e(N_X\Sigma)[\Sigma] \mod 2 \\ &= \Sigma \cdot \Sigma \mod 2. \end{aligned}$$

where $N_X\Sigma$ is the normal bundle of Σ in X.

If $w_2(X) = 0$, then the intersection form q_X is even, i.e.

$$q_X([\Sigma]) \equiv 0 \mod 2$$

for any $[\Sigma] \in H_2(X, \mathbb{Z})$. This direction does not require X to be simply connected. If $H^1(X, \mathbb{Z}) = 0$, then the other direction is also true, i.e. having an even intersection form q_X implies $w_2(X) = 0$.

Recall Rokhlin's theorem:

Theorem 21.2.1. *If X is a spin 4-manifold, then $\sigma(X) \equiv 0 \mod 16$.*

One way to think about this theorem is via the index theorem: in dimension 4,

$$\operatorname{Ind}_{\mathbb{C}} D := \dim_{\mathbb{C}}(\ker D) - \dim_{\mathbb{C}}(\operatorname{Coker} D) = -\frac{1}{8}\sigma(X),$$

where D is the Dirac operator associated to a spin manifold X. In fact, when dim $= 8k+4$, the Dirac operator is $\mathbb{H}$-linear, so

$$\operatorname{Ind}_{\mathbb{C}} D \equiv 0 \mod 2.$$

For instance, let $K^3 = \{x^4 + y^4 + z^4 + w^4 = 0\} \subset \mathbb{CP}^3$ be the K3 surface. Then $w_2(K^3) = 0$, $\sigma(K^3) = -16$. Its intersection form is given by

$$2(-E_8) \oplus 3 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

On the other hand, K^3 admits a fix-point-free involution τ. Then the intersection form of the quotient space $E = K^3/\tau$ is

$$(-E_8) \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

which is even. Moreover, $\sigma(E) = -8$, so E is not spin by Rokhlin's theorem. This doesn't violate our statement above, because $H_1(E, \mathbb{Z}) = \pi_1(E) = \mathbb{Z}_2$.

Back to our problem. In conclusion, assuming $H_1(X, \mathbb{Z}) = 0$, we have

$$\pi_0(\mathrm{Map}(X^4, S^3)) = \begin{cases} 0 & w_2(X) \neq 0 \\ \mathbb{Z}_2 & w_2(X) = 0. \end{cases}$$

We can cancel the obstruction class in (34) to extend a framing if and only if there exists an closed oriented surface $\Sigma' \subset X$ such that

$$w_2(TX|_{\Sigma'})[\Sigma'] \neq 0.$$

In the general case, if $H_1(X, \mathbb{Z}) = 0$, then

$$\pi_0(\mathscr{G}) = \begin{cases} 0 & w_2(X) \neq w_2(\operatorname{ad} P) \\ \mathbb{Z}_2 & w_2(X) = w_2(\operatorname{ad} P). \end{cases}$$

Indeed, we can find a surface $(\Sigma, \partial\Sigma = \gamma) \subset (I \times X, \{0\} \times X)$ with a bundle isomorphism:

$$\begin{tikzcd} N(\Sigma) \arrow[rr] \arrow[dr] & & \operatorname{ad} P|_\Sigma \arrow[dl] \\ & \Sigma & \end{tikzcd}$$

that extends a given isomorphism on the boundary:

$$\begin{tikzcd} N(\gamma) \arrow[rr] \arrow[dr] & & \operatorname{ad} P|_\gamma \arrow[dl] \\ & \gamma & \end{tikzcd}$$

if and only if there exists a closed oriented surface Σ' in X such that

$$w_2(TX|_{\Sigma'})[\Sigma'] \neq w_2(\operatorname{ad} P|_{\Sigma'})[\Sigma'].$$

To study $\pi_0(\mathscr{G}/Z(G))$, it remains to understand the map $Z(G) \to \pi_0(\mathscr{G})$. In particular, we focus on the case when $G = SU(2)$ and $w_2(TX) = w_2(\operatorname{ad} P)$,

- How does $Z(SU(2)) = \{\pm 1\}$ map into $\pi_0(\mathscr{G})$? Or equivalently,
- Is $-\operatorname{Id}$ homotopic to Id in $\mathscr{G}$?

In the reducible case, $P = L \oplus L^*$, then $\operatorname{ad} P = \mathbb{R} \oplus L^{\otimes 2}$. Then

$$\operatorname{Ad} P = \mathbb{S}(\operatorname{ad} P \oplus \mathbb{R}) = \mathbb{S}(L^{\otimes 2} \oplus \mathbb{R} \oplus \mathbb{R}).$$

where $\mathbb{S}$ denotes the unit sphere bundle.

(To be continued.)

Lecture 22. Orienting the Moduli Space

22.1. Orientability

Recall from the last lecture that the component group of $\mathscr{G}$ is

$$\pi_0(\mathscr{G}) = \begin{cases} 0 & w_2(X) \neq w_2(\operatorname{ad} P) \\ \mathbb{Z}_2 & w_2(X) = w_2(\operatorname{ad} P). \end{cases}$$

given that $H_1(X, \mathbb{Z}) = 0$. Since the $SO(3)$-bundle $\operatorname{ad} P$ is induced from an $SU(2)$-bundle, it is spin, so $w_2(\operatorname{ad} P) \equiv 0$. Moreover, $w_2(X) = 0$ if and only if the intersection form q_X of X is even. Hence,

$$\pi_0(\mathscr{G}) = \begin{cases} 0 & \text{if } q_X \text{ is odd (not even).} \\ \mathbb{Z}_2 & \text{if } q_X \text{ is even.} \end{cases}$$

In the first case (assuming $\pi_1(X) = \{0\}$), $\pi_0(\mathscr{G}/Z(G)) = \{0\}$ and the determinant line bundle $\lambda \to \mathscr{B}^*$ has to be trivial. In the second case, we take the connect sum $X \# \overline{\mathbb{CP}}^2$ to make the intersection form odd. In either case, we obtain Donaldson's diagonalization theorem.

Also, we learned from the last lecture that

$$Z(G) \to \pi_0(\mathscr{G})$$

is the zero map if and only if $\operatorname{ad} P$ splits as $\mathbb{R} \oplus L$, so

$$\pi_0(\mathscr{G}/Z(G)) = \begin{cases} 0 & \text{otherwise} \\ \mathbb{Z}_2 & \text{If } q_X \text{ is even and } c_2(P) = -c_1^2(\tilde{L}) \\ & \text{for some } c_1(\tilde{L}) \in H^2(X, \mathbb{Z}). \end{cases}$$

For other purpose, it is important to know whether the moduli space $\mathcal{M}_k(X)$ is orientable even if $\pi_0(\mathscr{G}/Z(G)) = \mathbb{Z}_2$ or $H_1(X, \mathbb{Z}) \neq 0$.

Theorem 22.1.1 (Donaldson). *The real line bundle* $\det(d^+ \oplus d^*) \to \mathscr{B}^*$ *is always trivial for* $SU(2)$*-bundles.*

We do not assume X is simply connected in this theorem.

22.2. Passing to $SU(3)$-bundles

In some cases $\pi_0(\mathscr{G}) \cong \mathbb{Z}_2$. This $\mathbb{Z}_2$ factor comes from

$$\pi_4(SU(2)) \cong \pi_4(S^3) \cong \mathbb{Z}_2.$$

by obstruction theory. However, when $n > 2$,

$$\pi_4(SU(3)) \cong 0,$$

so it is not a problem any more.

Remark. This is equivalent to the fact that the principal bundle

$$\begin{array}{ccc} SU(2) & \longrightarrow & SU(3) \\ & & \downarrow \\ & & S^5 \longrightarrow BSU(2) \end{array} \tag{35}$$

is not trivial. Consider the long exact sequence of homotopy groups:

$$\pi_5(S^5) \xrightarrow{j} \pi_5(BSU(2)) \cong \mathbb{Z}_2 \to \pi_4(SU(3)) \to 0.$$

The image of $j([1])$ classifies this bundle. □

The first step is to extend the structure group to $SU(3)$ and see how the determinant line bundle change. We embed $SU(2)$ into $SU(3)$ as

$$\left(\begin{array}{c|c} SU(2) & 0 \\ \hline 0 & 1 \end{array}\right).$$

Then the restriction of the adjoint representation $\mathrm{ad}_{SU(3)}|_{SU(2)}$ splits as

$$(3+4+1) = \mathfrak{su}(2) \oplus \mathbb{C}^2 \oplus \mathbb{R} = \left(\begin{array}{c|c} \mathfrak{su}(2) + c \cdot I_2 & \mathbb{C}^2 \\ \hline * & -2c \end{array}\right).$$

where $\mathbb{C}^2$ is the fundamental representation of $SU(2)$. Let $P' = P \times_{SU(2)} SU(3)$ be the $SU(3)$ bundle. Then the space of $SU(2)$ connections $\mathscr{A}_{SU(2)}$ becomes a subspace of $\mathscr{A}_{SU(3)}$. For $A \in \mathscr{A}_{SU(2)}$, consider the Fredholm operator:

$$d_A^+ \oplus d_A^* : \Omega^1(X, \mathrm{ad}_{SU(3)} P') \to \Omega^0(X, \mathrm{ad}_{SU(3)} P') \oplus \Omega^2_+(X, \mathrm{ad}_{SU(3)} P').$$

It splits as a direct sum of:

$$\begin{aligned} \Omega^1(X, \mathrm{ad}_{SU(2)} P) &\to \Omega^0(X, \mathrm{ad}_{SU(2)} P) \oplus \Omega^2_+(X, \mathrm{ad}_{SU(2)} P), \\ \Omega^1(X, E) &\to \Omega^0(X, E) \oplus \Omega^2_+(X, E), \\ \Omega^1(X, \mathbb{R}) &\to \Omega^0(X, \mathbb{R}) \oplus \Omega^2_+(X, \mathbb{R}). \end{aligned}$$

where E is the canonical 2-plane bundle associated to the fundamental representation of $SU(2)$. Note that the second operator is complex linear, so we have canonical orientations for the kernel and cokernel as complex vector spaces. The last operator is independent of A. In all, we have

$$\lambda_{SU(3)}|_{\mathscr{A}^*_{SU(2)}} \cong \lambda_{SU(2)}.$$

Therefore, it suffices to show the bundle

$$\lambda_{SU(3)} \to \mathscr{B}^*_{SU(3)}$$

is trivial.

Remark. Technically, $\mathscr{B}^*_{SU(3)}$ consists of irreducible $SU(3)$-connections and does not contain the image of $\mathscr{A}_{SU(2)}$. Since $\mathscr{A}$ or $\mathscr{A}^*$ is weakly contractible, the determine line bundle

$$\lambda \to \mathscr{A}_{SU(2)} \text{ or } \mathscr{A}_{SU(3)}$$

is always orientable. The problem is whether the monodromy

$$\pi_0(\mathscr{G}/Z(G)) \cong \pi_1(\mathscr{B}^*) \to \mathrm{Aut}(\lambda)$$

is non-trivial. It has nothing to do with the size of stabilizers. □

Compute $\pi_0(\mathscr{G}_{SU(3)})$. This is equivalent to the classification of bundles on $S^1 \times X$ which restricts to a given bundle on $\{s_0\} \times X$. We can analyze the problem via obstruction theory. Suppose we have a map $f : X \to BSU(n)$ ($n \geqslant 3$) which extends trivially to

$$S^1 \times X \to BSU(n).$$

One can ask if this extension is unique. Since

$$\pi_i(BSU(n)) = \begin{cases} 0 & 0 \leqslant i \leqslant 3 \\ \mathbb{Z} & i = 4 \\ 0 & i = 5 \ (n \geqslant 3). \end{cases}$$

For two extensions, the first obstruction to find a homotopy between them lies in

$$H^4(S^1 \times X, \{s_0\} \times X; \pi_4(BSU(n))) \cong H^3(X, \mathbb{Z}) \cong H_1(X, \mathbb{Z}).$$

The second obstruction is always zero:

$$H^5(S^1 \times X, \{s_0\} \times X; \pi_5(BSU(n))) \cong \{0\}.$$

Therefore, $\pi_0(\mathscr{G}) \cong H_1(X, \mathbb{Z})$. When $\pi_1(X) = \{0\}$, the component group is trivial. However, when $\pi_1(X) \neq \{0\}$, we need to work harder.

22.3. The grafting construction

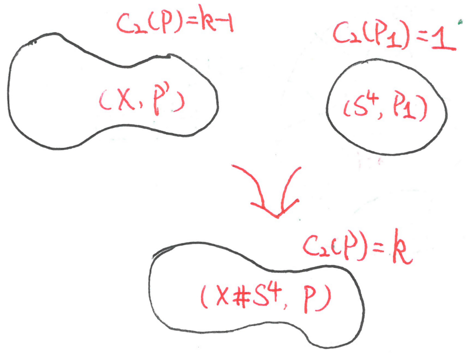
Let us describe a concrete realization of $\pi_0(\mathscr{G})$, by the grafting construction. Suppose $c_2(P) = k$ and $c_2(P') = k-1$. Let $P_1 \to S^4$ is the $SU(2)$-bundle with $c_2(P_1) = 1$. By taking the connected sum, we can obtain P from P' and P_1:

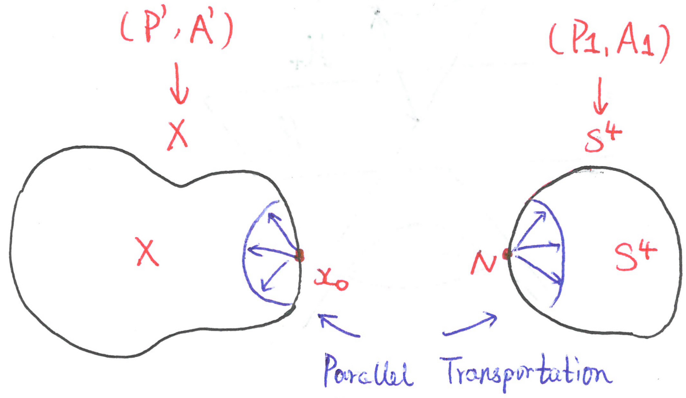
$$P \cong P' \# P_1 \to X \# S^4.$$

We can paste connections as well. Let (P', A') and (P_1, A_1) be connections on X and S^4 respectively. Take $x_0 \in X$ and the north pole $N \in S^4$ to be base points. Use the connection A' to trivialize the bundle $P' \to X$ near x_0 via parallel transportation along radial geodesics. Do the same thing near N.

We only need an identification of fibers

$$P'|_{x_0} \cong P_1|_N.$$





to get a bundle on $X\#S^4$. To paste connections, we interpolate connection 1-forms by cut-off functions:

$$A'|_{B_{x_0}} \cong \Gamma' + a', A_1|_{B_N} \cong \Gamma + a.$$

where B_{x_0} and B_N are some fixed small geodesic balls around $x_0 \in X$ and $N \in S^4$. Denote the outcome by $A'\#A_1$. This construction descends to the gauge equivalent classes, i.e.

$$A'\#A_1 \sim_{g'\#g_1} (g' \cdot A')\#(g_1 \cdot A_1),$$

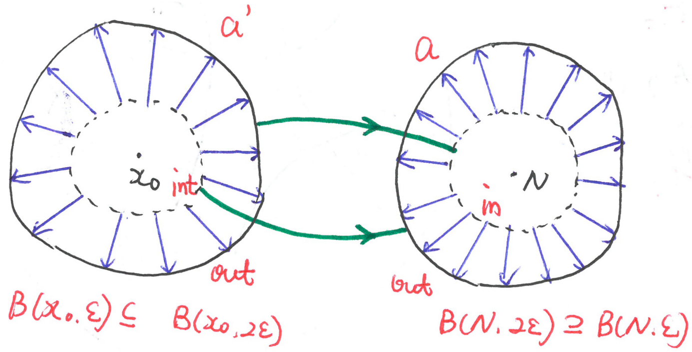
where $g'\#g_1$ is the connected sum of two gauge transformations. Note that the parallel transportation is equivariant under gauge transformations.

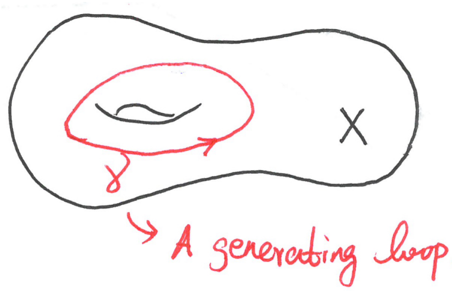
Now take a generating loop $\gamma \subset X$ such that $[\gamma] \neq 0 \in H_1(X, \mathbb{Z}) \cong \pi_0(\mathscr{G}) = \pi_1(\mathscr{A}^*/\mathscr{G})$. The idea is to do the grafting construction along the loop γ: keep $N \in S^4$ fixed and let x_0 travel along $\gamma \subset X$, so

$$\text{A loop } \gamma \Rightarrow \text{A loop } \gamma^* \in \mathscr{A}^*/\mathscr{G}.$$

How does the determinant line bundle looks like along this family γ^* of connections? We can understand this using the excision theorem for elliptic operators. Recall the setup from Lecture 18:

(1) $X_1 = U_1 \cup V_1$ and $X_2 = U_2 \cup V_2$.





(2) $W_1 = U_1 \cap V_1$ is diffeomorphic to $W_2 = U_2 \cap V_2$.
(3) We have elliptic differential operators:

$$D_1 : \Gamma(E_1) \to \Gamma(F_1) \text{ on } X_1$$
$$D_2 : \Gamma(E_2) \to \Gamma(F_2) \text{ on } X_2.$$

(4) There exist bundle isomorphisms

$$\phi : E_1|_{W_1} \to E_2|_{W_2}, \psi : F_1|_{W_1} \to F_2|_{W_2}$$

that cover the diffeomorphism in (2) and intertwine D_1 and D_2:

$$D_2 = \psi \circ D_1 \circ \phi^{-1} \text{ on } W_2.$$

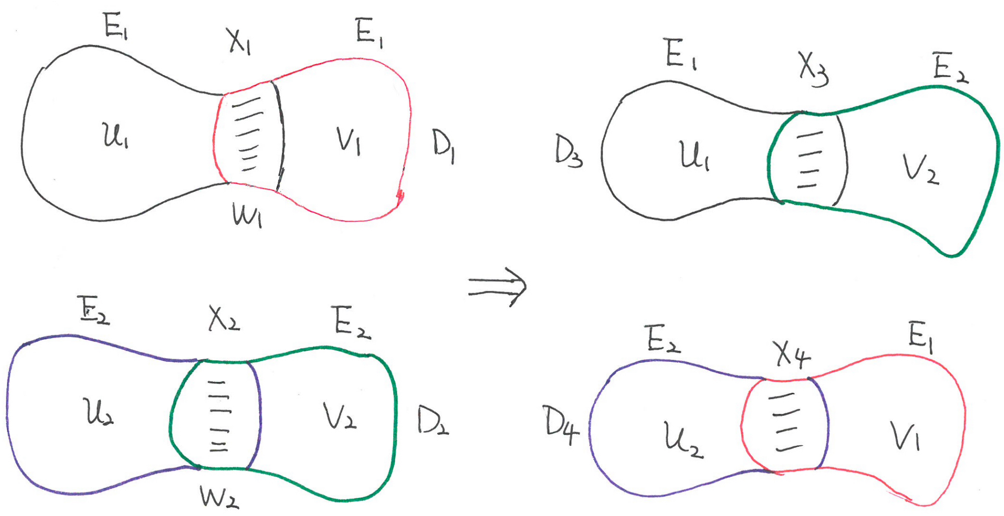
Under these assumptions, we can

(1) Define $X_3 = U_1 \cup V_2$ and $X_4 = U_2 \cup V_1$.
(2) Obtain $E_3 \to X_3$ by gluing $E_1|_{U_1}$ and $E_2|_{V_2}$.
(3) Obtain $E_4 \to X_4$ by gluing $E_2|_{U_2}$ and $E_1|_{V_1}$. Similarly, define $F_3 \to X_3$ and $F_4 \to X_4$.
(4) Define elliptic differential operators:

$$D_3 : \Gamma(E_3) \to \Gamma(F_3), D_4 : \Gamma(E_4) \to \Gamma(F_4).$$

The conclusion is that

$$\operatorname{Ind} D_1 + \operatorname{Ind} D_2 = \operatorname{Ind} D_3 + \operatorname{Ind} D_4.$$



The same construction can be carried out for a family of operators:

$$D_1^t : \Gamma(E_1) \to \Gamma(F_1) \text{ on } X_1$$

for $t \in T$, while D_2 is fixed. Here, T is a compact parameter space. We require that D_1^t is identified canonically on $V_1 \subset X_1$. In this case, the index defines an element in the K-theory

$$K^0(T).$$

The first Stiefel-Whitney class w_1 defines a group homomorphism that detects orientability:

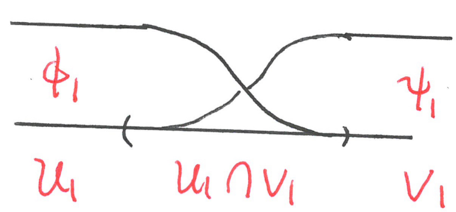
$$w_1 : K^0(T) \to H^1(T, \mathbb{Z}_2).$$

22.4. An Easy Proof of the Excision Theorem

Most proofs of the excision theorem rely on the index theorem, but it is silly. The theorem is just right. Choose a partition of unity on (ϕ_i, ψ_i) on X_i $(i = 1, 2)$ such that

$$\phi_i^2 + \psi_i^2 = 1.$$

and $\phi_1 = \phi_2, \psi_1 = \psi_2$ on the overlap $W_1 \cong W_2$.



Consider bundle maps:

$$\Gamma(E_1)\oplus\Gamma(E_2)\xrightarrow{\alpha}\Gamma(E_3)\oplus\Gamma(E_4)$$
$$\Gamma(F_1)\oplus\Gamma(F_2)\xleftarrow{\beta}\Gamma(F_3)\oplus\Gamma(F_4)$$

where

$$\alpha=\begin{pmatrix}\phi_1 & \psi_2\\ \psi_1 & \phi_2\end{pmatrix},\beta=\begin{pmatrix}\phi_1 & -\psi_1\\ \psi_2 & \phi_2\end{pmatrix}.$$

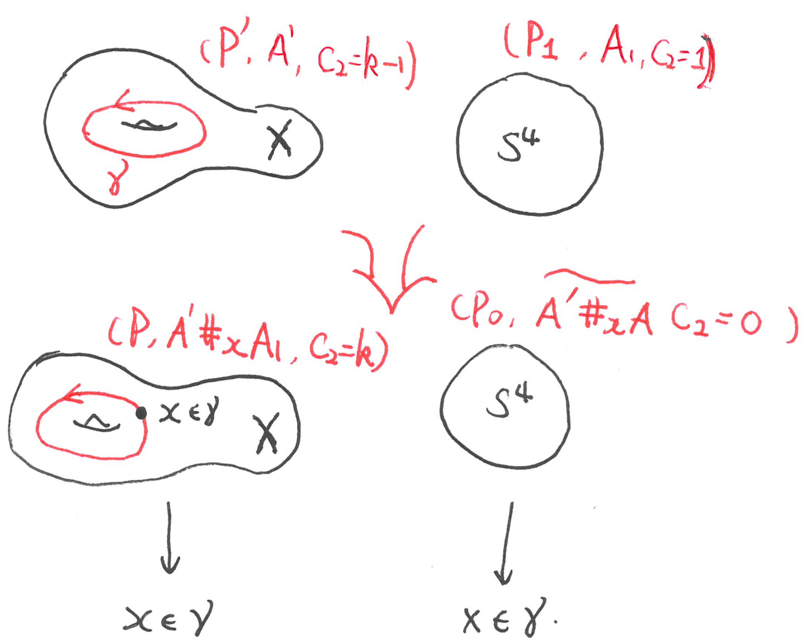
Formally, $\alpha\circ\beta=\beta\circ\alpha=\mathrm{Id}$. By direct computation,

$$\beta\circ\begin{pmatrix}D_3 & 0\\ 0 & D_4\end{pmatrix}\circ\alpha=\begin{pmatrix}D_1 & 0\\ 0 & D_2\end{pmatrix}+\text{ some error terms.}$$

Error terms involve commutators of D_i and multiplications of ϕ,ψ, which are zero-order operators, so indices add.

If we have a family of operators, then the same trick works.

To apply the excision theorem for families, let $X_1=X$ and $X_2=S^4$.



(To be continued.)

Exercise 22.4.1. *Verify that $\pi_4(SU(3))\cong\{0\}$.*

Proof. We prove that $SU(3)$ is not homotopy equivalent to $S^3\times S^5$, so the fiber bundle (1) is not trivial. These spaces have the same cohomology rings. However, using $\mathbb{Z}_2$-coefficient,

$$H^*(SU(3),\mathbb{Z}_2),H^*(S^3\times S^5,\mathbb{Z}_2)$$

are different as modules of the Steenrod algebra. Let u_3 and u_5 be the generators in dimension 3 and 5 respectively. Then

$$H^*(SU(3), \mathbb{Z}_2) \cong H^*(S^3 \times S^5, \mathbb{Z}_2) \cong \mathbb{Z}_2[u_3, u_5]/(u_3^2, u_5^2).$$

The difference is that

$$\operatorname{Sq}^2 u_3 = u_5 \in H^*(SU(3), \mathbb{Z}_2) \tag{36}$$

while $\operatorname{Sq}^2 u_3 = 0 \in H^*(S^3 \times S^5, \mathbb{Z}_2)$. To prove (36), we use the fact that the Steenrod operation Sq^2 is a stable operator, so it suffices to prove

$$\operatorname{Sq}^2 \Sigma u_3 = \Sigma u_5 \in H^6(\Sigma SU(3), \mathbb{Z}_2).$$

Consider the homotopy equivalence $SU(3) \cong \Omega BSU(3)$ and its adjunction

$$j : \Sigma SU(3) \to BSU(3).$$

Lemma 22.4.2. *Using either $\mathbb{Z}$ or $\mathbb{Z}_2$ coefficient, the pull-back map*

$$j^* : H^n(BSU(3)) \to H^n(\Sigma SU(3))$$

is always surjective. Moreover, it is an isomorphism when $0 \leqslant n \leqslant 7$.

Proof. Let us focus on $\mathbb{Z}_2$ coefficient. To prove surjectivity, note that

$$H^n(\Sigma SU(3), \mathbb{Z}_2) \cong [\Sigma SU(3), K(\mathbb{Z}_2, n)]_* \cong [SU(3), \Omega K(\mathbb{Z}_2, n)]_*.$$

Since $SU(3) \cong \Omega BSU(3)$, we have

$$\begin{tikzcd} \Omega BSU(3) \arrow[dr, dashed, "\tilde{f}"] & \\ SU(3) \arrow[u, "\cong"] \arrow[r, "f"] & \Omega K(\mathbb{Z}_2, n). \end{tikzcd}$$

Applying the adjunction again, we conclude that a map from $\Sigma SU(3)$ to $K(\mathbb{Z}_2, n)$ alway factorize through $j : \Sigma SU(3) \to BSU(3)$.

This map is also injective when $0 \leqslant n \leqslant 7$, because we can compute their dimensions in this range.

One can also prove this lemme using Serre's spectral sequence. ☐

This lemma implies that

$$j^*(c_2) = \Sigma u_3, j^*(c_3) = \Sigma u_5 \mod 2,$$

where c_2 and c_3 are the second and the third Chern classes. It remains to prove $Sq^2 c_2 = c_3$ and exploit the naturality of Steenrod operations:

$$\begin{tikzcd} H^6(BSU(3), \mathbb{Z}_2) \arrow[r, "j^*"] & H^6(\Sigma SU(3)) \\ H^4(BSU(3), \mathbb{Z}_2) \arrow[u, "\operatorname{Sq}^2"] \arrow[r, "j^*"] & H^4(\Sigma SU(3)). \arrow[u, "\operatorname{Sq}^2"] \end{tikzcd}$$

Consider the map $BSU(3) \to BU(3)$ and $i : BT^3 \to BU(3)$, where T^3 is the maximal torus of $U(3)$. We obtain maps on cohomology rings:

$$\begin{tikzcd} H^*(BU(3),\mathbb{Z}_2) \cong \mathbb{Z}_2[c_1,c_2,c_3] \arrow[r] \arrow[d, "i^*"] & H^*(BU(3),\mathbb{Z}_2)/\{c_1 = 0\} \arrow[d, equal] \\ H^*(BT^3,\mathbb{Z}_2) \cong \mathbb{Z}_2[x_1,x_2,x_3] & H^*(BSU(3),\mathbb{Z}_2). \end{tikzcd}$$

where $i^*c_2 = x_1x_2 + x_2x_3 + x_3x_1$ and $i^*c_3 = x_1x_2x_3$. Hence,

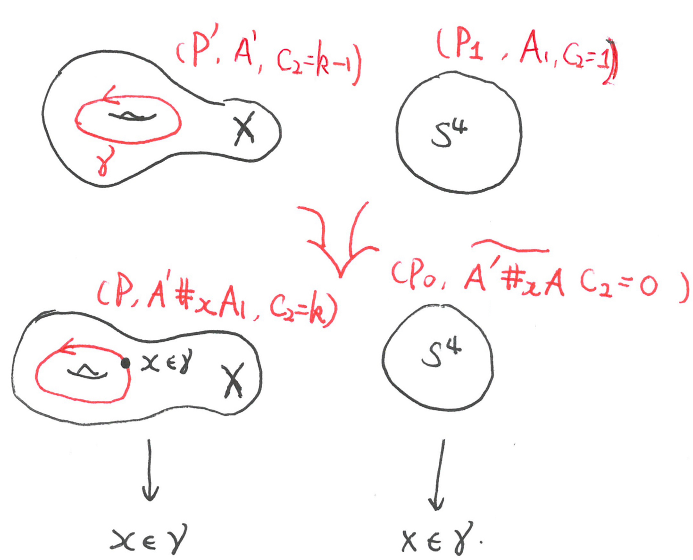
$$\begin{aligned} \operatorname{Sq}^2 c_2 &= \operatorname{Sq}^2(x_1x_2 + x_2x_3 + x_3x_1) \\ &= x_1^2x_2 + x_2^2x_3 + x_3^2x_1 + x_1x_2^2 + x_2x_3^2 + x_3x_1^2 \\ &= c_1c_2 - 3c_3 \\ &\cong c_3 \mod 2. \end{aligned}$$

$\square$

Lecture 23. The convention of orientations

23.1. The Grafting Argument

Recall the grafting argument from the last lecture. Let $P' \to X$ be the $SU(2)$ bundle with $c_2(P') = k-1$ and $P_1 \to S^4$ has $c_2(P_1) = 1$. By taking a base point $x_0 \in X$ and $N \in S^4$, we form the connected sum $X \# S^4$. Paste connections A' on P' and A_1 on P_1 along x_0. Let $A' \#_{x_0} A_1$ be the outcome. Take a loop $\gamma \subset X$ and let x_0 travel along γ. We obtain a path of connections on the bundle $P = P' \# P_1$ with $c_2(P) = k$. Similarly, we can "paste" operators by using the excision theorem:



We use γ to parametrize this family of operators. In the first picture, connections A' and A_1 are fixed, so are operators $d^* \oplus d^+$ (so the family is constant). We let x_0 move around the loop γ. In the second picture, connections form a loop in $\mathscr{B}_P^*$. By the excision theorem, two families of operators define the same element in

$$K^0(\gamma).$$

The first family is trivial, so this element is zero. This proves the orientability of $\lambda \to \mathscr{B}^*$ in general, even if $\pi_0(\mathscr{G}) \neq \{0\}$.

(This is a proof by pictures)

Consider the universal bundle

$$\begin{array}{ccc} SO(3) = SU(2)/Z(SU(2)) & \longrightarrow & \mathscr{U}^* \\ & & \downarrow \\ & & \mathscr{B}^* \times X. \end{array}$$

The grafting construction can be carried out at any point $x \in X$, so we obtain a map

$$\begin{aligned} \tau : X &\to \mathscr{B}^* \\ x &\mapsto A\#_x A_1. \end{aligned}$$

(Tom:"Let us use τ for Taubes. This is how it is pronounced.")

Consider $\tilde{\tau} = \tau \times \mathrm{Id}_X : X \times X \to \mathscr{B}^* \times X$ and the pull-back bundle

$$\tilde{\tau}^*(\mathscr{U}^*) \to X \times X.$$

Consider the slant product

$$\begin{aligned} \mu : H_i(X, \mathbb{Q}) &\to H^i(X, \mathbb{Q}) \\ a &\mapsto -\frac{1}{4} p_1(\tilde{\tau}^*(\mathscr{U}^*))/a. \end{aligned}$$

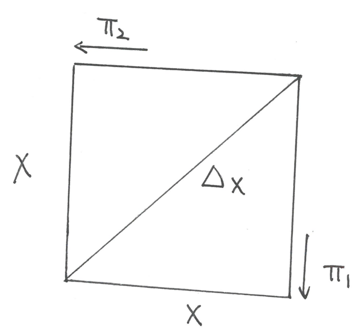
This map μ is just the Poincaré duality map. If $c_2(P) = k = 1$, then

$$PD[-\frac{1}{4} p_1(\tau^*(\mathscr{U}^*))] = \Delta_X \subset X \times X.$$

Thus, this cohomology class is supported near the diagonal. In general,

$$-\frac{1}{4} p_1(\tau^*(\mathscr{U}^*)) = PD[\Delta_X] - \frac{1}{4} p_1(\pi_1^*(\operatorname{ad} P')).$$

where $\pi_1 : X \times X \to X$ is the projection map onto the first factor and P' is the $SU(2)$-bundle with $c_2 = k - 1$.



Compare with the Abel-Jacobi map. Let $\mathrm{Jac}_d(\Sigma)$ be the moduli space of degree d holomorphic line bundles over a compact Riemann surface Σ. Then we have

$$\begin{aligned} \mu : \mathrm{Jac}_d \times \Sigma &\to \mathrm{Jac}_{d+1} \\ (\mathscr{L}, p) &\mapsto \mathscr{L}(p). \end{aligned}$$

If we carry out the grafting construction on Σ carefully and holomorphically, we will recover the Abel-Jacobi map.

23.2. Conventions for orienting moduli spaces

We extend the grafting construction to give a map

$$\begin{aligned} \mathscr{B}^*_k(X) \times \mathscr{B}^*_1(S^4) &\to \mathscr{B}^*_{k+1}(X) \\ (A_k, A_1) &\mapsto A_k \#_{x_0} A_1. \end{aligned} \tag{37}$$

By specifying the orientation for S^4, we wish to establish an orientation convention of $\mathcal{M}_k(X)$ such that when $c_2 = k$ or $k+1$, it is compatible with the grafting map (37).

Let $\mathcal{M}^o_1(S^4)$ be the framed moduli space on S^4 of charge 1. Equivalently, this is the moduli space of finite energy solutions to the ASD equation on $\mathbb{R}^4$, module gauge transformations whose limit is Id_G. We choose the base point to be the point at infinity. We have seen that

$$\mathcal{M}_1(S^4) = \mathring{B}^5$$

is the open 5-ball. Hence,

$$\mathcal{M}^o_1(S^4) \cong \mathcal{M}_1(S^4) \times SU(2) \cong \mathbb{R}^4 \times \mathbb{R}_+ \times SU(2).$$

Note that $\mathbb{R}^4 \times \mathbb{R}_+ \cong \mathrm{Conf}(\mathbb{R}^4)$ is the conformal group of $\mathbb{R}^4$ and $\mathbb{R}_+$ corresponds to dilations. We obtain an orientation of $\mathcal{M}^o_1(S^4)$ by orienting each factor and choosing a preferred order of factors. In fact, the framed ASD moduli space on a complex (hyper-Kahler) manifold is a complex (hyper-Kahler) manifold. Since $\mathbb{R}^4$ has the standard hyper-Kahler structure, $\mathcal{M}^o_1(S^4)$ has a canonical orientation.

When $k = 0$, consider the deformation complex of the trivial connection Γ on X. Its cohomology groups are

$$H^0(X, \mathbb{R}) \otimes \mathfrak{so}(3), H^1(X, \mathbb{R}) \otimes \mathfrak{so}(3), H^2_+(X, \mathbb{R}) \otimes \mathfrak{so}(3).$$

Definition 23.2.1. A homology orientation of X is an orientation of

$$H^0(X, \mathbb{R}) \oplus H^1(X, \mathbb{R}) \oplus H^2_+(X, \mathbb{R}).$$

Since maximal positive definite subspaces (with respect to the intersection form) of $H^2(X, \mathbb{R})$ form a connected set, this definition is independent of the choice of $H^2_+(X, \mathbb{R}) \subset H^2(X, \mathbb{R})$. An orientation of $\mathfrak{so}(3)$ is also needed in order to orient $\mathcal{M}_0(X)$.

Finally, to orient $SU(2)$-moduli spaces in general,

- Pick a cohomology orientation of X and an orientation of $\mathfrak{so}(3)$.
- Orient $\mathcal{M}_k(X)$ by grafting k-instantons. Choose a base point $x_0 \in X$ and $A_k \in \mathscr{A}_k(S^4)$. Use the grafting map:

$$\begin{aligned} \mu_k : \mathscr{B}^*_0 &\to \mathscr{B}^*_k \\ A &\mapsto A \#_{x_0} A_k. \end{aligned} \tag{38}$$

23.3. Orienting $SO(3)$ moduli spaces

For an $SO(3)$ principal bundle Q, the gauge group has more connected components.

For any simply connected compact Lie group G, it comes with its adjoint form $G^{ad} = G/Z(G)$. G^{ad} acts on G by conjugation. For any principal G^{ad} bundle Q, we have the adjoint bundle:

$$\begin{array}{ccccccc} G^{ad} & \longrightarrow & Q & \Longrightarrow & G^{ad} & \longrightarrow & \operatorname{Ad} Q \\ & & \downarrow & & & & \downarrow \\ & & X & & & & X. \end{array}$$

The adjoint bundle $\operatorname{Ad} Q$ has a natural covering space:

$$\widetilde{\operatorname{Ad} Q} = Q \times_{Ad} G = Q \times G/G^{ad}.$$

$\widetilde{\operatorname{Ad} Q}$ is a $|Z(G)|$-fold cover of $\operatorname{Ad} Q$:

$$\begin{array}{ccc} Z(G) & \longrightarrow & \widetilde{\operatorname{Ad} Q} \\ & & \downarrow{\scriptstyle \pi} \\ & & \operatorname{Ad} G. \end{array}$$

There are two different gauge groups:

$$\mathscr{G}_Q = \text{Sections of } \operatorname{Ad} Q$$
$$\widetilde{\mathscr{G}_Q} = \text{Sections of } \widetilde{\operatorname{Ad} Q}.$$

Life is simple with $\widetilde{\mathscr{G}_Q}$ as the gauge group. The problem comes with reducible solutions:

$$G \Rightarrow \text{The graph of commutants of } G \Leftarrow \widetilde{\mathscr{G}_Q}$$
$$G^{ad} \Rightarrow \text{A bigger graph than that of } G \Leftarrow \mathscr{G}_Q.$$

We wish to work with a smaller graph of commutants. Consider the exact sequence:

$$0 \to Z(G) \to \widetilde{\mathscr{G}_Q} \xrightarrow{\pi} \mathscr{G}_Q \to H^1(X, Z(G)) \to 0.$$

The middle map π is induced from the projection map $\widetilde{\operatorname{Ad} Q} \to \operatorname{Ad} Q$. To compute the cokernel, consider the lifting problem:

$$\begin{tikzcd} & \widetilde{\operatorname{Ad} Q} \arrow[d, "\pi"] \\ X \arrow[ur, dashed, "?"] \arrow[r, "s"] & \operatorname{Ad} Q \end{tikzcd}$$

The only obstruction class lies in $H^1(X, \pi_0(Z(G)))$, so

$$\mathscr{G}_Q / \operatorname{Im} \pi \cong H^1(X, Z(G)).$$

Definition 23.3.1. $\widetilde{\operatorname{Ad} Q}$ is called the determinant-1 gauge group.

The cokernel $H^1(X, Z(G))$ acts on $\mathscr{B}/\widetilde{\mathscr{G}}_Q$, so the size of stabilizers can increase.

23.4. The determinant-1 gauge group

Why is $\widetilde{\mathscr{G}_Q}$ called the determinant-1 gauge group? Consider $G = SU(n)$, then

$$G^{ad} = PU(n) = SU(n)/\mathbb{Z}_n = U(n)/U(1).$$

Here, $U(1)$ is the subgroup of $U(n)$ consisting of diagonal matrices. Let $P \to X$ be a $U(n)$ bundle:

$$\begin{array}{ccc} U(n) & \longrightarrow & P \\ & & \downarrow \\ & & X \end{array}.$$

Let $\mathscr{A}_P^1 = \{A \in \mathscr{A}_P : \det(A) = \lambda \text{ is a fixed connection in } \det(P)\}$. The subgroup $\mathscr{G}_P^1$ of $\mathscr{G}_P$ that acts on $\mathscr{A}_P^1$ consists of sections of $\operatorname{Ad} P$ that has determinant 1. Pick a section $s \in \mathscr{G}_P^1$, then

$$s_x \in SU(n)_x \subset U(n)_x = (\operatorname{Ad} P)_x$$

for any $x \in X$. The advantage of thinking this way is that a $U(n)$-bundle is easier to work with than a $PU(n)$-bundle.

Let $n = 2$, so $PU(2) = SO(3)$. Consider a complex 2-plane bundle:

$$\begin{array}{ccc} \mathbb{C}^2 & \longrightarrow & E \\ & & \downarrow \\ & & X^4 \end{array}.$$

and let P be the frame bundle of E. If $E = L_1 \oplus L_2$ splits, then

$$\operatorname{ad} P = \mathbb{R} \oplus \mathbb{R} \oplus (L_1^* \otimes L_2)_{\mathbb{R}}. \tag{39}$$

The first $\mathbb{R}$ factor corresponds to the determinant. Indeed, $\operatorname{ad} P \subset \operatorname{End}(E)$ consists of skew Hermitian maps and

$$\operatorname{End}(E) = \operatorname{End}(L_1) \oplus \operatorname{End}(L_2) \oplus (L_1 \otimes L_2^*) \oplus (L_2 \otimes L_1^*).$$

We define an orientation of the determinant line bundle $\lambda \to \mathscr{B}^*$ compatible with this splitting. The cohomology groups of the deformation complex are

$$H^0(X, \mathbb{R}), H^1(X, \mathbb{R}), H^2_+(X, \mathbb{R}), \text{some copies of } \mathbb{C}\text{'s}.$$

If $w_2(Q) = 0$, then Q is induced from an $SU(2)$-bundle. If $w_2(Q) = c \neq 0$, then we take $E = \mathbb{C} \oplus L_c$ with $w_2(L_c) = c$, and orient $\mathscr{A}_P^1/\mathscr{G}_P^1$ by the previous convention. At this point,

$$\operatorname{ad} Q \text{ and } \ker(\operatorname{ad} P = \mathbb{R} \oplus (\mathbb{R} \oplus L_c)_{\mathbb{R}} \to \mathbb{R})$$

have the same w_2, but $-\frac{1}{4}p_1$ could be different. To increase the instanton number, apply the grafting construction.

Remark. We need the Pontryagin square at the last step. Recall that the Pontryagin square is a cohomology operation:

$$\beta_n : H^n(X, \mathbb{Z}_2) \to H^{2n}(X, \mathbb{Z}_4).$$

such that

- $\beta_n(u) = u^2 \mod 2$.
- $\beta_n(u) = v^2 \mod 4$, if $v \in H^n(X, \mathbb{Z})$ is an integral lift of u.

In our case, it is important to know that for any $SO(3)$-bundle ξ,

$$\beta_2(w_2(\xi)) = p_1 \mod 4.$$

In particular, if $w_2(\xi_1) = w_2(\xi_2)$, then

$$\frac{1}{4}(p_1(\xi_1) - p_1(\xi_2)) \in H^4(X, \mathbb{Z})$$

is an integral class. Hence, we can use the grafting construction to compensate the difference.

For an axiomatic description of Pontryagin squares, see [6].

Lecture 24. Evaluation on the Fundamental Class

24.1. Reducible Solutions

Let us consider two cases:

(1) Let $Q \to X$ be an $SO(3)$ principal bundle. Consider the determinant-1 gauge group $\widetilde{\mathscr{G}_Q}$. Let
$$w = w_2(Q), k(Q) = -\frac{1}{4}p_1(Q).$$
(2) Suppose we have a complex 2-plane bundle
$$\begin{array}{ccc} \mathbb{C}^2 & \longrightarrow & E \\ & & \downarrow \\ & & X. \end{array}$$
Consider the determinant-1 gauge group $\mathscr{G}^1$ and the space of connections $\mathscr{A}^1$ with a fixed image in $\mathscr{A}(\det E)$. Let
$$w = w_2(E) = c_1(E) \mod 2,$$
$$k = k(E) := c_2(E) - \frac{1}{4}c_1^2(E).$$

Exercise 24.1.1. *Verify that $k(E \otimes L) = k(L)$ for any complex line bundle $L \to X$.*

The dimension of $\mathcal{M}_{k,w}(X)$ is $8k - 3(1 + b^+ - b^1)$. This moduli space $\mathcal{M}_{k,w}(X)$ is orientable and a cohomology orientation of X together with an orientation of $\mathfrak{so}(3)$ orients $\mathcal{M}_{k,w}$, so
$$\mathcal{M}^*_{k,w}(X) \subset \mathscr{B}^*_Q$$
is an oriented sub-manifold.

When does $\mathcal{M}_{k,w}(X)$ contains reducible solution? This amounts to finding a complex line bundle $L \to X$ such that
$$\operatorname{ad} Q = \mathbb{R} \oplus L_{\mathbb{R}}$$
and $c_1(L)$ has an ASD connections. By Fredholm alternatives, $c_1(L)$ has an ASD representative if and only if
$$\langle \theta, c_1(E) \rangle = 0, \ \forall \theta \in \mathcal{H}^2_+(X, g)$$
$$\Rightarrow \int_X \theta \wedge c_1(E) = 0, \ \forall \theta \in \mathcal{H}^2_+(X, g).$$
The last one is a cohomological condition. We would like to put some geometry into understanding this condition.

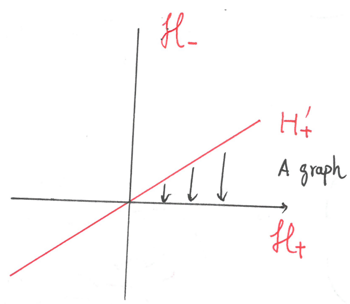
Remark. When $G = PU(2) = SO(3)$, the stabilizer of a reducible connection could be:
$$\mathbb{Z}_2, K_4, SO(2), O(2), SO(3).$$

Except for $\mathbb{Z}_2$, the structure group of Q is a subgroup of $SO(2)$, so $\operatorname{ad} Q$ has an $\mathbb{R}$-summand. □

Consider

$$\begin{aligned} \mathrm{Gr}_{b^+}(H^2(X,\mathbb{R})) &= \{b^+ \text{ dimensional subspaces of } H^2(X,\mathbb{R})\}. \\ \mathrm{Gr}^+_{b^+}(H^2(X,\mathbb{R})) &= \{H \in \mathrm{Gr}_{b^+}(H^2(X,\mathbb{R})) : H \text{ is positive definite with} \\ &\qquad \text{respective to the intersection form}\}. \end{aligned}$$

Therefore, $\mathrm{Gr}^+_{b^+} \subset \mathrm{Gr}_{b^+}$ and



$$\mathcal{H}^2_+(X,g) \in \mathrm{Gr}^+_{b^+}$$

is a point. For an integral class $c \neq 0 \in \mathrm{Im}(H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{R}))$, consider ($W$ stands for walls):

$$W_c = \{\mathcal{H} \in \mathrm{Gr}^+_{b^+} : c \cup h = 0, \forall h \in \mathcal{H}\}.$$

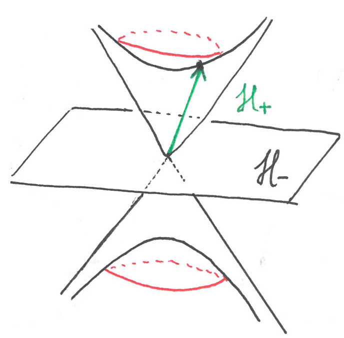
Then $W_c \subset \mathrm{Gr}^+_{b^+}$ is a smooth manifold of co-dimension b^+. Suppose $\mathcal{H} \in W_c$ and let $h_1, \cdots, h_{b^+}$ be a basis of $\mathcal{H}$. Then $\mathcal{H}^\perp$ is negative definite. Any other maximal positive definite subspace $\mathcal{H}'$ has to be graph over $\mathcal{H}$, so $\mathcal{H}'$ is characterized by a linear map:

$$f : \mathcal{H} \to \mathcal{H}^\perp.$$

For $\mathcal{H}'$ to be contained in W_c, we need

$$(h_i + f(h_i)) \cup c \Rightarrow f(h_i) \cup c = 0, \forall 1 \leqslant i \leqslant b^+.$$

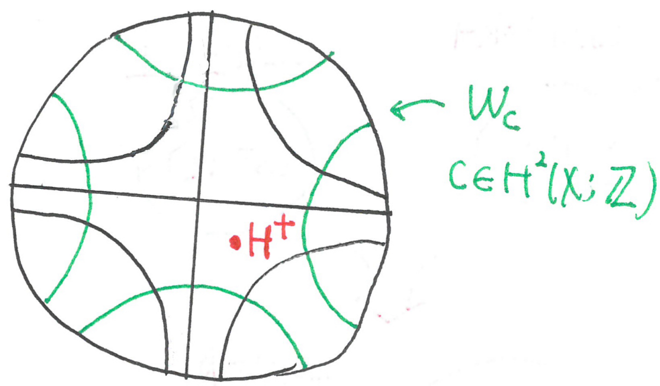
This is a co-dimensional b^+ constraint.



If $b^+ = 1$, then $\mathrm{Gr}^+_{b^+}$ is the hyperbolic space of dimensional $b^2 - 1$. The positive unit sphere of $H^2(X, \mathbb{R})$ has two connected components, and a cohomology orientation pick up a preferred component. Then the union

$$\bigcup_{c \in H^2(X,\mathbb{Z})} W_c$$

is locally finite in the interior of $\mathrm{Gr}^+_{b^+}$, but it becomes dense at the "boundary".



Theorem 24.1.2 (Taubes, Donaldson-Kronheimer)**.** *Let* Met *be the space of Riemannian metrics on X. Then the map*

$$P : \mathrm{Met} \to \mathrm{Gr}^+_{b^+}(H^2(X, \mathbb{R}))$$
$$g \mapsto \mathcal{H}^+(X, g)$$

is transverse to W_c for any $c \neq 0 \in H^2(X, \mathbb{Z})$.

This is a theorem first known to Taubes. A proof can be found in Donaldson and Kronheimer's book [7].

In particular, if $b^+ \geqslant 1$, then for a generic metric g, there are no g-ASD representative for any $c \neq 0 \in H^2(X, \mathbb{Z})$. If $b^+ \geqslant 2$, then for a pair of good metrics g_0, g_1 (no ASD representatives for any $c \neq 0 \in H^2(X, \mathbb{Z})$), they fits into an one-parameter family of good metrics $g_t, t \in [0, 1]$.

As a consequence, if $b^+ \geqslant 1$, then for a Baire set of Met, $\mathcal{M}_{k,w}(X, g) \subset \mathscr{B}^*_P$ is a finite dimensional smooth manifold. If $b^+ \geqslant 2$, then for a generic path (a Baire set of paths) g_t between two good metric g_0, g_1, the union

$$\bigcup_{t \in [0,1]} \mathcal{M}_{k,w}(X, g_t) \subset I \times \mathscr{B}^*(P)$$

is a smooth cobordism between $\mathcal{M}_{k,w}(X, g_0)$ and $\mathcal{M}_{k,w}(X, g_1)$.

The case when $c = 0$ only occurs when $w = 0$ and $k = 0$. The reducible solution is unique up to gauge. It is the trivial connection Γ in the trivial $SO(3)$-bundle.

24.2. Cohomology Classes

For $G = SO(3)$, we have seen that $H^*(\mathscr{B}^*, \mathbb{Q}) \cong \mathbb{A}(X)$.

Question. Does $[\mathcal{M}_{k,w}(X)]$ have a fundamental class in $H_*(\mathscr{B}^*, \mathbb{Q})$?

This is a non-trivial problem, because $\mathcal{M}_{k,w}(X)$ is not compact. How do we get a fundamental class?

Recall the Uhlenbeck's compactification of $\mathcal{M}_{k,w}(X)$:

$$\begin{aligned}
\overline{\mathcal{M}_{k,w}}^{Uhl}(X) = \mathcal{M}_{k,w}(X) & \qquad \dim_k = 8k - \frac{3}{2}(\chi + \sigma) \\
\bigcup \mathcal{M}_{k-1,w}(X) \times X & \qquad \dim = \dim_k - 4 \\
\bigcup \mathcal{M}_{k-2,w}(X) \times \operatorname{Sym}^2 X & \qquad \dim = \dim_k - 8 \\
\cdots & \\
\bigcup \mathcal{M}_{k^*,w}(X) \times \operatorname{Sym}^{k^*} X &
\end{aligned}$$

Note that $k = c_2(E) - \frac{1}{4}c_1(E)^2 \in \frac{1}{4}\mathbb{Z}$. If $k \not\equiv 0 \mod \mathbb{Z}$, then $k^* \neq 0$.

If $k \in \mathbb{Z}$, then $k^* = 0$ and $\mathcal{M}_{0,0}(X) = \{*\}$ consists of the unique trivial connection, if $\pi_1(X) = \{0\}$. The bottom stratum has dimension $4k$ independent of X.

There is a relative fundamental class for the top stratum

$$\mathcal{M}_{k,w} \text{ rel lower strata.}$$

For any $a \in H_*(X)$, we constructed $\mu(a) \in H^*(\mathscr{B}^*)$. The question is to whether $\mu(a)$ has a compact support:

$$\begin{tikzcd}
\mu(a) \in H^*(\mathscr{B}^*) \arrow[r] \arrow[dr, dashed, "?"] & H^*(\mathcal{M}_{k,w}(X)) \\
& H^*_{cpt}(\mathcal{M}_{k,w}(X)). \arrow[u]
\end{tikzcd}$$

The idea is to find a nice representative of $\mu(a)$ whose dual cycle will avoid each other on the end of $\mathcal{M}_{k,w}(X)$ for different $a \in H_*(X)$.

Let $V \subset X$ be an open subset of X. Define

$$\mathscr{B}^*_{k,w}(X, V) = \{[A] \in \mathscr{B}^*_{k,w}(X) : A|_V \text{ is irreducible}\} \subset \mathscr{B}^*_{k,w}(X).$$
$$\mathscr{B}^*_{k,w}(V) = \{[A] \in \mathscr{B}_{k,w}(V) : A \text{ is irreducible}\}.$$

Note that $\mathscr{B}^*_{k,w}(X, V)$ is a subspace of $\mathscr{B}^*_{k,w}(X)$, because some connections could be reducible on V, but they are irreducible on X. There is an obvious restriction map:

$$\text{res} : \mathscr{B}^*_{k,w}(X, V) \to \mathscr{B}^*_{k,w}(V).$$

Consider the universal over $\mathscr{B}^*_{k,w}(V) \times V$:

$$\mathscr{U}^*_V = \mathscr{A}^*_{k,w}(V) \times P|_V/\mathscr{G}(V) \to \mathscr{B}^*_{k,w}(V) \times V.$$

and we have a commutative diagram:

$$\begin{tikzcd}
 & & \mathscr{U}_V^* \arrow[d] & \\
\operatorname{res}^* \mathscr{U}_V^* = \iota^* \mathscr{U}^* \arrow[urr, bend left] \arrow[rrrd, bend left] \arrow[d] & & \mathscr{B}_{k,w}^*(V) \times V & \\
\mathscr{B}_{k,w}^*(X,V) \times V \arrow[urr, "\operatorname{res} \times 1_V"'] \arrow[rrrd, "\iota = i_1 \times i_2"] & & & \mathscr{U}^* \arrow[d] \\
 & & & \mathscr{B}_{k,w}^*(X) \times X
\end{tikzcd}$$

It induces a commutative diagram on the level of cohomology group:

$$\begin{tikzcd}
H_*(V) \arrow[r, "\mu_V"] \arrow[d, equal] & H^*(\mathscr{B}^*(V)) \arrow[d, "\operatorname{res}^*"] \arrow[dr, dashed, "?"] & \\
H_*(V) \arrow[r, "\mu_{X,V}"] \arrow[d] & H^*(\mathscr{B}^*(X,V)) \arrow[r, dashed, "?"] \arrow[dr, "j^*"] & H^*_{cpt}(\mathcal{M}_{k,w}(X)) \arrow[d] \\
H_*(X) \arrow[r, "\mu"] & H^*(\mathscr{B}^*(X)) \arrow[u, "i_1^*"] \arrow[r] & H^*(\mathcal{M}_{k,w}(X)).
\end{tikzcd}$$

The observation is that $\mathcal{M}_{k,w}(X)$ is contained in the subspace $\mathscr{B}^*(X,V)$, so the map j^* in the diagram is valid. This requires a unique continuation property of ASD connections.

Lemma 24.2.1. *If an ASD connection A is irreducible on X, then for any open subset $V \subset X$, $A|_V$ is irreducible.*

Let $\{\Sigma_i\}_{1 \leqslant i \leqslant n} \subset X$ be a finite collections of surfaces in X and let $V_i \subset X$ be a tubular neighborhood of Σ_i. We require that

$$V_i \cap V_j \cap V_k = \varnothing$$

whenever $i \neq j \neq k \neq i$. Finally, let $V = \bigcup V_i$. Each surface Σ_i defines a rational class:

$$\mu_V(\Sigma_i) \in H^2(\mathscr{B}^*(V_i), \mathbb{Q}),$$

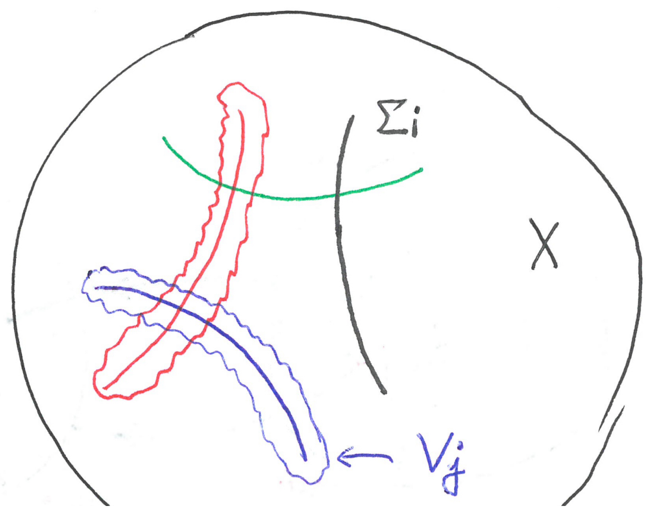
which determines a complex line bundle $L_i \to \mathscr{B}^*(V_i)$ (such that $c_1(L) = a_i \mu_V(\Sigma_i)$ for $a_i \in \mathbb{Q}$). Let s_i be the zero locus of a generic section of L_i, then formally $s_i = PD[\mu(\Sigma_i)]$. We wish to evaluate the product

$$\mu_V(\Sigma_1) \smile \cdots \smile \mu_V(\Sigma_n).$$

on $\mathcal{M}_{k,w}(X)$. Equivalently, this is counting the number of intersections with signs:

$$Z = \bigcap_{i=1}^{n} s_i \cap \mathcal{M}_{k,w}(X),$$

and divide out $\prod a_i \in \mathbb{Q}$. The idea is to show Z is compact, and hence finite.



Since each $\mu_V(\sigma_i)$ is in degree 2, it is necessary to assume $\dim_k = \dim \mathcal{M}_{k,w}(X)$ is even. For instance, let $b_1 = 0, b^+ = 3$ and $k = \frac{11}{4}$, then $\dim_k = 10$. Take $n = 5$. The compactification has two strata:

$$\overline{\mathcal{M}_{k,w}}^{Uhl}(X) = \mathcal{M}_{k,w}(X) \qquad \dim_k = 10$$
$$\bigcup \underbrace{\mathcal{M}_{k-1,w}(X)}_{\dim=2} \times \underbrace{X}_{\dim=4} \qquad \dim = 6$$

Since k is not an integer, the trivial connection Γ does not occur in the compactification. Suppose we can find a sequence $[A_n] \in Z$ such that they converge to an ideal connection $([A_\infty], x_\infty) \in \mathcal{M}_{k-1,w} \times X$. We claim that for each $1 \leqslant i \leqslant 5$, either

(1) $x_\infty \in V_i \subset X_i$, or
(2) $[A_\infty] \in s_i$.

If $x_\infty \notin V_i$, then $[A_n] \to [A_\infty]$ on V_i in $\mathcal{C}^\infty_{loc}$-topology. Since $\mathcal{M}_{k-1,w}$ does not contain reducible solutions or the trivial connection Γ, $A_\infty|_V$ is irreducible. Since s_i is a closed subset of $\mathscr{B}^*(V_i)$, $[A_\infty] \in s_i$.

If $x_\infty \neq V_i$ for any $1 \leqslant i \leqslant 5$, then $[A_\infty] \in \bigcap_{i=1}^5 s_i$. However, the formal dimension of

$$\bigcap_{i=1}^{5} s_i \cap \mathcal{M}_{k-1,w}(X).$$

is $2 - 5 \cdot 2 = -8$. This case is ruled out by taking generic sections s_i.

In general, x_∞ lies in at most two of V_i's. Hence, for at least three sections $s_\alpha, s_\beta, s_\gamma$,

$$[A_\infty] \in s_\alpha \cap s_\beta \cap s_\gamma \cap \mathcal{M}_{k-1,w}(X),$$

whose formal dimension is still negative. We conclude that Z is compact.

Lecture 25. Donaldson's Polynomial Invariants

25.1. The Polynomial Invariants

Recall the construction at the end of the last lecture. For each smoothly embedded oriented sub-manifold $\Sigma \subset X$, let V be a tubular neighborhood of Σ. Here Σ is allowed to be a point, a surface or a loop. There are two useful spaces of configurations:

$$\mathscr{B}^*(X,V) = \{[A] \in \mathscr{B}^*_{k,w}(X) : A|_V \text{ is irreducible}\} \subset \mathscr{B}^*(X).$$
$$\mathscr{B}^*(V) = \{[A] \in \mathscr{B}(V) : A \text{ is irreducible}\}.$$

There is an obvious restriction map:

$$r_V : \mathscr{B}^*(X,V) \to \mathscr{B}^*(V), A \mapsto A|_V.$$

Consider the universal bundle over $\mathscr{B}^*_V$:

$$\mathscr{U}^*_V = \mathscr{A}^*_V \times P|_V/\mathscr{G}_V \to V \times \mathscr{B}^*(V).$$

Using the slant product, we defined a rational cohomology class:

$$\mu_V([\Sigma]) := -\frac{1}{4}p_1(\mathscr{U}^*_V)/[\Sigma] \in H^{4-\dim\Sigma}(\mathscr{B}^*(V)).$$

By pulling back via the restriction map, we recover the class:

$$\mu_X([\Sigma]) = r_V^*(\mu_V([\Sigma])) \in H^*(\mathscr{B}^*(X,V),\mathbb{Q}).$$

Fixing a co-cycle representative $m_V([\Sigma])$ of $\mu_V([\Sigma])$, we obtain a representative:

$$m(\Sigma) = r_V^*(m_V(\Sigma)).$$

Now choose a collection of representatives of homology classes of X:

$$\Sigma_1, \cdots, \Sigma_n,$$

and construct $m(\Sigma_1), \cdots, m(\Sigma_n)$. We would like to show the cup product:

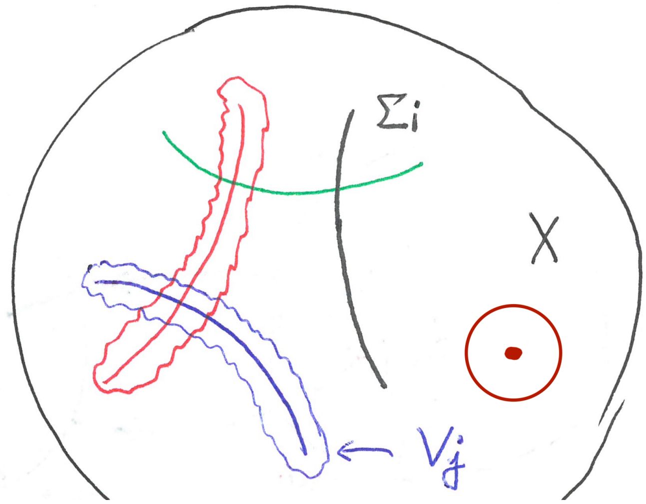
(*) $m(\Sigma_1) \smile \cdots \smile m(\Sigma_n)$ is a compactly supported class in $H^*(\mathcal{M}_P(X))$.

For (*) to hold, we need:

- $4n - \sum \dim \Sigma_i = 8k - \frac{3}{2}(\chi+\sigma) = \dim \mathcal{M}_p(X)$.
- Need neighborhoods V_i of Σ_i to have minimal intersections. For instance, any three surfaces do not meet; the neighborhood of a point is isolated from others.
- Need to worry about the "bottom stratum" where the instanton number $k = 0$ and the flat connection occur. This stratum has dimension equal $4k(P)$. We can find a situation when k is not an integer.

The invariants to be defined rely on the following data:

- $w \in H^2(X,\mathbb{Z}_2)$ such that $w_2(P) = w$. This class w is also used to orient the moduli space.
- A cohomology orientation o.



The polynomial invariant $\mathbb{D}^{w,o}$ is a $\mathbb{Q}$-linear map:

$$\mathbb{D}^{w,o} : \mathbb{A}(X) \to \mathbb{Q}$$
$$\mathbb{D}^{w,o}(\Sigma_1 \otimes \Sigma_2 \otimes \cdots \Sigma_n) = \langle m(\Sigma_1) \smile \cdots m(\Sigma_n), [\mathcal{M}_{k,w}, \partial \mathcal{M}_{k,w}] \rangle.$$

where $[\mathcal{M}_{k,w}, \partial \mathcal{M}_{k,w}]$ denotes the fundamental class of $\mathcal{M}_{k,w}$ relative to its end. The principal bundle P is chosen such that

- $8k(P) - \frac{3}{2}(\chi + \sigma) = 4n - \sum_{i=1}^{n} \dim \Sigma_i$.
- $w_2(P) \cong w \mod 2$. As $w_2(P)$ is fixed, only the Pontryagin class of P can be changed up to integers. The polynomial invariant $\mathbb{D}^{w,o}$ is defined using a sequence of bundles, packaging their information altogether.

This invariant $\mathbb{D}^{w,o}$ is well defined (independent of all choices) if $b^+(X) > 1$.

Remark. One may ask about the integrability of $\mathbb{D}^{w,o}$: does it send an integral class to an integer? Since $-\frac{1}{4}p_1$ is not necessarily a integral class, this is not clear in general.

However, to make this construction work, we need to verify the restriction of $\mathcal{M}_P(X, g)$ lies in $\mathscr{B}^*(V)$ for a generic metric g:

$$r_V(\mathcal{M}_P(X, g)) \subset \mathscr{B}^*(V).$$

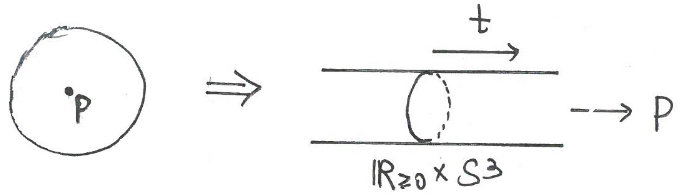
This follows from the unique continuation property of the ASD connections.

25.2. The unique continuation property

There are two classical unique continuation properties for harmonic functions. If f is a harmonic function on $\mathbb{R}^n$ (i.e. $\Delta_{\mathbb{R}^n} f = 0$) and f vanishes on a open subset U, then $f \equiv 0$ on $\mathbb{R}^n$.

The sharper version states that if f vanishes at a point $p \in \mathbb{R}^n$ at infinite order, i.e. $\nabla^{(k)} f(p) \equiv 0$ for any $k \geqslant 0$, then $f \equiv 0$ on $\mathbb{R}^n$.

The case for connections is tricker. There is **a global version** and **a local version** of unique continuation properties. If $\pi_1(X) \neq 0$, there will be some non-trivial flat connections which is locally trivial, and yet irreducible globally. Therefore, to conclude a locally trivial (flat) connection is trivial globally, we have to assume $\pi_1(X) = 0$.



The sharper version of the unique continuation property is easier to prove (strangely enough). For $p \in X$, an geodesic neighborhood U of P is conformally equivalent to

$$[0, \infty) \times S^3.$$

"The connection $A = \Gamma + a$ is flat at infinite order at p" means that a decays faster than any exponential functions on the half cylinder, i.e.

$$|a(t)| < \exp(-\delta t), \ \forall \delta > 0, \ t > 0.$$

If in addition A is ASD, i.e. $F_A^+ \equiv 0$, then the connection 1-form $a \equiv 0$ (Hint: look at the linearized ASD equation).

Suppose locally $A = \Gamma + a = \Gamma + b + c$ where $\Gamma + b$ is induced from $U(1)$-bundle $Q \to X$ with

$$P = Q \times_\rho SU(2),$$

and c is the difference. "The connection A is reducible to an infinite order at p" means that

$$|c(t)| < \exp(-\delta t), \ \forall \delta > 0, \ t > 0.$$

This will imply $c \equiv 0$ if A is ASD. Hence

$$\begin{aligned} &A \text{ is reducible to the infinite order at p} \\ \Rightarrow &A \text{ is reducible in a neighborhood of } p \\ \Rightarrow &A \text{ is reducible locally on } X. \end{aligned}$$

To pass from the local version to the global version, we need to assume X is simply connected. Finally, A is reducible globally.

25.3. Generating Functions

If $\pi_1(X) = 0$, then

$$\mathbb{A}(X) = \mathrm{Sym}(H_2(X; \mathbb{Q}) \oplus H_0(X, \mathbb{Q}).$$

It will be enlightening to look at the generating function of $\mathbb{D}^{w,o}$. Let $x \in H_0(X,\mathbb{Z})$ be the canonical generator (X is connected) and pick $h \in H_2(X,\mathbb{Q})$. Then as elements in $\mathbb{A}(X)$,

$$\deg x = 4, \deg h = 2.$$

Consider $\mathbb{D}^{w,o}(\exp(th + sx))$ where

$$\exp(th + sx) = \sum_{n=0}^{\infty} \frac{(th + sx)^n}{n!} \in \mathbb{A}[X] \otimes_{\mathbb{Q}} \mathbb{Q}[[s,t]]$$

and s, t are formal variables. All known 4-manifolds with $b^+ > 1$ and $b^1 \equiv 0$ satisfy **the simple type relation**:

$$\mathbb{D}^{w,o}_X(x^2 z) = 4\mathbb{D}^{w,o}_X(z).$$

Any 4-manifold with this property is said to have **simple type**. In this case, evaluating $\mu^a(h)x^b$ on $[\mathcal{M}_P]$ for $b = 0, 1$ incorporate all interesting information. Define

$$\mathcal{D}^{w,o}_X(th) = \mathbb{D}^{w,0}_X((1 + \frac{x}{2}) \exp(th)).$$

The main theorem is:

Theorem 25.3.1 (Kronheimer-Mrowka)**.** *If X^4 is a simply connected 4-manifold of simple type, then there exist classes $K_i \in H^2(X,\mathbb{Z})$ and numbers $a_i \neq 0 \in \mathbb{Q}$ for $1 \leqslant i \leqslant n$ such that*

$$\mathcal{D}^{w,o}_X(th) = e^{t^2 Q_X(h)}(\sum_{i=1}^{n} (-1)^{k_i \cdot w + w^2} a_i e^{t k_i \cdot h})$$

where Q_X denotes the intersection form of X, i.e. $Q_X(h) = h \cdot h$. These classes K_i are called basic classes. Moreover,

- $K_i \equiv w_2(X) \mod 2$.
- *If $\Sigma \subset X$ is any smoothly embedded surface with self-intersection $\Sigma^2 \geqslant 0$, then*

(40) $$2g(\Sigma) - 2 \geqslant K_i \cdot \Sigma + \Sigma \cdot \Sigma.$$

  for any $1 \leqslant i \leqslant n$. (This is a constraint of basic classes.)
- *If K_i is a basic class, then $-K_i$ is a basic class.*

Having simple type is closely related to whether or not

(41) $$K_i^2 - 2\chi(X) + 3\sigma(X) = 0.$$

Let us compare (40) with the adjunction equality. Suppose (X, J) is a complex manifold, define the canonical class as

$$K_X = -c_1(TX, J) \in H^2(X,\mathbb{Z}).$$

If $\Sigma \subset X$ is an embedded complex curve, the tangent bundle $T\Sigma$ is J-invariant. Since $TX|_\Sigma = T\Sigma \oplus N\Sigma$, we have

$$c_1(TX, J)|\Sigma = e(T\Sigma) + e(N\Sigma) = \chi(\Sigma) + \Sigma \cdot \Sigma.$$

Therefore, $2g(\Sigma) - 2 = -\chi(\Sigma) = K_X \cdot \Sigma + \Sigma \cdot \Sigma$.

As for (41), for an almost complex 4-manifold (X, J), we have

$$K(X, J)^2 = 2\chi(X) + 3\sigma(X). \tag{42}$$

Indeed, by the Hirzebruch signature formula,

$$\langle p_1(TX), X\rangle = -3\sigma(X).$$

The wedge bundle $\Lambda^2(X)$ splits as $\Lambda^+ \oplus \Lambda^-$, then both LHS and RHS of (42) are equal to

$$\langle p_1(\Lambda^+(X)), [X]\rangle.$$

To see the LHS equals, note that $\Lambda^+(X) = \mathbb{R}\omega \oplus (\Lambda^{0,2}(X))_{\mathbb{R}}$.

Exercise 25.3.2. *Show that for any closed oriented* 4*-manifold* X*,*

$$\langle p_1(\Lambda^+(X)), [X]\rangle = -2\chi - 3\sigma.$$

Lecture 26. The ASD equation on cylinders

26.1. The flowline equation

Consider an oriented closed 3-manifold Y and the trivial principal G-bundle $Q \to Y$. Let Γ be the trivial connection, with respect to a fixed trivialization. Recall that for $A = \Gamma + a$, the Chern-Simons functional is defined as

$$\mathcal{CS}(a) = \int \operatorname{tr}(a \wedge da + \frac{2}{3} a \wedge a \wedge a).$$

and its gradient at A is

$$\nabla \mathcal{CS}(A) = *F_A.$$

The downward gradient flow equation

$$-\frac{\partial}{\partial t} A(t) = *F_A$$

is equivalent to the ASD equation on the 4-manifold $\mathbb{R} \times Y$. We would like to transfer ideas from Morse functions to the Chern-Simons functional, and mimic the construction of Morse Homology. There are some potential problems:

- The functional $\mathcal{CS}$ is not single valued on the quotient space $\mathscr{B} = \mathscr{A}/\mathscr{G}$. We will probably need the Novikov-Morse homology.
- The space of connections on Y^3 can have complicated topology. $\mathscr{G}/Z(G)$ might not act freely on $\mathscr{A}$.
- Non-compactness due to bubbling.
- Constructing perturbation of $\mathcal{CS}$ to make it Morse.

Let us take a closer look at the ASD equation on a cylinder $Z = \mathbb{R} \times Y$. The trivial bundle $Q \to Y$ pulls back to the trivial bundle over Z:

$$P = \pi^* Q \to \mathbb{R} \times Y.$$

Fix a reference connection B in $Q \times Y$. Any connection A in P can be written as

$$A = B + b + cdt$$

with $b \in \Gamma(\mathbb{R} \times Y, \pi^*(T^*Y \otimes \operatorname{ad} Q))$ and $c \in \Gamma(\mathbb{R} \times Y, \operatorname{ad} P)$. We compute the curvature of A:

$$\begin{aligned} F_A &= F_{(B+b)} + dt \wedge (\frac{\partial}{\partial t} b - d_{(B+b)} c) \\ &= (F_B + d_B b + \frac{1}{2}[b \wedge b]) + dt \wedge (\frac{\partial}{\partial t} b - d_B c - [b \wedge c]). \end{aligned}$$

Moreover, the self dual part is given by

$$\begin{aligned} 2F_A^+ &= F_A + *_4 F_A \\ &= F_{(B+b)} + *_3(\frac{\partial}{\partial t} b - d_{(B+b)} c) + dt \wedge (*_3 F_{B+b} + \frac{\partial}{\partial t} b - d_{(B+b)} c) \end{aligned}$$

The ASD equation becomes

$$0 = \frac{\partial}{\partial t} b + *_3 F_{B+b} - d_{(B+b)} c. \tag{43}$$

$$= \frac{\partial}{\partial t} b + \nabla \mathcal{CS}(B+b) - d_{(B+b)} c. \tag{44}$$

Hence, $b(t)$ is a downward gradient flow up to an infinitesimal gauge transformation. The last part $d_{B+b}c$ is tangential to the gauge orbit at $B+b$.

The downward gradient flowline equation (43) itself does not form an elliptic system; we will need the Coulomb gauge fixing condition:

$$\begin{aligned} 0 &= -d_B^*(b + cdt) = *_4 d_B *_4 (b + cdt) \\ &= - *_4 d_B(dt \wedge *_3 b + *_3 c) \\ &= \frac{\partial}{\partial t} c(t) - d_B^* b. \end{aligned}$$

Together with the linearized ASD equation at B, we have:

$$0 = \frac{\partial}{\partial t} \begin{pmatrix} b \\ c \end{pmatrix} + \begin{pmatrix} *d_B & -d_B \\ -d_B^* & 0 \end{pmatrix} \begin{pmatrix} b \\ c \end{pmatrix}$$

The operator applied to (b, c) is cast in the form

$$\frac{\partial}{\partial t} + D_B$$

where D_B is a first order self-adjoint elliptic operator. This means

$$\ker D_B = \operatorname{Coker} D_B.$$

We would like to understand what this kernel means and when it is trivial.

26.2. The extended Hessian

A critical point of $\mathcal{CS}$ is a flat connection of $Q \to Y$, i.e. $F_B \equiv 0$. A flat connection is induced from a π_1-representation:

$$\rho : \pi_1(Y) \to G.$$

Consider the equation $D_B(b, c) = 0$:

$$\begin{cases} *d_B b - d_B c &= 0 \\ \qquad d_B^* b &= 0 \end{cases} \tag{45}$$

Note that $\operatorname{Im} d_B$ is orthogonal to $\operatorname{Im} d_B$ in $\Omega^1(Y, \operatorname{ad} Q)$. To see this, consider the twisted De Rham complex:

$$\begin{tikzcd} \Omega^0(Y, \operatorname{ad} Q) \arrow[r, "d_B"] & \Omega^1(Y, \operatorname{ad} Q) \arrow[r, "d_B"] \arrow[r, bend left, leftrightarrow, "*"] & \Omega^2(Y, \operatorname{ad} Q) \arrow[r, "d_B"] \arrow[l, bend left, "d_B^* = -*d_B*"] & \Omega^3(Y, \operatorname{ad} Q). \end{tikzcd}$$

This is a complex, because $d_B^2 = F_B = 0$.

Hence, (b, c) is a solution to (45) if and only if

$$d_B b = d_B^* b = 0, d_B c = 0,$$

i.e. b is a harmonic form in $\mathcal{H}^1(Y, \mathrm{ad}_B)$ and $c \in \mathcal{H}^0(Y, \mathrm{ad}_B)$. Here, $\mathcal{H}^*(Y, \mathrm{ad}_B)$ stands for harmonic forms with respect to d_B.

Definition 26.2.1. A critical points $[B]$ of $\mathcal{CS}$ is called non-degenerate if

(1) $\mathcal{H}^0(Y, \mathrm{ad}_B) = \{0\}$,
(2) $\mathcal{H}^1(Y, \mathrm{ad}_B) = \{0\}$.

The first condition implies that B is irreducible as a flat connection.

Later, we will explain how to achieve conditions (1)(2). If B is a non-degenerate critical point, then the operator

$$D_B : L_k^2(Y, (\Lambda^0 \oplus \Lambda^1) \otimes \mathrm{ad}\, Q) \to L_{k-1}^2(Y, (\Lambda^0 \oplus \Lambda^1) \otimes \mathrm{ad}\, Q)$$

is invertible. D_B is also called the extended Hessian at B.

Theorem 26.2.2. *The operator $\frac{\partial}{\partial t} + D_B$ is invertible as a bounded linear operator between*

$$L_k^2(\mathbb{R} \times Y, (\Lambda_Y^0 \oplus \Lambda_Y^1) \otimes \mathrm{ad}\, Q) \to L_{k-1}^2(\mathbb{R} \times Y, (\Lambda_Y^0 \oplus \Lambda_Y^1) \otimes \mathrm{ad}\, Q),$$

if and only if D_B is invertible.

In this theorem, the connection B is independent of the time t.

Proof. We prove the theorem when $k = 1$, by direct computation. Let $u \in \mathcal{C}_c^\infty(\mathbb{R} \times Y)$, then

$$\|(\frac{\partial}{\partial t} + D_B)u\|_{L^2(\mathbb{R}\times Y)}^2 = \int_{\mathbb{R}\times Y} |\frac{\partial}{\partial t}u|^2 + |D_B u|^2 + 2\langle \frac{\partial}{\partial t}u, D_B u\rangle.$$

Since D_B is self-adjoint, $\frac{\partial}{\partial t}\langle u, D_B u\rangle = \langle \frac{\partial}{\partial t}u, D_B u\rangle + \langle u, \frac{\partial}{\partial t}(D_B u)\rangle = 2\langle \frac{\partial}{\partial t}u, D_B u\rangle$. Note that u is compactly support, so

$$\int_{\mathbb{R}\times Y} 2\langle \frac{\partial}{\partial t}u, D_B u\rangle = \frac{\partial}{\partial t}\langle u, D_B u\rangle\Big|_{-\infty}^{+\infty} = 0.$$

On the other hand, by the invertibility of D_B,

$$\|D_B u\|_{L^2(Y)} \geqslant c\|u\|_{L_1^2(Y)}^2$$

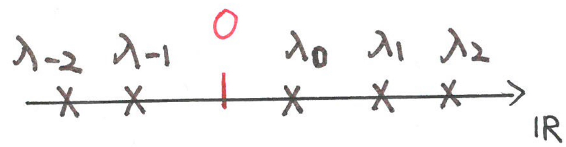
for some $c > 0$. Hence,

$$\|(\frac{\partial}{\partial t} + D_B)u\|_{L^2(\mathbb{R}\times Y)}^2 \geqslant \int_{\mathbb{R}\times Y} |\frac{\partial}{\partial t}u|^2 + \|u\|_{L_1^2(Y)}^2.$$

This estimate shows $\frac{\partial}{\partial t} + D_B : L_1^2 \to L^2$ is injective with closed range. It remains to show surjectivity. There are at least two ways to proceed:

(1) The formal adjoint of $\frac{\partial}{\partial t} + D_B$ is $-\frac{\partial}{\partial t} + D_B$. We prove it is injective using the same estimate.
(2) Prove the range of $\frac{\partial}{\partial t} + D_B$ is dense in L^2 by hand. D_B has a spectrum decomposition, i.e. there is an L^2-complete set of eigenvectors of D_B. Let them be

$$\{\phi_\lambda\}_{\lambda \in \mathrm{Spec}(D_B)}$$



The spectrum of D_B is discrete. The only accumulation point is at the infinity. In particular,

$$\|u\|^2_{L^2_1(Y)} = \sum_{\lambda \in \mathrm{Spec}\, D_B} \lambda^2 c_\lambda^2$$

if $u = \sum c_\lambda \phi_\lambda$ and $0 \notin \mathrm{Spec}\, D_B$. With respect to the spectrum decomposition, $\frac{\partial}{\partial t} + D_B$ is a diagonal matrix whose diagonal entries are given by

$$\frac{\partial}{\partial t} + \lambda,\ \lambda \in \mathrm{Spec}\, D_B.$$

We can show each $\frac{\partial}{\partial t} + \lambda$ has a dense range in

$$L^2(\mathbb{R}, \mathbb{R}).$$

The proof extends to higher k. $\square$

26.3. 4-manifolds with cylindrical ends

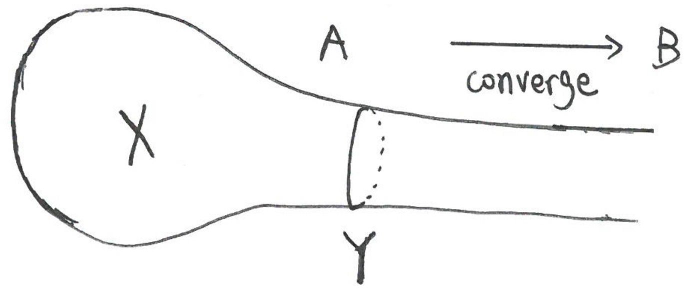
Suppose we have a compact 4-manifold X with a non-empty boundary $\partial X = Y$. We obtain a non-compact manifold by attaching cylindrical ends:

$$X^* = X \cup [0, \infty) \times Y.$$

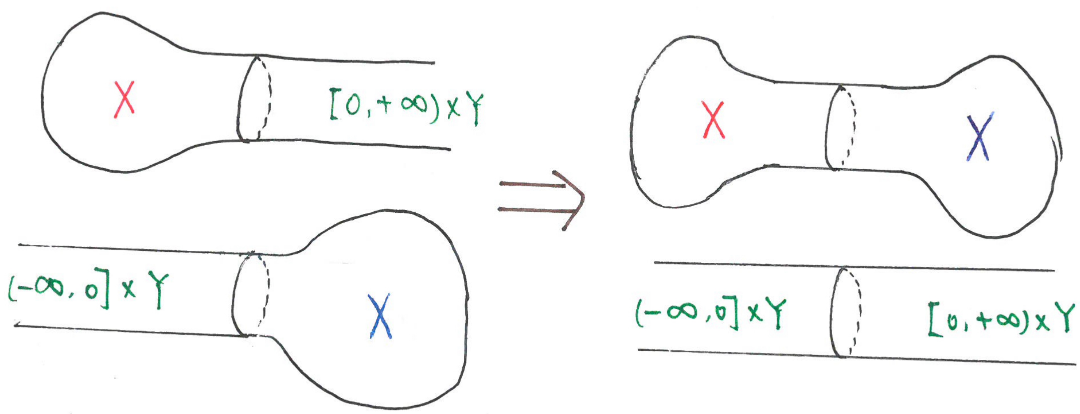
If we have a principal G-bundle P over X, then it extends to a bundle over X^*. The extension over the cylindrical end is the pull-back bundle:

$$\pi^* Q \text{ where } Q = P|_{\{0\}\times Y}.$$

Suppose A is a connection on X^* which converges exponentially to a connection B in $Q \to Y$ as $t \to \infty$. Provided that the extended Hessian D_B is invertible, we wish to check the linearized ASD equation together with the gauge fixing condition form



a Fredholm operator on X. This is done by the parametric patching argument. We apply the excision picture and obtain two manifolds with cylindrical ends.



It is important to know the error term in this patching argument is a compactly supported zeroth order operator, so the Sobolev embedding becomes compact.

If M is a finite dimensional manifold and $f : M \to \mathbb{R}$ is a Morse function, then $\operatorname{Hess} f = \nabla.(\nabla f)$ is a symmetric operator well-defined at any point in M. Similarly, the operator

$$D_B = \begin{pmatrix} *d_B & -d_B \\ -d_B^* & 0 \end{pmatrix}$$

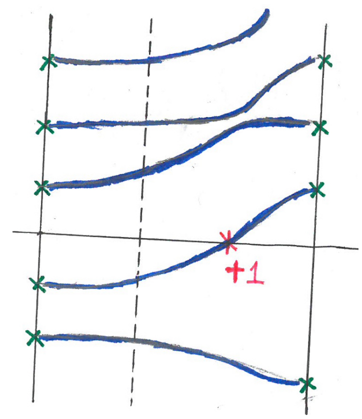
makes sense whether or not B is flat and it is the generalization of Hessians to self-adjoint operators. However, no notion of Morse indices could be defined, since the spectrum is unbounded in both the positive and negative directions.

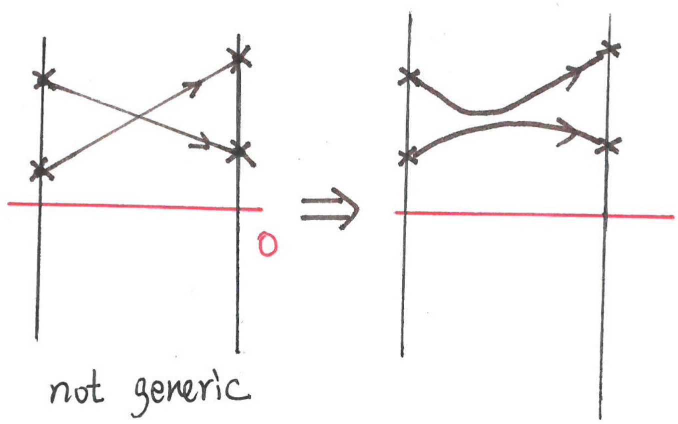
Suppose we have a family of operators $D_{B(t)}$ parametrized by $t \in [0, 1]$ and $D_{B(t)}$ is invertible when $t = 0, 1$ ($0 \notin \operatorname{Spec}(D_{B(t)})$). We define **the spectrum flow** as the signed count of intersections of

$$\operatorname{Spec} D_{B(t)}$$

with the zero-section.

The spectrum flow will replace the role of Morse indices and it equals the dimension of moduli spaces.





As t varies, $\mathrm{Spec}(D_{B(t)})$ might become non-simple, i.e some eigenvalues have multiplicities higher than 1 (the left picture). However, this is not a generic case. Having non-simple eigenvalues is a co-dimension 2 condition (for real operators), and it can be avoided for a generic path.

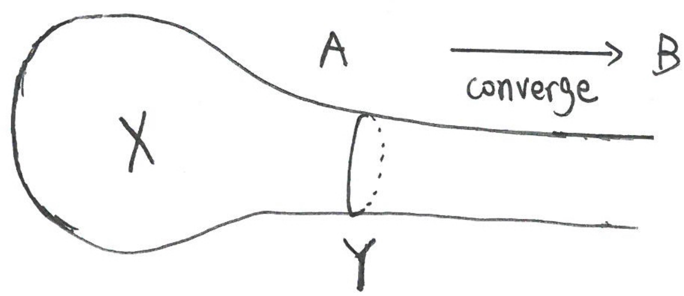
Lecture 27. The Spectrum Flow and the Index

27.1. The Sobolev Embedding Theorem

We discuss the Sobolev embedding theorem on non-compact manifolds with bounded geometry. The typical example is a 4-manifold with a cylindrical end:

$$X^* = X \cup [0, \infty) \times Y.$$

In general, we require the curvature tensor and any covariant derivatives of the curvature are uniformly bounded. Moreover, the injective radius $r(p)$ at any $p \in X^*$ has a uniform lower bound.



On the unit ball $B^n \subset \mathbb{R}^n$, we have:

$$L^p_k \hookrightarrow L^q_l(B^n) \text{ if } k - \frac{n}{p} \geqslant l - \frac{n}{q} \text{ and } k \geqslant l.$$

In the case when $k - \frac{n}{p} = l - \frac{n}{q}$, we require further that $q < +\infty$ and $p > 1$. We also have embedding theorems into Hölder spaces:

$$L^p_k \hookrightarrow \mathcal{C}^{l,\alpha}(B^n) \text{ if } k - \frac{n}{p} > l + \alpha \text{ and } \alpha > 0.$$

Theorem 27.1.1. *These embedding theorems continue to hold if (X, g) has uniformly bounded geometry.*

Proof. We will only sketch a proof. Take a countable cover $\{U_i\}$ of X by geodesic balls such that any point $p \in X$ lies in at most N balls for a constant $N > 0$. Therefore, we have restriction maps:

$$\begin{tikzcd}
L^p_k(X,g) \arrow[r, hook, "i_1"] \arrow[d, dashed, "?"] & \hat{\bigoplus}_{i\in\Lambda} L^p_k(U_i) \arrow[r, "\text{take differences}"] \arrow[d, "emb"] & \hat{\bigoplus} L^p_k(U_i \cup U_j) \arrow[d] \\
L^q_l(X,g) \arrow[r, hook] & \hat{\bigoplus}_{i\in\Lambda} L^q_l(U_i) \arrow[r, "\text{take differences}"] & \hat{\bigoplus} L^q_l(U_i \cup U_j)
\end{tikzcd}$$

Here, $\hat{\bigoplus}$ denotes the L^p or L^q completion of the direct sum. The horizontal sequences in the first and second rows are exact.

The vertical map *emb* is a direct sum of Sobolev embeddings whose constants are uniformly bounded, due to the boundedness of the geometry. Since $L^p_k(X,g)$ is realized as a closed subspace of $\hat{\bigoplus}_{i\in\Lambda} L^p_k(U_i)$, the Sobolev embedding takes $L^p_k(X,g)$ into $L^q_l(X,g)$. Hence, the embedding theorem holds in the same range for the non-compact manifold (X,g). □

Remark. Even in the case of strict inequalities, i.e. $k-\frac{n}{p} > l-\frac{n}{q}$ and $k>l$, we no longer have compact embeddings, unless X itself is.

We also have the Sobolev multiplication theorem, e.g.

$$\begin{aligned} L^p_k(X)\otimes L^q_l(X) &\hookrightarrow L^r_m(X)\\ (f,g) &\mapsto fg. \end{aligned}$$

is continuous if $(k-\frac{n}{p})+(l-\frac{n}{q}) \geqslant r-\frac{n}{m}$ and $k,l>m$.

Theorem 27.1.2. *If $(k-\frac{n}{p})+(l-\frac{n}{q}) > r-\frac{n}{m}$ and $k,l>m$, i.e. these inequalities are strict, then for a fixed $f\in L^p_k$, the multiplication operator*

$$\begin{aligned} M_f : L^q_l(X) &\to L^r_m(X)\\ g &\mapsto fg. \end{aligned}$$

is compact.

Proof. Approximate f by a sequence of compactly supported functions f_i in L^p_k. Then

$$M_{f_i} \to M_f$$

in the operator norm. Since each M_{f_i} factorizes through Sobolev spaces on compact manifolds, M_{f_i} is a compact operator. Hence, M_f is compact. □

For more details on this subject, we recommend [4, Section 13.2].

27.2. The Spectrum Flow and the Index

Recall from the last lecture that if

$$A = B + a = B + (b+c)$$

with $(b,c)\in L^p_k(\mathbb{R}\times Y, (\Lambda^1_Y\oplus\Lambda^0_Y)\otimes \operatorname{ad} Q))$, then the linearized operator

$$D_A = \frac{\partial}{\partial t} + D_B + [a\wedge\cdot]$$

is Fredholm of index 0. Indeed if $a\equiv 0$, D_A is invertible. The difference term $[a\wedge\cdot]$ is a compact operator, so F_A is Fredholm of index 0.

Here is another way to think about the Fredholmness. Since $a\to 0$ as $t\to\pm\infty$, the operator

$$D_B + [a(t)\wedge\cdot]$$

is close to D_B in the operator norm as $t\to\pm\infty$. Hence, $D_B+[a(t)\wedge\cdot]$ is invertible.

We would like to understand the relation between the spectrum flow and the index of D_A in general, when $a \notin L^p_k(Z)$. We assume that $a \in L^p_{k,loc}(Z)$ and there exist translation invariant forms a_+, a_- in $\mathcal{C}^\infty(\mathbb{R} \times Y)$ such that

$$a - a_+|_{[0,\infty)\times Y} \in L^p_k([0,\infty) \times Y)$$
$$a - a_-|_{(-\infty,0]\times Y} \in L^p_k((-\infty,0] \times Y).$$

Lemma 27.2.1. *If $\frac{\partial}{\partial t} + D_B + \alpha_+$ and $\frac{\partial}{\partial t} + D_B + \alpha_-$ are both invertible, then the operator*

$$D_A = \frac{\partial}{\partial t} + D_B + a : L^p_k(\mathbb{R} \times Y) \to L^p_{k+1}(\mathbb{R} \times Y)$$

is Fredholm.

Proof. This is proved by the standard parametric patching argument. □

We are interested in the index of D_A.

Example 27.2.2. Consider the operator

$$D_+ = \frac{\partial}{\partial t} + \tanh(t)$$

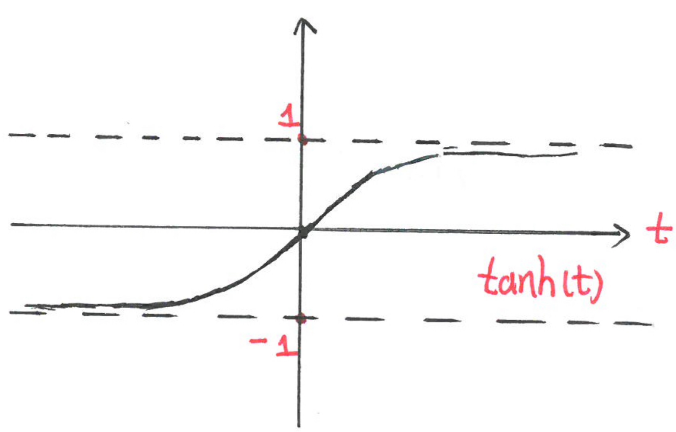
on the real line $\mathbb{R}$. As $t \to \pm\infty$, the operator approximates

$$\frac{\partial}{\partial t} + 1 \text{ and } \frac{\partial}{\partial t} - 1$$

respectively, which are invertible. We wish to compute its index as a Fredholm operator between spaces:

$$L^2_1(\mathbb{R}) \to L^2(\mathbb{R}).$$

Note that $\ker D_+$ is spanned by $\operatorname{sech}(t) \in L^2(\mathbb{R})$ while $\operatorname{Coker} D_+$ is spanned by



$\cosh(t) \notin L^2(\mathbb{R})$. Hence,

$$\operatorname{Ind} D_+ = 1$$

Similarly, we define

$$D_- = \frac{\partial}{\partial t} - \tanh(t)$$

and $\operatorname{Ind} D_- = -1$.

In general, the index is the net number of eigenvalues going from negative to positive. To prove "SF=Ind" in general, we use the fact that the index is a homotopy invariant. When fixing $a_\pm$, we are allowed to homotopy the path $a(t)$. The limits $a_\pm$ are allowed to change as long as they stay invertible during a homotopy. We apply perturbations of $a_\pm$ such that

$$D_B + a_\pm$$

have the same eigenvectors. Then we reduce the problem to model examples above.

Example 27.2.3. The spectral flow along a closed loop of operators might not be zero. Consider the operator

$$D_\alpha = i\frac{\partial}{\partial t} + \alpha : L^2_1(S^1, \mathbb{C}) \to L^2(S^1, \mathbb{C}).$$

for $\alpha \in [0, 1]$. Then $\operatorname{Spec} D_\alpha = \mathbb{Z} + \alpha$. When $\alpha = 0, 1$, D_1 is conjugate to D_0,

$$i\frac{\partial}{\partial t} + 1 = e^{it}(i\frac{\partial}{\partial t})e^{-it}.$$

So far, $\{D_\alpha\}_{\alpha\in[0,1]}$ does not form a closed loop. Both $g_1 = \mathrm{Id}$ and

$$g_0(f(t)) = f(t)e^{it}, t \in S^1.$$

are unitary operators on $L^2(S^1, \mathbb{C})$. Here, we need Kuiper's theorem:

Theorem 27.2.4 (Kuiper)**.** *For any separable infinite dimensional Hilbert space H, the unitary group $U(H)$ is contractible.*

Proof. The original proof of Kuiper is hard to follow. Tom recommended [**?**]:

- Mitjagin, B. S.
  The homotopy structure of a linear group of a Banach space.
  Uspehi Mat. Nauk 25 (1970), no. 5 (155), 63106.

in which the Kuiper's theorem was generalized to other suitable Banach spaces. This theorem is however, not valid for arbitrary Banach spaces. □

By Kuiper's theorem, g_0 and g_1 is connected by a path of unitary operators g_β. Then the family of operators

$$D'_\beta = g_\beta(\frac{\partial}{\partial t})g_\beta^{-1}$$

will take D_1 back to D_0 without changing the spectrum. By composing D_α with this family, we obtain a closed loop with non-trivial spectrum flow.

27.3. The Spectrum Flow in the Instanton Floer Homology

Over the trivial principal bundle $Q = SU(2)\times Y \to Y$, the Chern-Simons functional defines a circle valued functional:

$$\mathcal{CS} : \mathscr{A}/\mathscr{G} \to \mathbb{R}/c\mathbb{Z}\ (c > 0).$$

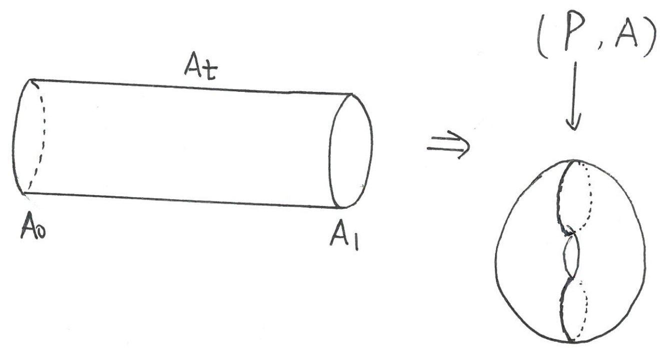
In this case, $\mathscr{G} = \mathrm{Map}(Y, SU(2))$ and $\pi_0(\mathscr{G})$ is classified by the degree of the map:

$$u : Y \to SU(2).$$

Therefore, $\pi_1(\mathscr{A}/\mathscr{G}) = \mathbb{Z}$. It is important to understand what is the spectral flow around a generator of π_1.

Let $\gamma : [0,1] \to \mathscr{A}/\mathscr{G}$ to be a closed loop. Let $A_0 = \gamma(0)$ and $A_1 = \gamma(1)$. Then $u \cdot A_0 = A_1$ for a gauge transformation $u : Y \to SU(2)$. Consider the mapping torus:

$$\begin{array}{c} P = [0,1] \times Q/(u\cdot p, 0) \sim (p,1) \\ \Big\downarrow{\scriptstyle \pi} \\ X = S^1 \times Y. \end{array}$$



The index of the operator on the cylinder $\mathbb{R}\times Y$:

$$\frac{\partial}{\partial t} + D_{A(t)}$$

equals the expected dimension of the moduli space on P. Since $\sigma(X) = \chi(X) = 0$,

$$SF = \mathrm{Ind} = 8c_2(P) = 8\deg(u).$$

The constant 8 will remain unchanged if we replace $SU(2)$ by any other compact Lie group G.

Another important fact is that $\mathcal{CS}$ detects π_1, i.e.

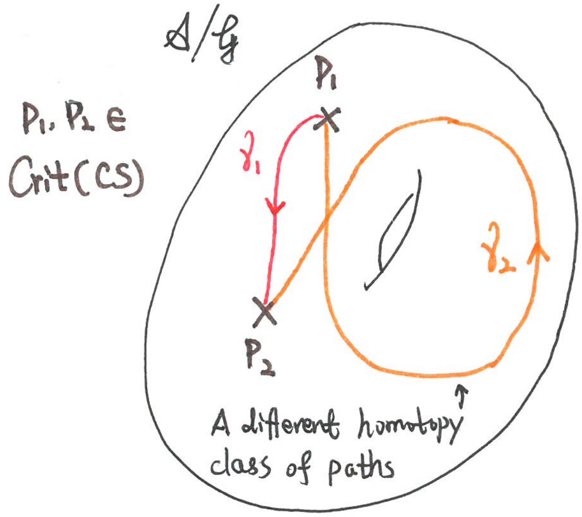
$$\mathcal{CS}_* : \pi_1(\mathscr{A}/\mathscr{G}) \to \pi_1(S^1),$$

is an isomorphism.

Remark. It becomes more complicated when it is a $PU(n)$-bundle. However, the picture is more or less the same if we stick to the determinant-1 gauge group. □

When counting differentials in the Floer theory, we focus on moduli spaces of formal dimension 1. Hence,

$$\begin{aligned} &\text{Pinning down the dimension,} \\ \Rightarrow &\text{Pinning down the homotopy classes of paths,} \\ \Rightarrow &\text{Pinning down the change of } \mathcal{CS}. \end{aligned}$$



Remark. The total change $\Delta_\gamma \mathcal{CS}$ of the Chern-Simons functional is proportional to the spectrum flow.

As a consequence, we do not need Novikov rings for the Floer homology.

This phenomenon is comparable with the monotonicity condition in the symplectic Floer theory, where

$$\mathcal{CS} \xrightarrow{\text{replaced by}} \text{the symplectic action}$$

$$SF \xrightarrow{\text{replaced by}} c_1(M, J).$$

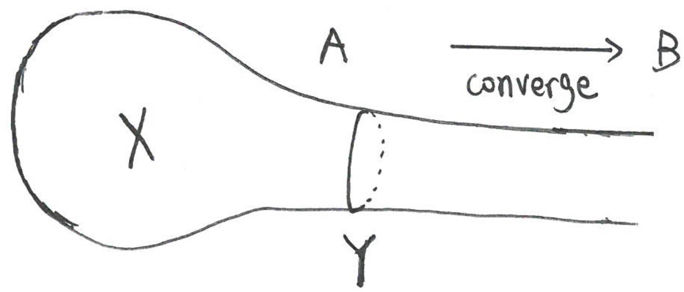
The monotonic condition then states that $[\omega] = \lambda c_1(M, J) \in H^2(M, \mathbb{Z})$ for some $\lambda \in \mathbb{R}$.

Lecture 28. 4-Manifolds with Cylindrical Ends

28.1. 4-Manifolds with Cylindrical Ends

Let $(X, \partial X = Y)$ be a compact 4-manifold with boundary. We obtain a manifold with cylindrical end by attaching $[0,\infty) \times Y$:

$$X^* = X \cup [0,\infty) \times Y.$$



We study instantons on X^*.

Fix a reference connection A_0 in $P \to X^*$ such that

$$A_0\Big|_{[0,\infty)\times Y} = \pi^* B$$

where B is a fixed flat connection in $Q \to Y$, i.e. $F_B = 0$. Let

$$\mathscr{A} = A_0 + L^2_{k,A_0}(X, T^*X \otimes \operatorname{ad} P)$$
$$\mathscr{G} = \{u \in \operatorname{End}(V) : 1 - u \in L^2_{k+1,A_0}(X, \operatorname{End} V), u^*u = 1, \det(u) = 1\},$$

where V is the vector bundle induced from the fundamental representation of $G = SU(2)$. To define the L^2_k-norm, we need a covariant derivative of $\operatorname{ad} P \to X$. We used the reference connection A_0:

$$\|a\|^2_{L^2_k(X)} = \int_X |\nabla^k_{A_0} a|^2 + \cdots + |a|^2$$

The gauge group $\mathscr{G}$ acts on $\mathscr{A}$ smoothly, if $k > 1$. Assume B is a non-generate critical point of $\mathcal{CS}$, i.e.

(1) $H^0(Y, \operatorname{ad}_B) = \{0\}$, which means B is irreducible as a connection.
(2) $H^1(Y, \operatorname{ad}_B) = \{0\}$, which means B is isolated as a critical point.

Recall that $u \cdot A = A + u d_A u^{-1}$ for $u \in \mathscr{G}$ and $A \in \mathscr{A}$.

Exercise 28.1.1. *If $d_{A_0} u \in L^2_k$, then we can find $u_0 \in \mathscr{G}(Y)$ such that $d_B u_0 = 0$ and $u - \pi^* u_0 \in L^2_{k+1,A_0}|_{[0,\infty)\times Y}$.*

Since B is irreducible, in fact, $u_0 \equiv \pm 1 \in Z(G)$.

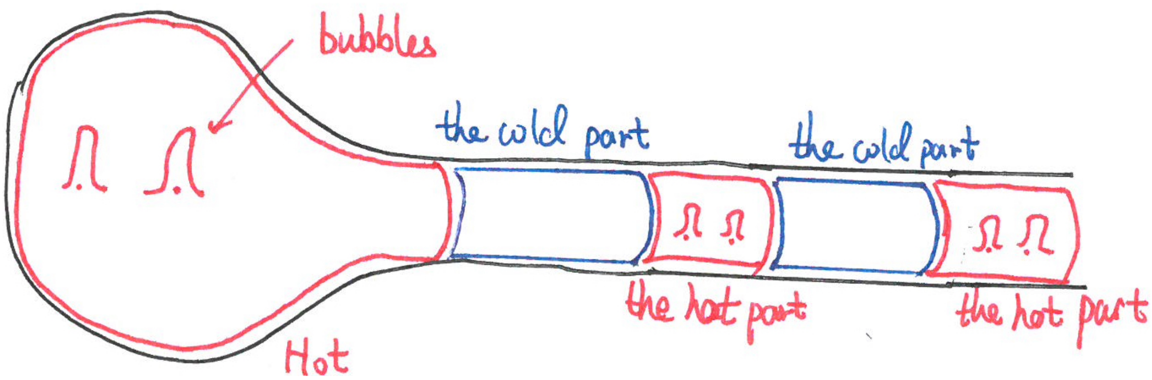
The map

$$A \mapsto F_A^+$$
$$L_k^2 \to L_{k-1}^2$$

is a smooth map. Provided that B is non-degenerate,

$$(d_A^+, d_A^*) : L_k^2 \to L_{k-1}^2 \oplus L_{k-1}^2$$

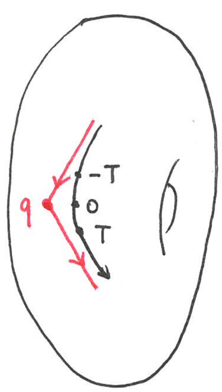
is a Fredholm map. To pursue analogue of Morse homology, we have to work with a new phenomenon: bubbling. The energy can slide off along the cylindrical end for a sequence of ASD connections A_i:



Need to understand small energy solutions along a long cylinder $Z = [-T, T] \times Y$, over which we have

$$\int_Z |F_{A_i}|^2 \to 0,$$

so the sequence $[A_i]$ converges to a flat connection $\pi^* B'$. Given this, the bubbling phenomenon can not occur on Z. A small-energy solution on a long cylinder means a new critical point (in our case, $[B']$) is nearby.



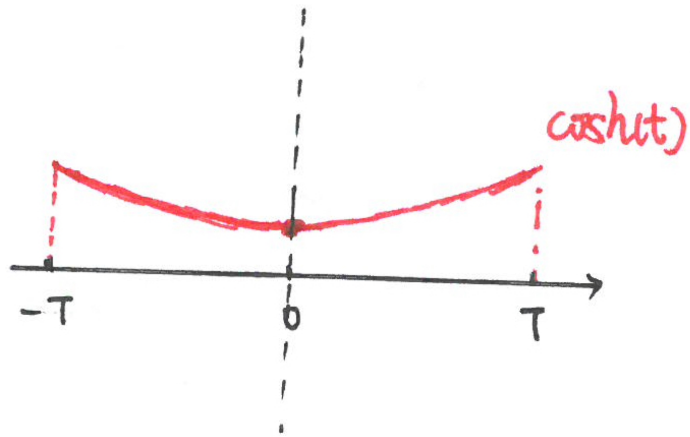
Recall the estimate from the finite dimensional Morse theory. If $\gamma(t)$ is a downward gradient flowline of a Morse function $f : M \to \mathbb{R}$ and $\gamma(t)$ is close to a critical point c when $t \in [-T, T]$, then

$$dist(\gamma(t), c) \leqslant C \cdot \frac{\cosh(\lambda t)}{\cosh(\lambda T)}$$

for some $C, \lambda > 0$. It follows from the differential inequality

$$\frac{\partial}{\partial t}(f \circ \gamma)(t) = -|\nabla f \circ \gamma|^2 \leqslant -\lambda f.$$

for some $\lambda > 0$ when $\gamma(t)$ is close to c.



The same thing holds for the Chern-Simons functional. Over the cylinder $[-T, T] \times Y$, write

$$A(t) = \check{A}(t) + \frac{\partial}{\partial t} + cdt.$$

where $\check{A}(t) = A(t)|_{\{t\}\times Y}$ is the restriction of A at the slice $\{t\} \times Y$. Then

$$\begin{aligned} &- 2\mathcal{CS}(\check{A}(T)) + 2\mathcal{CS}(\check{A}(-T)) \\ =&\int_Z \mathrm{tr}(F_A \wedge F_A) = -\int_Z \mathrm{tr}(F_A \wedge *F_A) = \int_Z \|F_A\|^2 \\ =&2\int_{[-T,T]} \int_{\{t\}\times Y} |F_{\check{A}(t)}|^2. \end{aligned} \tag{46}$$

At the last step, we used the fact that

$$2\|F_{\check{A}(t)}\|^2 = \|F_A\|^2,$$

because F_A is ASD, so $F_A = F_{\check{A}(t)} + *_4 F_{\check{A}(t)}$. The differential version of (46) states that

$$\frac{\partial}{\partial t}\mathcal{CS}(\check{A}(t)) = -\int_Y |F_{\check{A}(t)}|^2.$$

Assume the critical connection B' is **non-generate**. Write

$$\check{A}(t) = B' + b(t)$$

for $b(t) \in \Gamma(Y, \operatorname{ad} Q)$. Then by choosing a proper gauge, we have

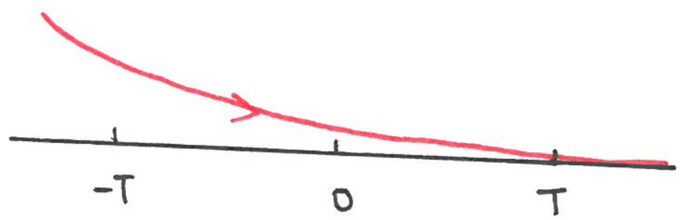
$$\int_Y \|F_B\|^2 \geqslant \lambda^2 \|b\|^2_{L^2_1(Y)} \tag{47}$$

If $\|b\|_{L^2_1(Y)}$ is small enough, then

$$|\mathcal{CS}(B' + b) - \mathcal{CS}(B')| = |\int_Y \operatorname{tr}(b \wedge db + \frac{2}{3} b \wedge b \wedge b)| \leqslant c\|b\|^2_{L^2_1(Y)}.$$

In conclusion, we have

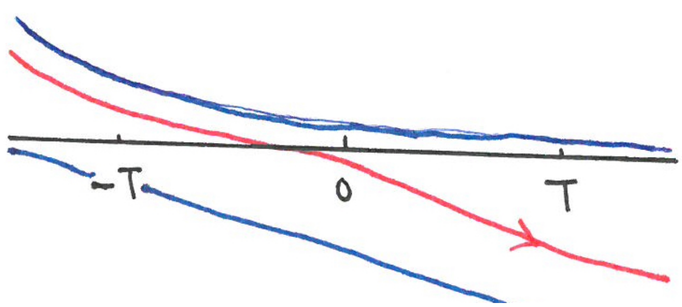
$$\frac{\partial}{\partial t}(\mathcal{CS}(\check{A}(t)) - \mathcal{CS}(B')) \leqslant -\frac{\lambda^2}{c}|\mathcal{CS}(\check{A}(t)) - \mathcal{CS}(B')|. \tag{48}$$



If it is an infinite half cylinder $Z_+ = [-T, \infty) \times Y$, then $\mathcal{CS}(\check{A}(t)) - \mathcal{CS}(B') \geqslant 0$ for any $t \in [0, \infty)$. The differential inequality (48) implies

$$0 \leqslant \mathcal{CS}(\check{A}(t)) - \mathcal{CS}(B') \leqslant (\mathcal{CS}(\check{A}(-T)) - \mathcal{CS}(B'))e^{-(t+T)\lambda^2/c}.$$

for any $t \in [-T, \infty)$. For a cylinder Z of finite length, the value $\mathcal{CS}(\check{A}(t)) - \mathcal{CS}(B')$ is in the sandwich between two exponential decay functions.



The exponential decay of $\mathcal{CS}(\check{A}(t))$ allows us to conclude

$$A \to B'$$

in $L^2_{k,loc}$-topology for any $k > 0$. Usually, we can not do bootstrapping directly from (L^2_1-connection, L^2_2-gauge transformations) in dimension 4. However, this is allowed when the energy is sufficiently small.

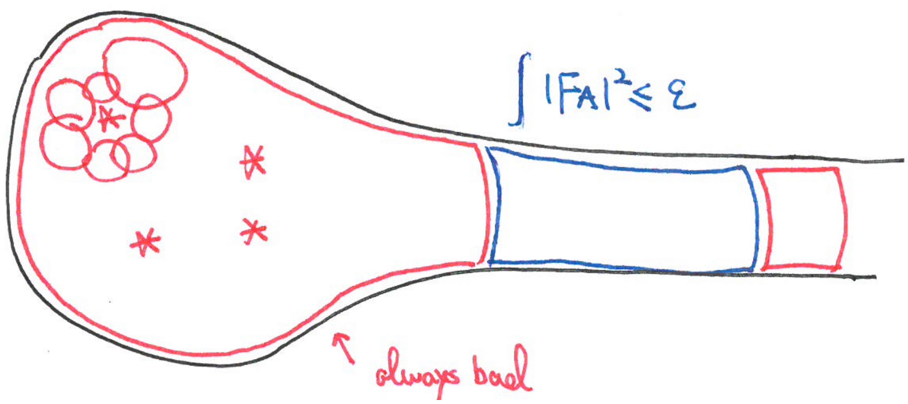
Remark. The estimate (48) is gauge invariant. However, its proof invokes a good choice of gauge such that (47) holds.

28.2. Uhlenbeck's Compactness Theorem on X^*

Fix the total energy on X^*:

$$\int_{X^*} |F_A|^2 < N.$$

Let $I_r = [r-1, r+1]$ for each $r \in \mathbb{Z}$. For any sequence of ASD connections $[A_i]$ on X^*, we can find a finite collection of points $\{x_i\}_{1 \leqslant i \leqslant m} \subset X$, a finite collections of integers $\{r_i\}_{i \in J}$ and a subsequence of $\{[A_i]\}$ such that



(1) Both m and $|J|$ are bounded by a constant multiple of N.
(2) Away from $\{x_i\}$, X is covered by a countable collection of geodesic balls $\{B_k\}$ such that

$$\int_{B_k} |F_{A_i}|^2 \leqslant \epsilon$$

for any A_i in that subsequence.
(3) For any $r \neq r_i$ and $r \geqslant 0$, we have an estimate:

$$\int_{I_r \times Y} |F_{A_i}|^2 \leqslant \epsilon.$$

for any A_i in that subsequence.

The constant ϵ comes from Uhlenbeck's fundamental lemma. The union

$$\{x_1, \cdots, x_m\} \bigcup_{j \in J} I_{r_j} \times Y \subset X$$

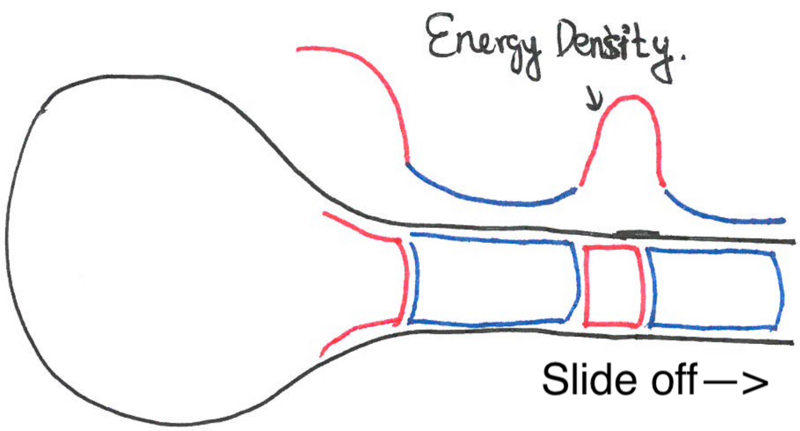
is the bad region, where the bubbling could happen. However, away from the bad region, we have convergences in $L^2_{k,loc}$-topology or convergence to a broken flowline of $\mathcal{CS}$.

To make it more concrete, let $Z = \mathbb{R} \times Y$. For any $[B_-], [B_+] \in \mathrm{Crit}(\mathcal{CS})$, define

$$\mathcal{M}([B_-],[B_+]) = \{[A] \text{ on } \mathbb{R} \times Y : \int |F_A|^2 < \infty, F_A + *_4 F_A = 0, \lim_{t \to \pm\infty} \check{A}(t) = [B_\pm]\}.$$

In fact, $\mathcal{M}([B_-],[B_+])$ is a disjoint union

$$\begin{aligned}\mathcal{M}([B_-],[B_+]) &= \bigcup_{k \geqslant 0} \mathcal{M}_k([B_-],[B_+]) \\ &= \bigcup_{k \geqslant 0} \{[A] \in \mathcal{M}([B_-],[B_+]) : \int_Z \mathrm{tr}(F_A \wedge F_A) = k\}.\end{aligned}$$



Let $\widetilde{\mathcal{M}}_k([B_-],[B_+]) = \mathcal{M}_k([B_-],[B_+])/\mathbb{R}$. Define the moduli space of ideal instantons:

$$\begin{aligned}\mathcal{IM}_k([B_-],[B_+]) = \mathcal{M}_k([B_-],[B_+]) &\\ \bigcup \mathcal{M}_{k-1}([B_-],[B_+]) \times Z &\\ \bigcup \mathcal{M}_{k-2}([B_-],[B_+]) \times \mathrm{Sym}^2 Z &\\ \bigcup \cdots .&\end{aligned}$$

Let $\widetilde{\mathcal{IM}}_k([B_-],[B_+]) = \mathcal{IM}_k([B_-],[B_+])/\mathbb{R}$.

(Tom did not have enough sleep last night, so he did not finish this part. He will write down a compactification of $\mathcal{M}(X,[B])$ in the next lecture).

Lecture 29. Instanton Floer Homology

29.1. Compactification of the Moduli space

Let $Q = G \times Y \to Y$ be the trivial principal G-bundle over Y and $Z = \mathbb{R} \times Y$ be the cylinder. Suppose the Chern-Simons functional $\mathcal{CS}$ descends to a non-generate Morse function on the quotient space:

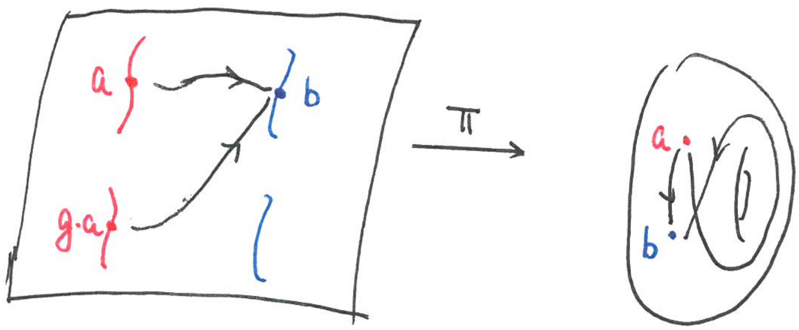
$$\mathscr{B}_Q = \mathscr{A}_Q / \mathscr{G}_Q.$$

Let $\alpha, \beta \in \mathrm{Crit}(\mathcal{CS})$ and $a, b \in \mathscr{A}_Q$ be their lifts, i.e.

$$[a] = \alpha, [b] = \beta.$$

Fix a reference connection A_0 on Z such that

$$A_0\Big|_{(-\infty,-T]\times Y} = \pi^* a,\ \ A_0\Big|_{[T,\infty)\times Y} = \pi^* b.$$



Consider the configuration space $\mathscr{A}_{P,A_0}(a,b)$ and its quotient space

$$\mathscr{A}_{P,A_0}(a,b)/\mathscr{G} = \mathscr{B}_\gamma(\alpha,\beta).$$

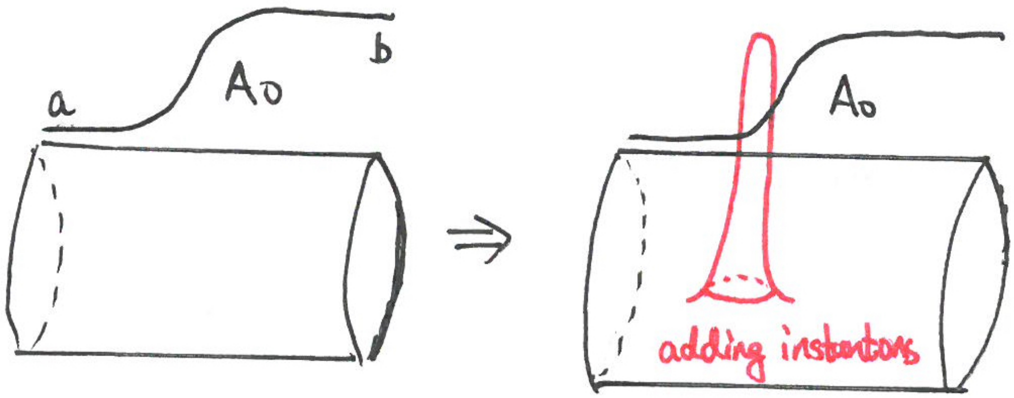
Any element in $\mathscr{A}_{P,A_0}(a,b)$ determines the same homotopy class of paths in $\mathscr{B}_Q$ that connects α and β. Use γ to denote the homotopy class of path determined by A_0:

$$\gamma \in \pi_1(\alpha, \mathscr{B}_Q, \beta).$$

Remark. Note that $\pi_1(\alpha, \mathscr{B}_Q, \beta)$ is a $\mathbb{Z}$-torsor, i.e. it admits an action by integers. It is defined by "adding instantons" at some point x in Z. For the principal bundle $P_k \to S^4$ with $c_2(P_k) = k$, pick any connection A_k. By taking the connected sum of connections $A_0 \#_x A_k$, we obtain a new homotopy class of paths: $\gamma + k$. Here, k is allowed to be any integer and A_k is not necessarily an ASD connection. When $k > 0$ and A_k is ASD, this helps us construct new instantons out of a given ASD connection when Z is closed. $\square$

In particular, we look at finite energy solutions to the ASD equation inside $\mathscr{B}_\gamma(\alpha,\beta)$:

$$\mathcal{M}_\gamma(\alpha,\beta) \subset \mathscr{B}_\gamma(\alpha,\beta).$$



Define the moduli space of ideal instantons of homotopy class γ:

$$\begin{aligned}\mathcal{IM}_\gamma(\alpha,\beta) = \mathcal{M}_\gamma(\alpha,\beta)&\\ &\bigcup \mathcal{M}_{\gamma-1}(\alpha,\beta)\times Z\\ &\bigcup \mathcal{M}_{\gamma-2}(\alpha,\beta)\times \operatorname{Sym}^2 Z\\ &\bigcup \cdots,\end{aligned}$$

and its unparameterized version $\widetilde{\mathcal{IM}}_\gamma(\alpha,\beta) = \mathcal{IM}_\gamma(\alpha,\beta)/\mathbb{R}$. This space is not compact so far. We need to add broken trajectories. Define

$$\overline{\mathcal{IM}}_\gamma(\alpha,\beta) = \bigcup_{r\geqslant 0}\ \bigcup_{\substack{\alpha_1,\cdots,\alpha_r\\ \in \operatorname{Crit}(\mathcal{CS})}}\ \bigcup_{\substack{\gamma_0,\cdots\gamma_r\\ \gamma_i\in\pi_1(\alpha_i,\mathscr{B}_Q,\alpha_{i+1})\\ \sum\gamma_i=\gamma}} \widetilde{\mathcal{IM}}_\gamma(\alpha,\alpha_1)\times\widetilde{\mathcal{IM}}_\gamma(\alpha_1,\alpha_2)\times\cdots\times\widetilde{\mathcal{IM}}_\gamma(\alpha_r,\beta).$$

When r is fixed, we assumed $\alpha_0=\alpha$ and $\alpha_{r+1}=\beta$. The union is over all homotopy classes of chains:

$$\alpha=\alpha_0\xrightarrow{\gamma_0}\alpha_1\xrightarrow{\gamma_1}\alpha_2\cdots\xrightarrow{\gamma_r}\alpha_{r+1}=\beta.$$

The first union is over the length of chains. The second union is over possible critical points α_i that appear in this chain. The last is over all possible homotopy classes of path between two adjacent critical points. Finally, we require $\sum\gamma_i=\gamma$.

When $r=0$, this union contains a single piece: $\widetilde{\mathcal{IM}}_\gamma(\alpha,\beta)=\mathcal{IM}_\gamma(\alpha,\beta)/\mathbb{R}$, which corresponds to unbroken trajectories. It is also the top stratum of the stratified space $\overline{\mathcal{IM}}_\gamma(\alpha,\beta)$. The length r records the number of breaks in a broken trajectory. The expected dimension will drop by 1 for each break. The top dimension of spaces in

$$\bigcup_{\substack{\alpha_1,\cdots,\alpha_r\\ \in\operatorname{Crit}(\mathcal{CS})}}$$

is $\dim \widetilde{\mathcal{IM}}_\gamma(\alpha,\beta) - r$. It contains lower dimensional strata from the bubbling phenomenon:

$$\mathcal{M}_{\gamma_i - k}(\alpha_i, \alpha_{i+1}) \times \mathrm{Sym}^k(Z)/\mathbb{R}.$$

Each bubble will drop the dimension by 4.

This huge union $\overline{\mathcal{IM}}_\gamma(\alpha,\beta)$ is compact.

29.2. A criterion to avoid reducible flat connections

So far we have made no perturbation of the Chern-Simons functional and have assumed that all critical points are non-generate.

In general, a perturbation is alway needed. When there is none, critical points come from representations of the fundamental group, so they form a compact set. However, the compactness of critical points becomes an issue if a perturbation is made.

If our critical points are non-degenerate, then finite energy flowlines always have finite length and $\breve{M}_\gamma(\alpha,\beta)$ is dense in $\overline{\mathcal{IM}}_\gamma(\alpha,\beta)$.

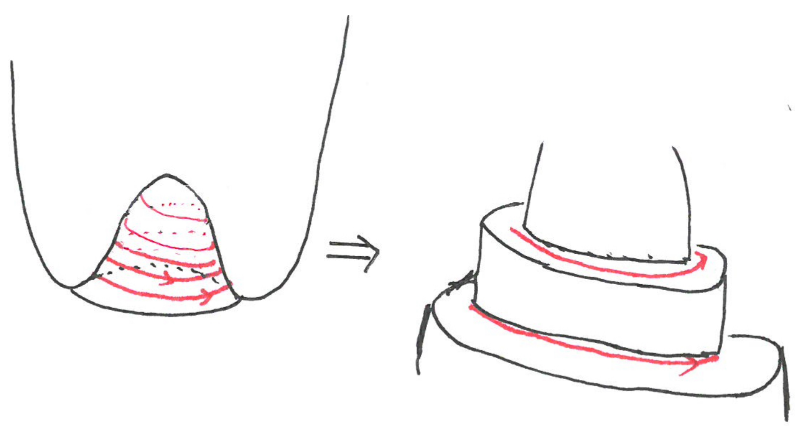
When a Morse function f is degenerate, yet real analytic on a finite dimensional compact manifold M, a flowline alway has a finite length, by Lojasiewicz's inequality. A flowline might not have exponential decay to the critical locus; only a power law decay is guaranteed.

Lojasiewicz's inequality states that if f real analytic and $\gamma(t)$ is close to a critical point c, then

$$|\nabla f(\gamma(t))|^\theta \geqslant |f(\gamma(t)) - f(crit)|$$

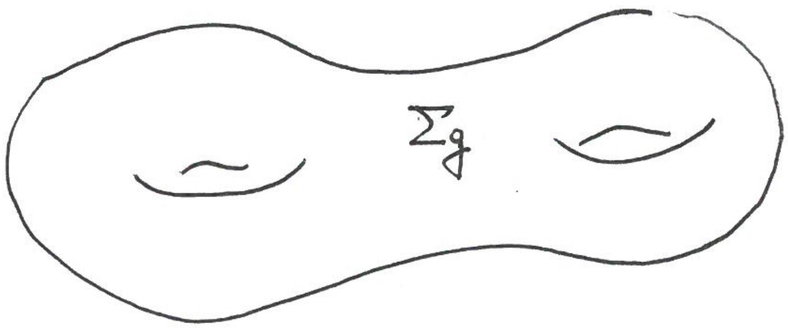
for some fixed $1 < \theta \leqslant 2$. When f is Morse, then $\theta = 2$.

It becomes worse if f is not real analytic or Morse. An example is illustrated as follows. Starting with a surface of revolution, we drill a groove all the way down to the critical circle, spiral around the hill. The height function f is certainly not real analytic on this surface. A downward gradient flow of f travels along the groove and does not converge to any point on the circle.



When $G = SU(2)$, $\mathrm{Crit}(\mathcal{CS})$ always contains reducible connections (the trivial connection). The non-degeneracy condition is never satisfied for $\mathcal{CS}$.

If $G = PU(2) = SO(3)$, then it becomes possible. Consider a genus-g ($g \geqslant 1$) surface Σ_g and an $SO(3)$ bundle $Q \to \Sigma_g$ with $w_2(Q) \neq 0$. We work with the determinant-1 gauge group.



A flat connection is a group homomorphism $\rho : \pi_1(\Sigma) \to SO(3)$. We need to determine when the induced bundle

$$\begin{array}{ccc} \mathfrak{so}(3) & \longrightarrow & \operatorname{ad}\rho \\ & & \downarrow \\ & & \Sigma_g. \end{array}$$

has $w_2(\operatorname{ad}\rho) \neq 0$. Note that

$$\pi_1(\Sigma_g) = \langle a_1, b_1, \cdots, a_g, b_g \big| \prod_{i=1}^{g}[a_i, b_i] = 1 \rangle.$$

Let $A_i = \rho(a_i) \in SO(3)$ and $B_i = \rho(b_i) \in SO(3)$. Then $\prod_{i=1}^{g}[A_i, B_i] = 1 \in SO(3)$. Let

$$\tilde{A}_i \in \pi^{-1}(A_i), \tilde{B}_i \in \pi^{-1}(B_i)$$

be their lift in $SU(2)$. Then

$$w(\rho) := \prod_{i=1}^{g}[\tilde{A}_i, \tilde{B}_i] = \pm 1 \in Z(SU(2)).$$

Apparently, $w(\rho)$ does not depends on the choice of $\tilde{A}_i$ and $\tilde{B}_i$. In fact,

$$w(\rho) = w_2(\operatorname{ad}\rho) \in \mathbb{Z}/2\mathbb{Z}.$$

Let us define

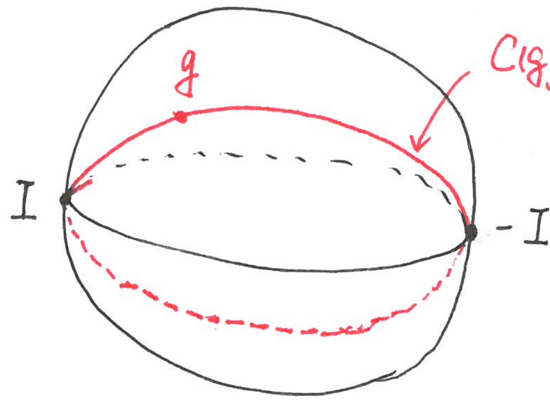
$$\mathcal{R}^{odd}(\Sigma_g) = \{\{\tilde{A}_i, \tilde{B}_i\}_{1 \leqslant i \leqslant g} : \tilde{A}_i, \tilde{B}_i \in SU(2), \prod_{i=1}^{g}[\tilde{A}_i, \tilde{B}_i] = -1\}/SU(2).$$

Note that $\mathcal{R}^{odd}(\Sigma_g)$ is a cover over

$$\{\rho : \pi_1(\Sigma) \to SO(3) : w_2(\operatorname{ad}\rho) \neq 0\}.$$

Theorem 29.2.1. *$\mathcal{R}^{odd}(\Sigma_g)$ contains no reducible representations.*

Proof. If $h \in SU(2)\backslash Z(SU(2))$, then the commutant of h is the great circle S_h passing through h and ± 1. If h commutes with $\tilde{A}_i, \tilde{B}_i$ for each $1 \leqslant i \leqslant g$, then $\tilde{A}_i, \tilde{B}_i$ lie in S_h.



Hence, $\{\tilde{A}_i, \tilde{B}_i\}_{1\leqslant i\leqslant g}$ has non-trivial stabilizers (larger than $Z(SU(2))$) if and only if $\tilde{A}_i, \tilde{B}_i$ are simultaneously diagonalizable. In particular,

$$[\tilde{A}_i, \tilde{B}_i] = 1$$

for each $1 \leqslant i \leqslant g$. This contradicts the assumption that $\prod_{i=1}^{g}[\tilde{A}_i, \tilde{B}_i] = -1$. $\square$

This theorem gives a convenient criterion to exclude reducible connections:

- An $SO(3)$ bundle $Q \to Y$ with $w_2(Q) \neq 0$ is called good if we can find a oriented embedded surface $\Sigma_g \subset Y$ with $g \geqslant 1$ such that $w_2(Q)[\Sigma] \neq 0$. Then Q does not admit reducible flat connections.

Remark. The construction of $w(\rho)$ generalizes to any $PU(n)$ bundles, where the invariant is defined in

$$\beta(\rho) \in H^2(\Sigma, \mathbb{Z}/n\mathbb{Z}).$$

Using the short exact sequence

$$1 \to Z(SU(n)) = \mathbb{Z}/n\mathbb{Z} \to SU(n) \to PU(n) \to 0,$$

we choose the lift of A_i in $SU(n)$.

Example 29.2.2. Let $g = 1$ and $\Sigma_g = \mathbb{T}^2$. Take $G = SU(2) = Sq(1)$. If $\tilde{A}_1 = I$ and $\tilde{B}_1 = J$, then

$$[\tilde{A}_1, \tilde{B}_1] = IJI^{-1}J^{-1} = -1.$$

29.3. A pre-definition of Floer Homology

Define the chain group as

$$C(Y,Q) = \bigoplus_{\alpha \in \mathrm{Crit}(\mathcal{CS})} \mathbb{Z}_2 \alpha$$

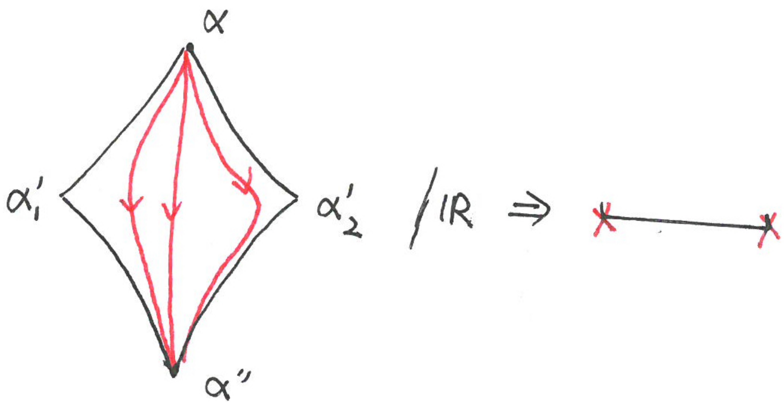
and the differential by

$$\begin{aligned} \partial : C(Y,Q) &\to C(Y,Q) \\ \alpha &\mapsto \sum_{\beta \in \mathrm{Crit}(\mathcal{CS})} \sum_{\substack{\gamma \in \pi_1(\alpha, \mathscr{B}_Q, \beta) \\ sf_\gamma(\alpha,\beta)=1}} \# \check{M}_\gamma(\alpha,\beta) \cdot \beta. \end{aligned}$$

We have to verify $\partial^2 = 0$. This is proved by examining the compactification of the moduli space when $sf_\gamma(\alpha,\alpha') = 2$. In this case,

$$\breve{\mathcal{I}\mathcal{M}}_\gamma(\alpha,\alpha') = \breve{\mathcal{M}}_\gamma(\alpha,\alpha') \bigcup_{\substack{\gamma_1 \in \pi_1(\alpha, \mathscr{B}_Q, \beta) \\ \gamma_2 \in \pi_1(\beta, \mathscr{B}_Q, \alpha') \\ sf_{\gamma_1} = sf_{\gamma_2} = 1}} \breve{\mathcal{M}}_{\gamma_1}(\alpha,\beta) \times \breve{\mathcal{M}}_{\gamma_2}(\beta,\alpha').$$

A gluing theorem states that this is a compact 1-manifold with boundaries, the number of boundary components has to be even. Hence, $\partial^2 = 0$.



Lecture 30. **Functoriality**

30.1. Instanton Floer Homology

Let $Q = G \times Y \to Y$ be the trivial principal G-bundle over Y. Assume there is no reducible flat connections and all flat critical points are non-generate. Take

$$\alpha, \beta \in \mathrm{Crit}(\mathcal{CS}).$$

In the last lecture, we defined the union

$$\overline{\mathcal{IM}}_\gamma(\alpha,\beta) = \bigcup_{r \geqslant 0} \bigcup_{\substack{\alpha_1,\cdots,\alpha_r \\ \in \mathrm{Crit}(\mathcal{CS})}} \bigcup_{\substack{\gamma_0,\cdots\gamma_r \\ \gamma_i \in \pi_1(\alpha_i, \mathscr{B}_Q, \alpha_{i+1}) \\ \sum \gamma_i = \gamma}} \widecheck{\mathcal{IM}}_\gamma(\alpha,\alpha_1) \times \widecheck{\mathcal{IM}}_\gamma(\alpha_1,\alpha_2) \times \cdots \times \widecheck{\mathcal{IM}}_\gamma(\alpha_r,\beta).$$

For any $\gamma \in \pi_1(\alpha, \mathscr{B}_Q, \beta)$, $\overline{\mathcal{IM}}_\gamma(\alpha,\beta)$ is a topological space stratified by manifolds.

Definition 30.1.1. A topology space X is stratified by manifolds if we can find a closed subset $X_i \subset X$ for each $i \geqslant 0$ such that $X_i \subset X_{i+1}$, $X = \bigcup_{i \geqslant 0} X_i$ and

$$S_i := X_i \backslash X_{i-1} \text{ (might be empty)}$$

is a manifold of pure dimension i.

A stratum of $\overline{\mathcal{IM}}$ is a union of products of

$$\mathrm{Sym}^{k_i} Z \times \check{M}_{\gamma_i - k_i}(\alpha_i, \alpha_{i+1})$$

for each $0 \leqslant i \leqslant r$ and some $k_i \geqslant 0$. The dimension of such a product is

$$\dim = SF_\gamma(\alpha,\beta) - (r+1) - 4(k_0 + k_1 + \cdots + k_r)$$

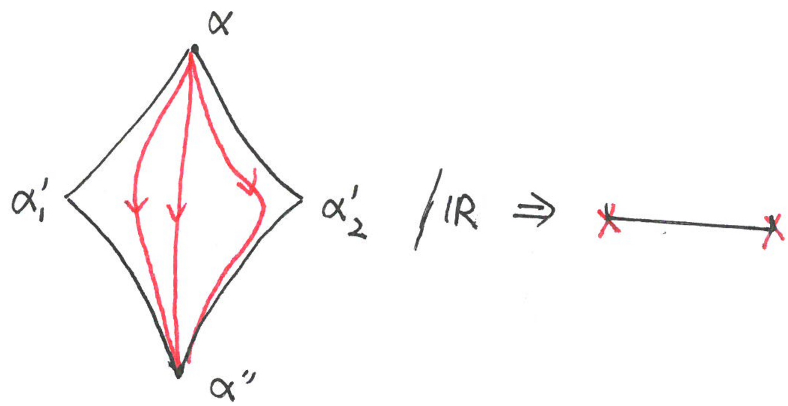
assuming transversality. Strictly speaking, we also need to stratify each symmetric product $\mathrm{Sym}^k Z$, which is omitted here.

Recall that the Floer Chain complex is define by

$$\begin{aligned} C(Y,Q) &= \bigoplus_{\alpha \in \mathrm{Crit}(\mathcal{CS})} \mathbb{Z}_2 \alpha \\ \partial : C(Y,Q) &\to C(Y,Q) \\ \alpha &\mapsto \sum_{\beta \in \mathrm{Crit}(\mathcal{CS})} \sum_{\substack{\gamma \in \pi_1(\alpha, \mathscr{B}_Q, \beta) \\ sf_\gamma(\alpha,\beta) = 1}} \# \check{M}_\gamma(\alpha,\beta) \cdot \beta. \end{aligned}$$

We prove $\partial^2 = 0$ by considering the moduli space when $sf_\gamma(\alpha,\alpha'') = 2$. In this case, the compactification $\overline{\mathcal{IM}}_\gamma(\alpha,\alpha'')$ contains two pieces: $r = 0, 1$. The bubbles can not occur for dimension reasons. Hence,

$$\widecheck{\mathcal{IM}}_\gamma(\alpha,\alpha') = \widecheck{\mathcal{M}}_\gamma(\alpha,\alpha') \bigcup_{\substack{\gamma_1 \in \pi_1(\alpha, \mathscr{B}_Q, \beta) \\ \gamma_2 \in \pi_1(\beta, \mathscr{B}_Q, \alpha') \\ sf_{\gamma_1} = sf_{\gamma_2} = 1}} \widecheck{\mathcal{M}}_{\gamma_1}(\alpha,\beta) \times \widecheck{\mathcal{M}}_{\gamma_2}(\beta,\alpha').$$



A gluing theorem states that $\widetilde{\mathcal{IM}}_\gamma(\alpha, \alpha')$ is a compact manifold with boundaries, so the number of boundary components has to be even. If no bubbles occur, then a neighborhood of the lower stratum looks like

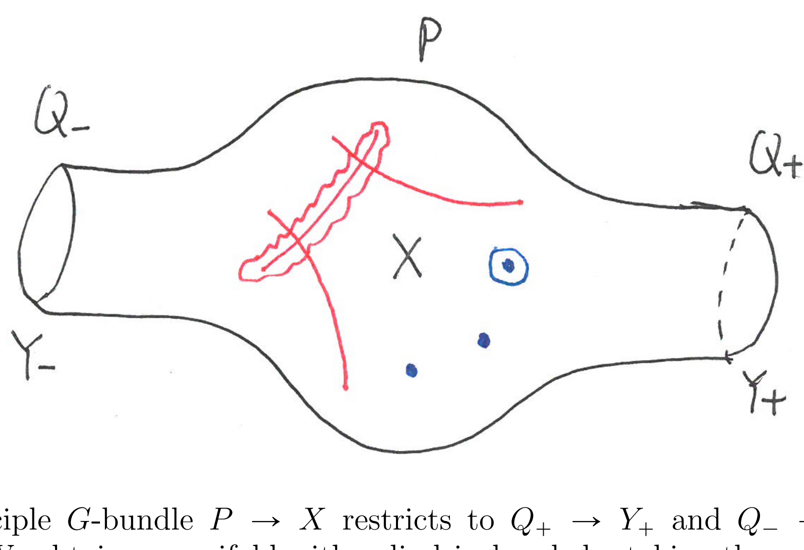
$$[0, \infty)^{n-i} \times S_i$$

where n is the dimension of the top stratum.

30.2. Functoriality

Suppose that we have a cobordism between 3-manifolds Y_- and Y_+:

$$X : Y_- \to Y_+.$$



The principle G-bundle $P \to X$ restricts to $Q_+ \to Y_+$ and $Q_- \to Y_-$ at the boundary. We obtain a manifold with cylindrical ends by taking the union:

$$X^* = (-\infty, 0] \times Y_- \bigcup X \bigcup [0, \infty) \times Y_+.$$

Choose

$$\alpha \in \mathrm{Crit}(CS|_{Q_-}) \text{ and } \beta \in \mathrm{Crit}(CS|_{Q_+})$$

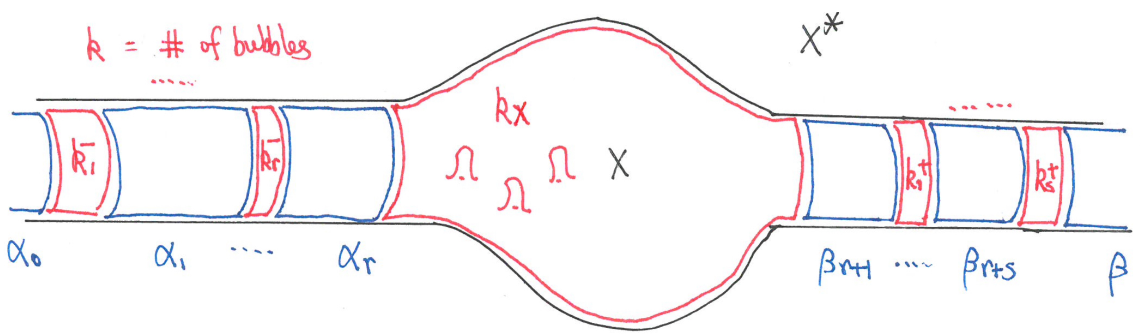
and a homotopy class of connections on P:

$$\gamma \in \pi_1(\alpha, \mathscr{B}_P, \beta).$$

Assume all moduli spaces contain only irreducible connections and are cut out transversely. Let

$$\mathcal{M}_\gamma(\alpha, X, \beta)$$

be the moduli space of finite energy instantons on X^* in the homotopy class γ. Tom left it as exercise to write down a compactification $\overline{\mathcal{IM}}_\gamma(\alpha, X, \beta)$ of $\mathcal{M}_\gamma(\alpha, X, \beta)$ by using one's imagination to describe a picture as follows:



Suppose no bubbles occur in the compactification for some dimension reasons, then $\overline{\mathcal{IM}}_\gamma(\alpha, X, \beta)$ will be a compact space stratified by manifolds. Moreover, by the unique continuation property of ASD connections, we have a restriction map

$$r : \overline{\mathcal{IM}}_\gamma(\alpha, X, \beta) \to \mathscr{B}^*(X).$$

Over each stratum, this map is

$$\begin{aligned} r : \breve{M}_{\gamma_X - k}(\alpha, X, \beta) \times \mathrm{Sym}^k X^* &\to \mathscr{B}^*(X) \\ ([A]; x_1, \cdots, x_k) &\mapsto [A]|_X. \end{aligned}$$

The underlying bundle of $\mathscr{B}^*(X)$ might change for different pieces.

This restriction map r is continuous away from the part with bubbles ($k \geqslant 1$). If we take a cohomology class of $\mathscr{B}^*(X)$ defined by a "good" collection of neighborhoods of cycles in X (let $c_1, \cdots, c_t \in C_*(X)$ be cycles), we hope the cup product

$$r^*(\mu(c_1) \smile \cdots \smile \mu(c_t))$$

is compactly supported on $M_\gamma(\alpha, X, \beta)$. In this case, we do not have the $\mathbb{R}$-translation on the X^*, so we do not need to quotient out the $\mathbb{R}$-action. Note that

$$\dim r^*(\mu(c_1) \smile \cdots \smile \mu(c_t)) = \sum_{i=1}^{t} (4 - \dim c_i).$$

It makes sense to evaluate this class on the top stratum of $\overline{\mathcal{IM}}_\gamma(\alpha, X, \beta)$, or even some boundary strata if

$$sf_\gamma(\alpha, \beta) \geqslant \sum_{i=1}^{t}(4 - \dim c_i) \geqslant sf_\gamma(\alpha, \beta) - 7.$$

These boundary strata can not come from bubbling, so the constraint

$$\geqslant sf_\gamma(\alpha, \beta) - 7.$$

is imposed.

Another scenario is when we have a parametrized moduli space. The definition of $\mathcal{M}(\alpha, X, \beta)$ depends choices of a metric of X and a perturbation of the Chern-Simons functional $\mathcal{CS}$. Let C be a finite dimensional manifold with boundaries that parameterizing these auxiliary data. Define

$$\mathcal{M}_C(\alpha, X, \beta) = \bigcup_{c \in C} \mathcal{M}_c(\alpha, X, \beta).$$

Each $\mathcal{M}_c(\alpha, X, \beta)$ may or may not be a smooth manifold. However, we assume $\mathcal{M}_C(\alpha, X, \beta)$ is cut out transversely as a manifold. Hence,

$$\dim \mathcal{M}_C(\alpha, X, \beta) = \dim c + \dim \mathcal{M}(\alpha, X, \beta).$$

We can evaluate the cohomology class

$$r^*(\mu(c_1) \smile \cdots \smile \mu(c_t))$$

on the top stratum of $\overline{\mathcal{IM}}_C(\alpha, X, \beta)$ or its boundary strata if the difference of dimensions is within 7.

30.3. The Invariance of Floer Homology

Let c_0 and c_1 be two generic choices of metrics and perturbations and let $C = \{c_t : t \in [0, 1]\}$ be a generic path between c_0 and c_1. Take $X = [0, 1] \times Y$ and

$$X^* = Z = \mathbb{R} \times Y.$$

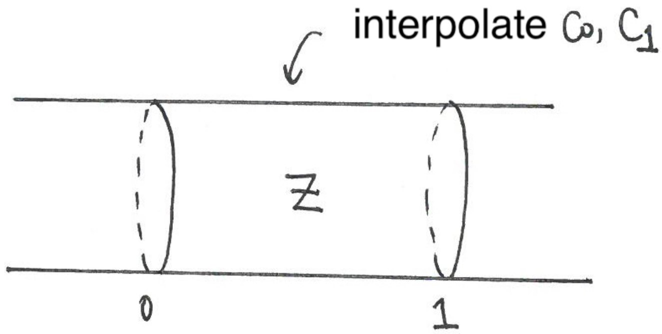
We interpolate the metric on X such that over $\{0\} \times Y$ the metric agrees with c_0 while the metric over $\{1\} \times Y$ agrees with c_1. Similarly, we interpolate the perturbation ϕ_0 and ϕ_1.

By counting 0-dimensional moduli spaces, define:

$$\begin{aligned} \Phi : C(Y, Q)_{c_0} &\to C(Y, Q)_{c_1} \\ \alpha &\mapsto \sum_{\substack{\beta \in \mathrm{Crit}(\mathcal{CS} + \phi_1) \\ sf_\gamma(\alpha, \beta) = 0}} \#\mathcal{M}(\alpha, Z, \beta) \cdot \beta \end{aligned}$$

To prove this is a chain map, we need to verify

$$\partial_1 \circ \Phi + \Phi \circ \partial_0 = 0.$$



This is done by analyzing 1-dimensional moduli space between α and β. If $sf_\gamma(\alpha, \alpha'') = 1$, then

$$\overline{\mathcal{IM}}_\gamma(\alpha, \alpha'') = \mathcal{M}_\gamma(\alpha, \alpha'') \bigcup_{\alpha' \in \mathrm{Crit}(CS+\phi_0)} \bigcup_{\substack{\gamma_0 \in \pi_1(\alpha, \mathscr{B}_Q, \alpha') \\ \gamma' \in \pi_1(\alpha', \mathscr{B}_X, \alpha'') \\ sf_{\gamma_0}(\alpha, \alpha')=1 \\ \gamma_0+\gamma'=\gamma}} \breve{\mathcal{M}}_\gamma(\alpha, \alpha') \times \mathcal{M}_{\gamma'}(\alpha', Z, \alpha'') \bigcup_{\alpha' \in \mathrm{Crit}(CS+\phi_1)} \bigcup_{\substack{\gamma' \in \pi_1(\alpha, \mathscr{B}_X, \alpha') \\ \gamma_1 \in \pi_1(\alpha', \mathscr{B}_Q, \alpha'') \\ sf_{\gamma_1}(\alpha', \alpha'')=1 \\ \gamma'+\gamma_1=\gamma}} \mathcal{M}_{\gamma'}(\alpha, Z, \alpha') \times \breve{\mathcal{M}}_\gamma(\alpha', \alpha'').$$

We will use the parametrized moduli space to show the induced map by Φ on Floer homology is independent of the generic path C, so the cobordism

$$[0,1] \times Y : (Y, c_0) \to (Y, c_1)$$

induces a well-defined map between Floer homology groups:

$$\Phi_* : I(Y, c_0) \to I(Y, c_1).$$

If we can further prove this assignment defines a functor, then we can prove Φ_* is a canonical isomorphism of $\mathbb{Z}_2$-modules.

Lecture 31. **Functoriality II**

31.1. Evaluating Cohomology Classes

Recall from the last lecture that $\overline{\mathcal{IM}}_\gamma(\alpha, X, \beta)$ is a topological space stratified by manifolds. A topology space X is stratified by manifolds if

- $X = X_d \supseteq X_{d-1} \supseteq \cdots \supseteq X_0$.
- $S_i = X_i - X_{i-1}$ is a manifold of dimension i (it could be empty).
- X_i is compact.

Choose a cover $\{U_\alpha\}_{\alpha\in\Lambda}$ of X.

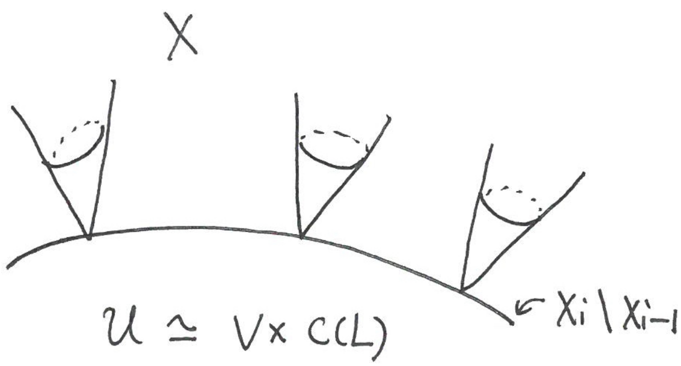
Definition 31.1.1. A cover of X is transverse to the stratification if for any $i+2$ tuple $\alpha_1, \cdots, \alpha_{i+2} \in \Lambda$,

$$U_{\alpha_1} \cap U_{\alpha_2} \cap \cdots \cap U_{\alpha_{i+2}} \cap X_i = \varnothing.$$

Usually, we will find ourselves in the situation when a neighborhood of $S_i = X_i \backslash X_{i-1}$ in X is homeomorphic to a cone-bundle over $S_i = X_i \backslash X_{i-1}$, and we take $U_i \cong S_i \times c(L)$ to be such a neighborhood. Then,

$$U_i, i = 0, 1, \cdots, d$$

form a cover transverse to the stratification.



Suppose $d = \dim \mathcal{M}_\gamma(\alpha, X, \beta)$ and

$$c = \mu(\Sigma_1) \smile \cdots \smile \mu(\Sigma_t) \smile \mu(x_1) \smile \cdots \mu(x_s) \in H^d(\mathscr{B}^*(X)).$$

To evaluate c on $\mathcal{M}_\gamma(\alpha, X, \beta)$, we realize c as a Cech cohomology class. Take a cover of $\overline{\mathcal{IM}}_\gamma(\alpha, X, \beta)$ transverse to the stratification. For any distinct indices

$$\alpha_0, \cdots, \alpha_d \in \Lambda$$

the intersection $U_{\alpha_1} \cap U_{\alpha_2} \cap \cdots \cap U_{\alpha_d}$ will only meet the top stratum of $\overline{\mathcal{IM}}_\gamma(\alpha, X, \beta)$. We assume the cover is fine enough such that $\overline{\mathcal{IM}}_\gamma(\alpha, X, \beta)$ is homeomorphic to the nerve of $\{U_\alpha\}_{\alpha\in\Lambda}$. Each $(d+1)$-tuple $(U_{\alpha_0}, \cdots, U_{\alpha_d})$ determines a d-simplex in the nerve.

Suppose $d = 10$ and c is given by

$$c = \mu(\Sigma_1) \smile \cdots \smile \mu(\Sigma_5).$$

In this case, we have only one stratum coming from bubbles in dimension 6:

$$X \times \mathcal{M}_{\gamma-1}(\alpha, X, \beta).$$

Then the evaluation pairing becomes:

$$\begin{aligned} \mu(\Sigma_1) \smile \cdots \smile \mu(\Sigma_5)(U_{\alpha_0}, U_{\alpha_1}, \cdots, U_{\alpha_{10}}) &= \mu(\Sigma_1)(U_{\alpha_0}, U_{\alpha_1}, U_{\alpha_2}) \\ &\times \mu(\Sigma_2)(U_{\alpha_2}, U_{\alpha_3}, U_{\alpha_4}) \\ &\times \cdots \end{aligned}$$

Each $\mu(\Sigma_j)(U_{\alpha_{2j-2}}, U_{\alpha_{2j-1}}, U_{\alpha_{2j}})$ is an evaluation on a 2-simplex. We have the following alternatives:

(1) Either some (at most two) of $U_{\alpha_{2j-2}} \cap U_{\alpha_{2j-1}} \cap U_{\alpha_{2j}}$ meet at lower strata, while others are zeros. Hence, the evaluation of c is zero.
(2) All of them lie in the top strata $\mathcal{M}_\gamma(\alpha, X, \beta)$, so we get a finite evaluation on the relative fundamental class.

Tom failed to recall a proof. He promised to say it better next time, but he did not.

31.2. Functoriality

Given a homogeneous element z of $\mathbb{A}(X)$, we define a map:

$$\begin{aligned} \Phi(z) : C(Y_-) &\to C(Y_+) \\ a &\mapsto \sum_{\substack{b,\gamma \\ sf_\gamma = \deg z}} \langle \mu(z), [\mathcal{M}_\gamma(a, X, b)] \rangle b. \end{aligned}$$

To show this is a chain map, consider $\alpha \in \mathrm{Crit}(Y_-), \alpha'' \in \mathrm{Crit}(Y_+)$ and $\gamma \in \pi_1(\alpha, \mathscr{B}_X, \alpha'')$ such that

$$sf_\gamma = \deg z + 1.$$

The first and the second stratum of $\overline{\mathcal{IM}}_\gamma(\alpha, X, \alpha')$ looks like

$$\begin{aligned}\overline{\mathcal{IM}}_\gamma(\alpha, X, \alpha'') = \mathcal{M}_\gamma(\alpha, X, \alpha'') \\ \bigcup_{\alpha' \in \mathrm{Crit}(CS+\phi_0)} \bigcup_{\substack{\gamma_0 \in \pi_1(\alpha, \mathscr{B}_{Y^-}, \alpha') \\ \gamma' \in \pi_1(\alpha', \mathscr{B}_X, \alpha'') \\ sf_{\gamma_0}(\alpha,\alpha')=1 \\ \gamma_0+\gamma'=\gamma}} \breve{\mathcal{M}}_\gamma(\alpha, Y^- \alpha') \times \mathcal{M}_{\gamma'}(\alpha', X, \alpha'') \\ \bigcup_{\alpha' \in \mathrm{Crit}(CS+\phi_1)} \bigcup_{\substack{\gamma' \in \pi_1(\alpha, \mathscr{B}_X, \alpha') \\ \gamma_1 \in \pi_1(\alpha', \mathscr{B}_{Y^+}, \alpha'') \\ sf_{\gamma_1}(\alpha',\alpha'')=1 \\ \gamma'+\gamma_1=\gamma}} \mathcal{M}_{\gamma'}(\alpha, X, \alpha') \times \breve{\mathcal{M}}_{\gamma_1}(\alpha', Y^+, \alpha''). \\ \cdots .\end{aligned}$$

Since $\delta\mu(z)$ is zero in the $(d+1)$-th cohomology group, we have

$$\begin{aligned} 0 &= \langle \delta\mu(z), [\mathcal{M}_\gamma(\alpha, X, \alpha'')] \rangle \\ \text{(by integration by parts)} &= \langle \mu(z), \partial[\mathcal{M}_\gamma(\alpha, X, \alpha'')] \rangle. \\ &= \sum_{\gamma_0, \gamma'} \langle \partial_{-,\gamma_0} a, a' \rangle \langle \mu(z), [\mathcal{M}_{\gamma'}(\alpha', X, \alpha'')] \rangle \\ &\quad + \sum_{\gamma_1, \gamma'} \langle \mu(z), [\mathcal{M}_{\gamma'}(\alpha, X, \alpha')] \rangle \langle \partial_{+,\gamma_1} a', a'' \rangle. \end{aligned}$$

This proves that $\partial_+ \circ \Phi(z) + \Phi(z) \circ \partial_- = 0$.

31.3. The composition law

The next step is to compose cobordisms. Suppose we have

$$X_1 : Y_0 \to Y_1,\ X_2 : Y_1 \to Y_2.$$

and elements $z_1 \in \mathbb{A}(X_1)$ and $z_2 \in \mathbb{A}(X_2)$. Do we have

$$\Phi_{X_2}(z_2) \circ \Phi_{X_1}(z_1) = \Phi_{X_1 \#_{Y_1} X_2}(z_2 \cdot z_1)?$$

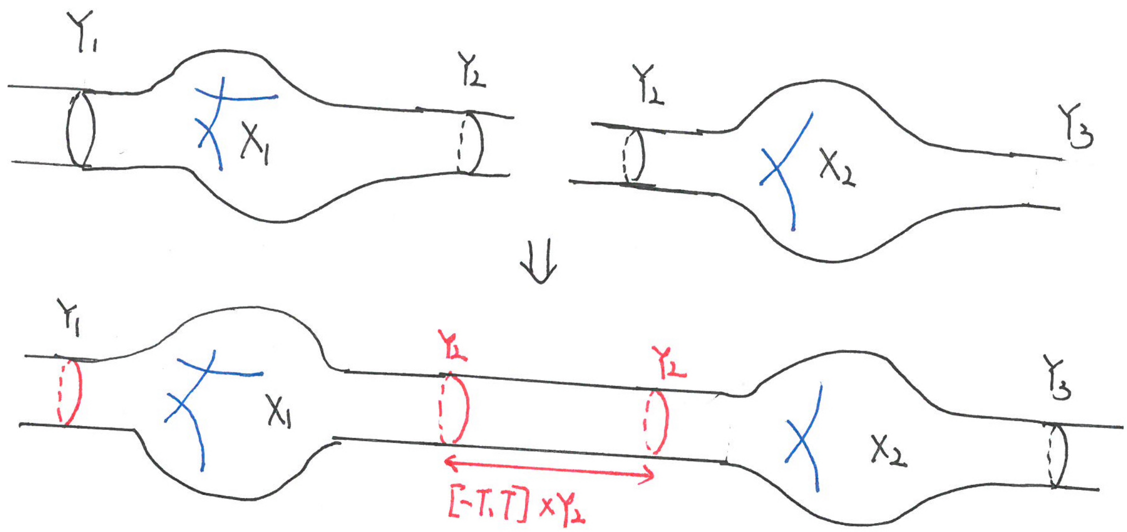
This can be proved by introducing parametrized moduli spaces by changing the length T of cylinders.

A more subtle question is to set up the category. Gluing on the level of manifolds is well-defined up to diffeomorphisms of 3-manifolds Y_i. We pin down the diffeomorphism

$$\partial X_i \cong -Y_{i-1} \coprod Y_i.$$

as part of the definition of cobordisms. The problem is to keep track of bundles.

When G is simply connected (e.g. $G = SU(n)$), the dimension of $z = z_2 \cdot z_1$ will determine the Chern number of the bundle on $X_1 \#_{Y_1} X_2$. The bundle $Q_2 \to Y_2$ is necessarily trivial, and $\pi_1(\mathscr{G}_{Q_2}) \cong \mathbb{Z}$. Hence, the connected component of $\mathscr{G}_{Q_2}$ is pinned down by z.



When G is not simply connected (e.g. $G = PU(n)$), it is more delicate. There could be more than one principal G-bundle over Y_2.

There are two ways to formulate the functoriality for non-simply connected gauge groups. We start with the first one:

- Pick a $U(n)$-bundle P.
- Fix the determinant line bundle of P by fixing an isomorphism: $\psi : \det(P) \cong L$.
- Choose a connection Δ in $\det(P)$.
- Use the determinant-1 gauge group.

To setup a category, let an object to be a quadruple

$$(Y, L, \Delta, \psi).$$

The bundle P is recovered by taking $\mathbb{C} \oplus L$ with a reference connection (d, Δ). A morphism is another quadruple

$$(X, \tilde{L}, \Delta, \psi)$$

such that when restricted to $\partial X = -Y_- \cup Y_+$, it recovers $(Y_-, L_-, \Delta_-, \psi_-)$ and $(Y_+, L_+, \Delta_+, \psi_+)$.

The second way is to think of L as an divisor. Choose a oriented 1-manifold $\omega \subset Y$ and consider $SU(2)$ connections on $Y \backslash \omega$ so that the link holonomy along ω is (-1) (in fact, it is the asymptotic linking holonomy).

To compare (1) with (2), we pick a singular connection in L that is flat on $Y \backslash \omega$ and the holonomy is (-1) around ω.

An object is a pair (Y, ω). A cobordism is a pair (X, S) such that $S \subset X$ is an properly embedded 2-surface such that

$$X : Y_- \to Y_+,$$
$$S : \omega_- \to \omega_+.$$

The composition is defined by requiring ω to match.

Lecture 32. Flat Connections on Σ_g

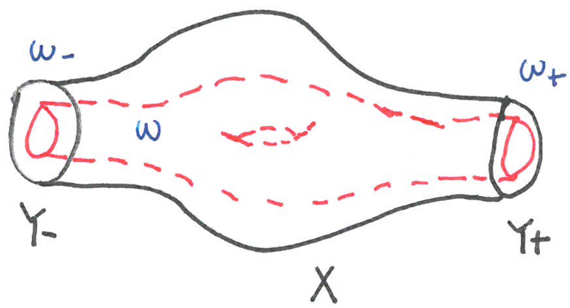
32.1. The cobordism map

Consider a compact 4-manifold X^4 with boundary

$$\partial X^4 = -Y^- \cup Y^+.$$

Let $\omega^- \subset Y^-, \omega^+ \subset Y^+$ be embedded closed oriented 1-submanifolds. Moreover, suppose there is a oriented surface $\omega \subset X$ such that

$$\partial\omega = -\omega^- \cup \omega^+.$$



We can view (X, ω) as a cobordism between pairs:

$$(X, \omega) : (Y^-, \omega^-) \to (Y^+, \omega^+).$$

Choose some auxiliary data $c_\pm$ on $Y^\pm$ (metrics, perturbations, etc.), we defined a chain complex

$$(C(Y^\pm, \omega^\pm, c_\pm), \partial_\pm).$$

Choosing suitable cycles $\sigma_1, \cdots, \sigma_t$ on X, then a cobordism (X, ω) induces a chain map

$$\Phi_{(X,\omega)}(\sigma_1, \cdots, \sigma_t) : C(Y^-, \omega^-, c_-) \to C(Y^+, \omega^+, c_+).$$

Different choices on X give rise to chain homotopic maps. These chain maps satisfy the composition law, up to chain homotopy.

Example 32.1.1. Suppose $X = [0, 1] \times Y^+$, $\omega = [0, 1] \times \omega^+$ and there are no cycles. Then

$$\Phi_{prod} = \Phi_{[0,1]\times Y^+,[0,1]\times\omega^+} : C(Y^+, \omega^+, c_+) \to C(Y^+, \omega^+, c_+)$$

is the identity map.

Example 32.1.2. Suppose $X = [0, 1] \times Y^+$ and there are no cycles. w^- and w^+ are allowed to be different. Then we have an induced map:

$$\Phi_{(X,\omega)} : C(Y^+, \omega^-, c_-) \to C(Y^+, \omega^+, c_+).$$

By reversing the direction of (X, ω), we obtain a map in the opposite direction:

$$\Phi'_{(X,\omega)} : C(Y^+, \omega^+, c_+) \to C(Y^+, \omega^-, c_-).$$

The composition

$$\Phi'_{(X,\omega)} \circ \Phi_{(X,\omega)}$$

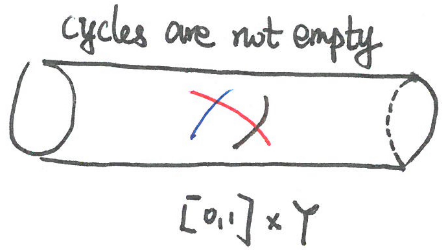
is chain homotopy equivalent to Φ_{prod}, and so is $\Phi_{(X,\omega)} \circ \Phi'_{(X,\omega)}$. Hence, $\Phi_{(X,\omega)}$ and $\Phi'_{(X,\omega)}$ are chain homotopy inverses to each other.

Therefore, up to chain homotopy equivalence, $(C(Y,\omega,c),\partial)$ is an invariant of (Y,ω) and the assignment

$$(Y,\omega) \mapsto I(Y,\omega) := H_*(C(Y,\omega,c),\partial)$$

is a functor. Moreover, $\mathbb{A}(Y)$ acts on $I(Y,\omega)$. We allow cycles $\sigma_1,\cdots,\sigma_t$ to be non-empty and define the action by the chain map:

$$\Phi_{prod}(\sigma_1,\cdots,\sigma_t).$$



Not any 3-manifold has a nice smooth space of flat connections. A perturbation is needed to make the construction work in genera.

(Tom did not talk about perturbations seriously. He made the digression to flat connections on a surface.)

32.2. Flat Connections on Σ_g

For a closed oriented surface Σ_g of genus g, $H^1(\Sigma^2,\mathbb{R})$ admits a symplectic pairing:

$$(\alpha,\beta) \mapsto \int_\Sigma \alpha\wedge\beta.$$

If $\partial Y^3 = \Sigma^2$, then the image

$$i^* : H^1(Y,\mathbb{R}) \to H^1(\Sigma,\mathbb{R})$$

is a Lagrangian subspace, i.e. a maximal subspace on which ω_Σ vanishes. Indeed, if $\alpha = i^*(\alpha')$ and $\beta = i^*(\beta')$, then by Stokes' theorem,

$$\int_\Sigma \alpha\wedge\beta = \int_Y d(\alpha'\wedge\beta') = 0.$$

This proves ω_Σ vanishes on $\operatorname{Im} i^*$. The maximality follows from the Poincaré duality. Consider the long exact sequence and its dual:

$$\begin{tikzcd} H^1(Y,\Sigma) \arrow[r] \arrow[d] & H^1(Y) \arrow[r, "i^*"] \arrow[d] & H^1(\Sigma) \arrow[r, "\delta"] \arrow[d, "PD"] & H^2(Y,\Sigma). \arrow[d, "PD"] \\ H_1(Y) \arrow[r] & H_2(Y,\Sigma) \arrow[r, "\partial"] & H_1(\Sigma) \arrow[r, "i_*"] & H_1(Y). \end{tikzcd}$$

If $(\alpha, i^*\beta) = 0$ for any $\beta \in H^1(Y)$, or equivalently $i_* \circ PD(\alpha) = 0$ in $H_1(Y)$, then

$$PD \circ \delta(\alpha) \Rightarrow \delta(\alpha) = 0 \Rightarrow \alpha \in \operatorname{Im} i^*.$$

Given a compact Lie group G, consider a principal bundle

$$G \to P \to \Sigma_g$$

over a surface Σ. Then we have a symplectic form on the configuration space $\mathscr{A}_P$. Indeed,

$$T_A\mathscr{A}_P = \Omega^1(\Sigma, \operatorname{ad} P).$$

For tangent vectors $\alpha, \beta \in \Omega^1(\Sigma, \operatorname{ad} P)$, let

$$\omega_\Sigma(\alpha, \beta) = \int_\Sigma \operatorname{tr}(a \wedge b).$$

Then $\mathscr{A}_P$ becomes a "formal" symplectic manifold (one needs to define and verify $d\omega = 0$ in this infinite dimensional setting). Moreover, the gauge group $\mathscr{G}$ acts on $\mathscr{A}_P$ by symplectomorphisms, i.e. the symplectic form ω_σ is invariant under $\mathscr{G}$.

Is there a moment map? Recall that for a finite dimensional symplectic manifold (M, ω) with a symplectic group action G, we have a Lie algebra homomorphism:

$$\mathfrak{g} \to \operatorname{Vect}(M), \xi \mapsto \hat{\xi}.$$

The vector field $\hat{\xi}$ corresponds to the infinitesimal group action along ξ and

$$\mathscr{L}_{\hat{\xi}}\omega = 0.$$

By Cartan's formua, $\mathscr{L}_{\hat{\xi}}\omega = d \circ \iota_{\hat{\xi}}\omega + \iota_{\hat{\xi}} \circ d\omega = d \circ \iota_{\hat{\xi}}\omega$ since $d\omega = 0$. This shows that $\iota_{\hat{\xi}}\omega$ is closed. We require it to be exact and

$$\iota_{\hat{\xi}}\omega = d\langle \mu, \xi \rangle$$

for a smooth function $\mu : M \to \mathfrak{g}^*$. This means $\hat{\xi}$ is generated by a Hamiltonian function $\langle \mu, \xi \rangle$. The function μ is called a moment map if it is G-equivariant.

A moment map may not exist in general; an obstruction exists.

In our situation, $G = \mathscr{G}$. $\mathfrak{g} = \Omega^0(\Sigma, \operatorname{ad} P)$ and $\mathfrak{g} = \Omega^2(\Sigma, \operatorname{ad} P)$. The non-degenerate pairing is given by

$$(\alpha, \beta) = \int_\Sigma \operatorname{tr}(\alpha \wedge \beta).$$

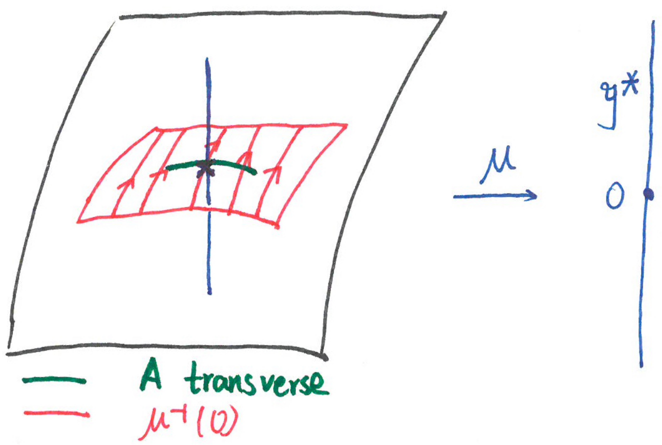
for $\alpha \in \mathfrak{g}$ and $\beta \in \mathfrak{g}^*$.

Exercise 32.2.1. *Show that $\mu(A) = F_A$ is a moment map for the gauge action.*

32.3. Symplectic Reduction

When 0 is a regular value of μ, then $\mu^{-1}(0)/G$ is a symplectic manifold (orbifold). This is the contend of symplectic reduction, first realized by Guillemin-Sternberg and Maslov-Weinstein in the context of Noether's theorem.

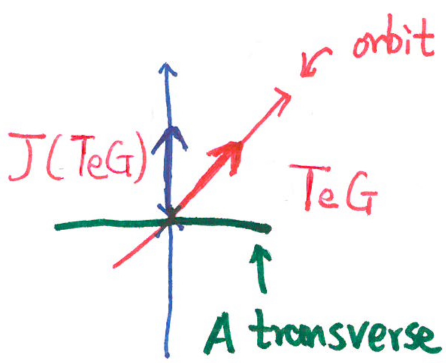
The observation is that the infinitesimal action on $\mu^{-1}(0)$ is free, so the stabilizer (at any point of $\mu^{-1}(0)$) must be a discrete subgroup of G . If the action of G happens to be free, then the quotient is a genuine manifold. It has a symplectic structure because we can restrict ω to a transverse of G-orbits, and it is still non-degenerate.



If T is a transverse at $x \in \mu^{-1}(0)$, then

$$T_xM = T_x(G \cdot x) \oplus JT_x(G \cdot x) \oplus T_xT.$$

In general, if G is a compact Lie group, we can only have a symplectic orbifold.



Back to our case, the representation variety

$$\mathcal{R}_G(\Sigma) = \{A : F_A = 0\}/\mathscr{G}$$

is a symplectic manifold away from reducible representations. Moreover, the restriction map

$$\mathcal{R}_G(Y) \to \mathcal{R}_G(\Sigma)$$

is a Lagrangian immersion.

Lecture 33. Symplectic Reductions

33.1. Symplectic Reduction

Let (X, ω) be a symplectic manifold with a symplectic group action G and a moment map:

$$\mu : X \to \mathfrak{g}^*.$$

The group action generates a Lie algebra homomorphism:

$$\mathfrak{g} \to \mathrm{Vect}(M),\ \xi \mapsto \hat{\xi}.$$

The moment map is characterized by the property:

$$d\langle \mu, \xi \rangle = \iota_{\hat{\xi}} \omega.$$

For any element t in the center of $\mathfrak{g}^*$, the quotient

$$\mu^{-1}(t)/G$$

is called the symplectic reduction at t. If t is regular value of μ, then $\mu^{-1}(t)/G$ is a symplectic manifold (orbifold).

In the infinite dimensional setting, for a principle G bundle $P \to \Sigma$, let $M = \mathscr{A}_P$ with symplectic form

$$\omega(a, b) = \int_\Sigma \mathrm{tr}(a \wedge b).$$

The moment map μ is defined as

$$\mu(A) = F_A.$$

Ideally, we would have $\mu^{-1}(0)/G$ is a (finite dimensional) symplectic manifold.

Suppose (X, J, ω, g) is a Kähler manifold. The metric g is compatible with ω, i.e.

$$\omega(X, Y) = g(JX, Y).$$

This implies that $\hat{\xi} = -J\nabla\langle \mu, \xi \rangle$. Indeed, we have.

$$g(\nabla\langle \mu, \xi \rangle, Y) = \iota_{\hat{\xi}}\omega(Y) = g(J\hat{\xi}, Y).$$

We assume G preserves both J and g, i.e. G acts on M isometrically and holomorphically.

In this case, we can often allow the complexified group $G_{\mathbb{C}}$ to act on M.

Example 33.1.1. Consider $U(n)$ acting on $\mathrm{End}(\mathbb{C}^n)$ by conjugation. The complexified group is $GL(n, \mathbb{C})$. The action extends to $GL(n, \mathbb{C})$ in an obvious way.

Note that the Lie algebra of $G_{\mathbb{C}}$ is $\mathfrak{g} \otimes \mathbb{C}$. We get an action of $\mathfrak{g} \otimes \mathbb{C}$ using J:

$$\mathfrak{g} \otimes \mathbb{C} \to \mathrm{Vect}(X),\ \xi + i\eta \mapsto \hat{\xi} + J\hat{\eta}.$$

Whether this infinitesimal action is induced from an actual group action of $G_{\mathbb{C}}$ is asking whether these vector fields are integrable.

The complexified group $G_{\mathbb{C}}$ is not a compact group. It is a delicate question to examine its action on X. For instance, $X/G_{\mathbb{C}}$ need not be Hausdorff, even if X is a finite dimensional manifold.

Example 33.1.2. Let $G = U(1)$ and $G_{\mathbb{C}} = \mathbb{C}^*$. Consider their actions on $\mathbb{C}$:

$$(x, z) \mapsto x \cdot z, x \in \mathbb{C}^*, z \in \mathbb{C}.$$

Then $\mathbb{C}/\mathbb{C}^* = \{[0], [1]\}$ is a set of two elements. The orbit $[0]$ of the origin is closed, but $[1]$ is not. In fact,

$$\overline{[1]} = \{[0], [1]\}.$$

To avoid this pathology, the idea is to identify the semi-stable part X^{ss} of X such that the quotient

$$X^{ss}/G_{\mathbb{C}}$$

is Hausdorff.

Example 33.1.3. In the example above, $X^{ss} = \{0\}$.

Another important question is when the orbit of a point $x \in X$ closed. The motivates the definition of the stable region of X:

$$X^s = \{x \in X : G_{\mathbb{C}} \cdot x \text{ is closed in } X\}.$$

Example 33.1.4. In the example above, $0 \in \mathbb{C}$ has a closed orbit, but the orbit of $1 \in \mathbb{C}$ is not closed.

Example 33.1.5. Let $GL(n, \mathbb{C})$ act on $\mathrm{End}(\mathbb{C}^n) = \mathfrak{gl}(n, \mathbb{C})$ by conjugation. A matrix A has a closed orbit if and only if A is diagonalizable. It is easy to see if A is not diagonalizable, then the orbit is not closed. For instance,

$$\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \lambda & t \\ 0 & \lambda \end{pmatrix}$$

and t is allowed to be any number in $\mathbb{C}^*$. Moreover, if A has distinct eigenvalues, then A can only be approximated by stable elements. If A has repeated eigenvalues, then A can be approximated by unstable elements.

Symplectic reductions are comparable with GIT quotients in algebraic geometry. For a given algebraic variety with a group action, consider the action on the algebra and we can identify the invariant sub-algebra.

Kempf-Ness figured out when it is possible to identify

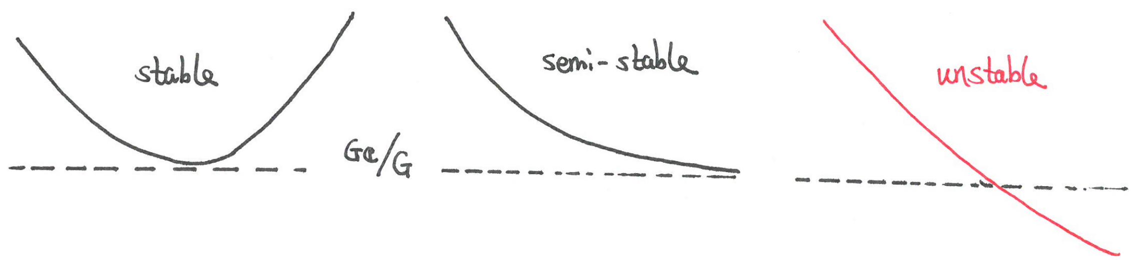
$$\mu^{-1}(0)/G \Leftrightarrow \text{the GIT quotient}.$$

Consider the G-invariant function $\bar{m} = \log|\mu|^2 : X \to \mathbb{R}$. Consider its restriction at a $G_{\mathbb{C}}$ orbit:

$$\begin{aligned} m : G_{\mathbb{C}}/G &\to \mathbb{R} \\ x &\mapsto \bar{m}(x). \end{aligned}$$

The quotient $G_{\mathbb{C}}/G$ has a complete metric of non-negative curvatures. We can figure out whether $[x]$ is stable or non-stable by looking at $\bar{m}$. A theorem states that m is convex along any geodesics in $G_{\mathbb{C}}/G$.

- If m achieves a minimum along any geodesics on $G_{\mathbb{C}}/G$, then the $G_{\mathbb{C}}$-orbit of x is closed, or equivalently $[x]$ is stable.
- If m has a lower bound along any geodesics, and for some geodesics the minimum is not achieved, then the $G_{\mathbb{C}}$-orbit of $[x]$ is not closed. $[x]$ is called semi-stable in this case.
- If m does not have a lower bound along some geodesics, then $[x]$ is called unstable, and the orbit is not closed.



The stable region X^s might not be closed. To compactify X^s, we need to include semi-stable orbits.

33.2. The Yang-Mills Functional

Apply the general theory to a compact Riemann surface. Let

$$YM(A) = |\mu(A)|^2 = \int_\Sigma |F_A|^2.$$

If A is a critical point of $|\mu(A)|^2$, then for any path $A_t, t \in [0,1]$ with $A_0 = A$, $\frac{\partial}{\partial t} A_t|_{t=0} = a$, we have

$$\frac{\partial}{\partial t}|\mu(A_t)|^2 = 0.$$

This implies

$$0 = \frac{\partial}{\partial t}\int_\Sigma \langle F_{A_t}, F_{A_t}\rangle = 2\int_\Sigma \langle d_A a, F_A\rangle = 2\int_\Sigma \langle a, d_A^* F_A\rangle.$$

for any $a \in \Omega^1(\Sigma, \mathrm{ad}\, P)$. In particular,

$$0 = d_A^* F_A = - * d_A * F_A \Leftrightarrow d_A * F_A = 0 \Leftrightarrow *F_A \text{ is } A\text{-parallel}.$$

In conclusion, A is a critical point of $|\mu(A)|^2$ if and only if the holonomy of A commutes with $*F_A$. In higher dimension, we do not such a nice criterion for Yang-Mills connections.

If $G = U(n)$, let $E = P \times_\rho \mathbb{C}^n$. Then $*F_A$ is viewed as an endomorphism of E with constant eigenvalues. Indeed,

$$\operatorname{tr}(*F_A)^n$$

is a constant function on Σ, since $d_A(*F_A) \equiv 0$ and $d_A(*F_A)^n \equiv 0$. Eigenvalues are imaginary since $*F_A$ is skew-symmetric. Decompose E according to eigenvalues of $*F_A$:

$$E = E_{i\lambda_1} \oplus E_{i\lambda_2} \oplus E_{i\lambda_3} \oplus \cdots \oplus E_{i\lambda_k}.$$

Here, $k \leqslant n$ because repeated eigenvalues might occur. The connection A preserves the decomposition:

$$A = (A_1, \cdots, A_k).$$

We normalize the volume of Σ so that

$$\int_\Sigma 1 = 2\pi.$$

Then

$$c_1(E_{i\lambda_j})[\Sigma] = \frac{i}{2\pi}\int_\Sigma \operatorname{tr}(*F_{A_j}) = \frac{i}{2\pi}\int_\Sigma (i\lambda_j)\operatorname{rank} E_{i\lambda_j} = -\lambda_j \operatorname{rank}(E_{i\lambda_j})$$

Remark. If we instead work with $PU(n)$-bundles, we will assume it comes from a $U(n)$ bundle. The morphism

$$U(n) \xrightarrow{\pi\times\det} PU(n) \times U(1)$$

induces

$$P \to P_{PU(n)} \times \det P.$$

Then we fix a constant curvature connection A_* on $\det P$ (this space is $2g$-dimensional torus for fixed $c_1(P)$). We look at the subspace of $\mathscr{A}_P$ whose induced connection is A_* on $\det P$, with determinant-1 gauge groups. □

The upshot is

$$\langle c_1(E), [\Sigma]\rangle = \sum_{j=1}^k \langle c_1(E_{i\lambda_j}), [\Sigma]\rangle$$

which is a topological constraint, and

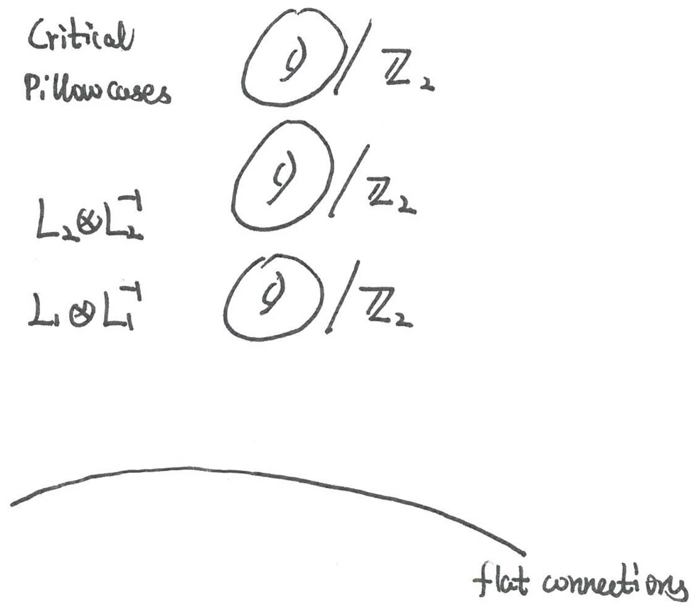
$$YM(A) = \sum_{j=1}^k \lambda_i^2 \operatorname{rank}(E_{i\lambda_j}).$$

When $YM(A) = 0$, A is a flat connection.

Example 33.2.1. When $G = SU(2)$, $c_1[E] = 0$. The flat connections gives the global minimum. Any higher critical point corresponds to a splitting

$$L \oplus L^{-1}$$

which contributes to a critical pillowcase $\mathbb{T}^2/\mathbb{Z}_2$ in $\mathscr{A}_P/\mathscr{G}_P$.


Critical Pillowcases
$/\mathbb{Z}_2$
$L_2 \otimes L_2^{-1}$
$/\mathbb{Z}_2$
$L_1 \otimes L_1^{-1}$
$/\mathbb{Z}_2$
flat connections

Lecture 34. The Yang-Mills equation over Riemann Surfaces

34.1. Yang-Mills Connections

Let $P \to \Sigma$ be a principle G-bundle, then $\mathscr{A}_P$ is a symplectic manifold:

$$\omega(a, b) = \int_\Sigma [a \wedge b].$$

This symplectic form is independent of metrics and depends only on the orientation of the surface Σ. The definition of the bracket

$$[\cdot \wedge \cdot] : \Omega^i(X, \operatorname{ad} P) \otimes \Omega^j(X, \operatorname{ad} P) \to \Omega^{i+j}(X, \mathbb{R}).$$

relies on an ad-invariant bilinear form $\mathfrak{g} \otimes \mathfrak{g} \to \mathbb{R}$. By choosing a metric g on Σ, we obtain a metric on $\mathscr{A}_P$:

$$\langle a, b\rangle = \int_\Sigma [a \wedge *b].$$

The moment map is defined as

$$\begin{aligned} \mu : \mathscr{A}_p &\to \Omega^2(\Sigma, \operatorname{ad} P) = \Omega^0(\Sigma, \operatorname{ad} P)^* \\ A &\mapsto F_A. \end{aligned}$$

Then the symplectic reduction

$$\mathcal{R}_G(\Sigma) = \mu^{-1}(0)/\mathscr{G} = \{\rho : \pi_1(\Sigma) \to G\}/\text{conjugations}$$

is a symplectic manifold (orbifold). Since $\mathscr{A}_P$ is also a Kähler manifold, this makes $\mathcal{R}_G(\Sigma)$ into a Kähler manifold.

Critical points of $YM(A) = |\mu(A)|^2$ are Yang-Mills connections. Only in dimensional 2, we obtain a simple criterion of Yang-Mills connections:

$$d_A^* F_A = 0 \Leftrightarrow *F_A \text{ is } A\text{-parallel}.$$

Consider the bundle map:

$$\begin{aligned} \Lambda : \operatorname{ad} P &\to \operatorname{ad} P \\ s &\mapsto i[*F_A, s]. \end{aligned}$$

Then Λ is a self-adjoint endomorphism at each fiber. Suppose $\Lambda = \lambda \operatorname{Id}$ and P is a $U(n)$-bundle associated to a complex vector bundle E. Then

$$c_1(E) = \frac{i}{2\pi} \operatorname{tr}(F_A) = \frac{\lambda \operatorname{rank}(E)}{2\pi} dvol_\Sigma.$$

We normalize the volume so that $\int_\Sigma dvol_\Sigma = 2\pi$. Then

$$\deg E = \langle c_1(E), [\Sigma]\rangle = \lambda \operatorname{rank}(E).$$

Equivalently, $\lambda = \deg(E)/\operatorname{rank}(E)$. The ratio $\lambda = \deg(E)/\operatorname{rank}(E)$ is also called the slope of E. In general, decompose E according to eigenvalues of $i * F_A$:

$$E = E_1 \oplus \cdots \oplus E_k$$
$$E_j \leftarrow \lambda_j$$
$$\lambda_j = \frac{\deg(E_j)}{\operatorname{rank}(E_j)}$$

The connection A preserves this decomposition: $A = (A_1, \cdots, A_N)$. Then

$$YM(A) = \sum_{j=1}^{k} \operatorname{rank}(E_k)(\frac{\deg E_i}{\operatorname{rank} E_i})^2.$$

The configuration space $\mathscr{A}_P$ admits an action by $\mathscr{G}_{\mathbb{C}}$. We need to define it suitably, since the usual formula

$$A \mapsto u \circ A \circ u^{-1}$$

does not work for a general bundle automorphism $u : E \to E$ (the result is not a unitary connection in general).

Suppose $G = U(n)$ and P is induced from a complex vector bundle E with a Hermitian inner product. Given a connection A, we have covariant derivative:

$$d_A : \Omega^0(\Sigma, E) \to \Omega^1(\Sigma, E).$$

Since the Hodge star operator $*$ acts on $\Omega^1(\Sigma, E)$ and $*^2 = -1$,

$$\Omega^1(\Sigma, E) \otimes \mathbb{C} = \Omega^{1,0}(\Sigma, E) \oplus \Omega^{0,1}(\Sigma, E) \sim (dz, d\bar{z}).$$

Write $d_A = \partial_A + \bar{\partial}_A$. The $\bar{\partial}$-operator $\bar{\partial}_A$ defines a holomorphic structure of E: a section $s \in \Omega^0(\Sigma, E)$ is holomorphic if and only if

$$\bar{\partial}_A s = 0,$$

and we can construct a local holomorphic frame of E. In general, $\bar{\partial}_A^2 \neq 0$ and

$$F_A^{0,2} = \partial_A^2 \in \Omega^{0,2}(X, \operatorname{End}(E))$$

is the obstruction for $\bar{\partial}_A$ to define a holomorphic structure. However, when $\dim X = 2$,

$$\Omega^{0,2}(\Sigma) = 0$$

for dimension reasons. Hence, we get implication in one direction:

$$\text{Unitary connections} \Rightarrow \text{Holomorphic Structures}$$

A holomorphic structure on E is specified by $\bar{\partial}$-operators:

$$\bar{\partial}_E : \Omega^{0,0}(E) \to \Omega^{0,1}(E)$$
$$\bar{\partial}_E(fs) = f\bar{\partial}_E s + (\bar{\partial} f)s.$$

Let $\mathscr{A}_E^{0,1}$ be the space of $\bar{\partial}$-operators on E. Two $\bar{\partial}$-operators differs by a (0,1)-form:

$$(\bar{\partial}_E - \bar{\partial}'_E)(fs) = f(\bar{\partial}_E - \bar{\partial}'_E)(s)$$

so $(\bar\partial_E - \bar\partial'_E)(s) = a^{0,1} \otimes s$ for some $a^{0,1} \in \Omega^{0,1}(\Sigma, \mathrm{End}(E))$. The vector space $\Omega^{0,1}(\Sigma, \mathrm{End}(E))$ characterizes all deformations of holomorphic structures on E.

The complexified gauge group $\mathscr{G}_{\mathbb{C}} = \Omega^0(\Sigma, GL(E))$ acts on $\mathscr{A}_E^{0,1}$ by the formula:

$$g(\bar\partial_E) = g \circ \bar\partial_E \circ g^{-1}.$$

Hence, if s is a holomorphic section of $\bar\partial_E$, then $g(s)$ is a holomorphic section of $g(\bar\partial_E)$.

In fact, there is a bijection between

$$\mathscr{A}_P \Leftrightarrow \mathscr{A}_E^{0,1},$$

and we have seen the implication in one direction. On the other hand, if $\bar\partial_E$ is a holomorphic structure on E, let $s_1, \cdots, s_n$ be a local holomorphic frame. Define

$$h_{ij} = \langle s_i, s_j \rangle.$$

If there is a unitary connection A with $\bar\partial_A = \bar\partial_E$, then

$$\partial h_{ij} = \langle \partial_A s_i, s_j \rangle + \langle s_i, \underbrace{\bar\partial_A s_j}_{=0} \rangle = \langle \partial_A s_i, s_j \rangle.$$

Suppose $\partial_A s_i = a_{ij} \otimes s_j$, then

$$\partial h_{ij} = a_{ik} h_{kj}.$$

In short, we have $a = h^{-1}\partial h$. One verifies that these two maps are indeed inverse to each other.

We arrive at the following picture:

$$\mathscr{G} = \mathscr{G}_{U(n)} \rightharpoonup \mathscr{A}_P \Leftrightarrow \mathscr{A}_E^{0,1} \leftharpoondown \mathscr{G}_{\mathbb{C}} = \mathscr{G}_{GL(n,\mathbb{C})}.$$

Groups actions on LHS and RHS are compatible under the bijection. Extend the action of $\mathscr{G}$ by the action of $\mathscr{G}_{GL(n,\mathbb{C})}$ on $\mathscr{A}_E^{0,1}$.

34.2. The Yang-Mills Flow

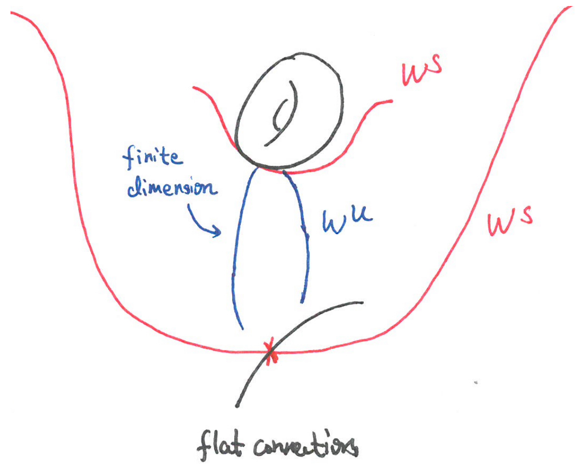
The symplectic quotient

$$\mu^{-1}(0)/\mathscr{G}$$

is the minimal critical set of $YM(A)$. In some good situation (e.g $(\operatorname{rank} E, \deg E) = 1$ (co-prime)), it is a smooth symplectic manifold and the Yang-Mills functional YM is a Morse-Bott function. It is possible to look at the stable manifold of this minimal critical set. In fact,

$$W^s = \mathscr{G}_{\mathbb{C}}\text{-orbits of the minimal critical set.}$$

For higher critical points, we have both stable and unstable manifolds, W^s and W^u. W^s still comes from $\mathscr{G}_{\mathbb{C}}$-orbit of the critical set. The downward gradient flow of YM always preserves the $\mathscr{G}_{\mathbb{C}}$-orbit.



Example 34.2.1. Take $G = SU(2)$, then higher critical points come from decompositions

$$E = L \oplus L^{-1}.$$

Suppose $\deg L = 1$. Then we have a short exact sequence:

$$0 \to L^{-1} \to E \to L \to 0.$$

Extensions of holomorphic structures are parametrized by

$$H^1_{\bar{\partial}_E}(\Sigma, L^{\otimes 2}).$$

This space is isomorphic to the negative eigenspace of the Hessian of YM at ∂_E (a critical point).

For higher critical points, the eigenspace of the Hessian that flows down to intermediate critical points are related to the iterated extension problem.

Let us think about the Hessian more carefully. Recall that

$$F_{A+a} = F_A + d_A a + \frac{1}{2}[a \wedge a].$$

Then

$$YM(F_{A+ta}) = YM(F_a) + 2t\langle F_A, d_A a\rangle + t^2(\langle d_A a, d_A a\rangle + \langle F_A, [a \wedge a]\rangle)$$

The Hessian is given by

$$\begin{aligned} a &\mapsto \langle d_A a, d_A a\rangle + \langle F_A, [a \wedge a]\rangle \\ &\to \langle d_A^* d_A a + *[*F_A, a], a\rangle. \end{aligned}$$

We obtain an self-adjoint elliptic operator by adding the gauge fixing equation

$$d_A d_A^* a = 0.$$

It turns out that on $H^{1,0}_A(\Sigma, \operatorname{ad} P)$, $*[*F_A, \cdot]$ is a positive operator and $H^{0,1}(\Sigma, \operatorname{ad} P)$ will give us the negative eigenspace.

Lecture 35. **Orbifolds**

Tom answered a few questions before the class. Over a Riemann surface Σ, we have a symplectic structure on the configuration space $\mathscr{A}_\Sigma$. In general, given a symplectic manifold (X, ω), we define the symplectic structure on $\Omega^1(X, \operatorname{ad} P)$ by

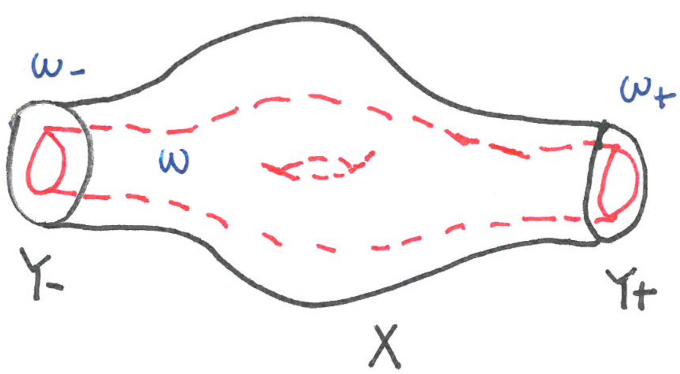
$$\omega_P(a, b) = \int_X \operatorname{tr}(a \wedge b) \wedge \omega^{n-1}.$$

Moreover, if X is a Kähler manifold, then the moduli space inherits a Kähler structure. Recall that in the ADHM construction of instantons on $\mathbb{R}^4$, we have a trivial Kähler structure on $\mathbb{R}^4$, so there is a Kähler structure on the moduli space of the framed ASD connections.

35.1. Orbifolds

For an oriented 3-manifold Y, fix a complex line bundle $L \to Y$ with $w_2(L) \neq 0$. A pair (Y, ω) is called admissible if

- ω is an embedded 1-submanifold of Y and represents the Poincaré dual of $c_1(L)$.
- We can find a closed oriented surface $\Sigma \subset Y$ such that $c_1(L)[\Sigma] = \#(\omega \cap \Sigma) \neq 0 \mod 2$.



We defined the cobordism map:

$$I_*(X, \omega) : I_*(Y_-, \omega_-) \to I_*(Y_+, \omega_+),$$

and showed that it satisfies the composition law. The theory depends only on the pair (Y, L) so far. To extend the construction to a relative version for a pair

$$(Y, K)$$

where K is a knot, a link or even a trivalent graph in Y, we need to work with orbifolds.

Definition 35.1.1. An orbifold structure on a topological space X is given by a collections of orbifold charts (V, Γ, ψ) where

- $\Gamma \subset O(n)$ is a finite subgroup.
- V is an open subset of $\mathbb{R}^n$ invariant under Γ.
- $\psi : V/\Gamma \to X$ is a homeomorphism onto its image.

For two orbifold charts $(V_\alpha, \Gamma_\alpha, \psi_\alpha)$ and $(V_\beta, \Gamma_\beta, \psi_\beta)$ with an inclusion $\phi_\alpha(V_\alpha/\Gamma_\alpha) \subset \phi_\beta(V_\beta/\Gamma_\beta)$, we require an embedding $i_\alpha : \Gamma_\alpha \hookrightarrow \Gamma_\beta$.

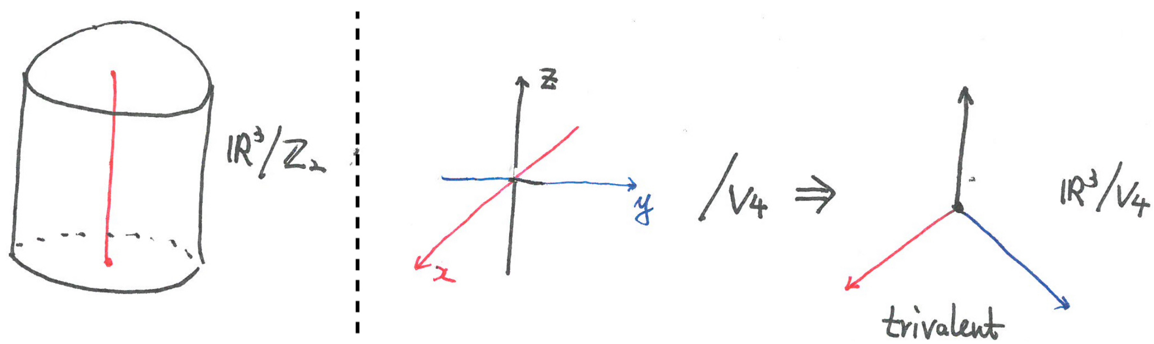
Remark. In general, Γ is allowed to be any finite group acting on V via a group homomorphism $\rho : \Gamma \to O(n)$. The morphism ρ need not to be injective. In the definition above, we assume ρ is injective, so the action is effective. □

Let us look at some simple orbifolds. When $n = 3$, we focus on the case when

$$\Gamma = 1,\ \mathbb{Z}_2 = \{\begin{pmatrix} \pm I_2 & 0 \\ 0 & 1 \end{pmatrix}\},$$

$$V_4 = \mathbb{Z}_2 \times \mathbb{Z}_2 = \{A = \mathrm{diag}(\pm 1, \pm 1, \pm 1) : \det A = 1\}.$$

Topologically, $\mathbb{R}^3/\mathbb{Z}_2 \cong \mathbb{R}^3/V_4 \cong \mathbb{R}^3$, but they have some orbifold loci:



We will go back and forth between two objects:

(1) (Y^3, K) where K is a trivalent graph embedded in Y^3.
(2) Y^3 is a 3-dimensional orbifold with restrictive isotropy groups: $\Gamma = \{e\}, \mathbb{Z}_2, V_4$.

They are called webs in the physical literature.

In dimension 4, we have so-called "foams". The isotropy group Γ is allowed to be

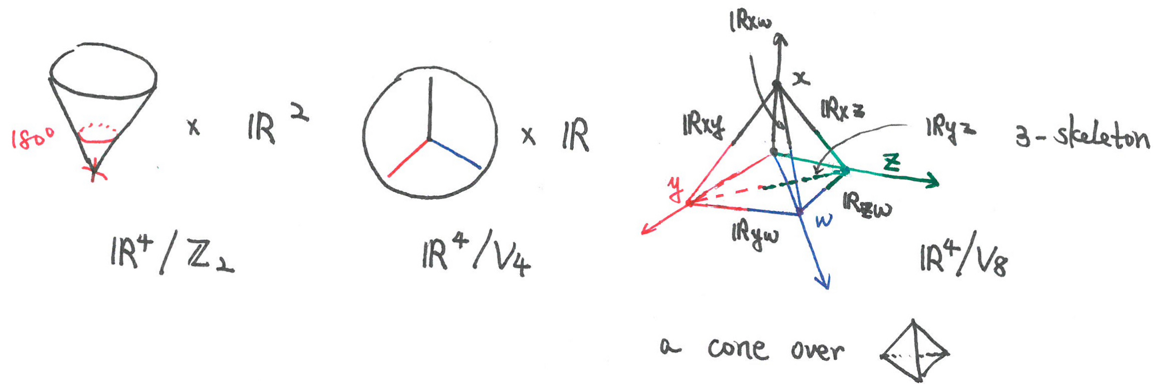
- $\{e\}$
- $\mathbb{Z}_2 = \mathrm{diag}(\pm I_2, I_2)$.
- $V_4 = \{\mathrm{diag}(\pm 1, \pm 1, \pm 1, I) : \det = 1\}$.
- $V_8 = \{\mathrm{diag}(\pm 1, \pm 1, \pm 1, \pm 1) : \det = 1\}$.

Their local pictures look like:

In the last case, the 3-skeleton of $\mathbb{R}^4/V_8$ near the origin is a cone over a tetrahedron. In all cases, $\mathbb{R}^4/\Gamma \cong \mathbb{R}^4$. For instance, one may start with the quotient

$$\mathbb{R}^4/\{\mathrm{diag}(\pm 1, \pm 1, \pm 1, \pm 1)\}$$

and think of $\mathbb{R}^4/V_8 \cong \mathbb{R}^4$ as a union of two fundamental domains.



35.2. Orbifold Principal Bundles

An orbifold principal bundle

$$\begin{array}{ccc} G & \longrightarrow & P \\ & & \downarrow \\ & & (X,F) \end{array}$$

is defined by requiring the action of Γ on V to be extendable over $V \times G$ in a compatible way:

$$\begin{tikzcd} & V \times G \arrow[d] \\ \Gamma \arrow[ur, dashed, "action"] \arrow[r, "action"] & V \end{tikzcd}$$

We assume that

- the action of Γ on $V \times G$ is free.

In particular, this implies the total space P is a manifold with a non-free (right) G-action. The base space X is realized as "a global quotient":

$$P/G.$$

Seifert fibered manifolds form an interesting class of examples. Suppose an oriented 3-manifold Y admits an S^1-action such that

- The stabilizer at a point $y \in Y$ is always finite.
- For generic points, the stabilizer is trivial.

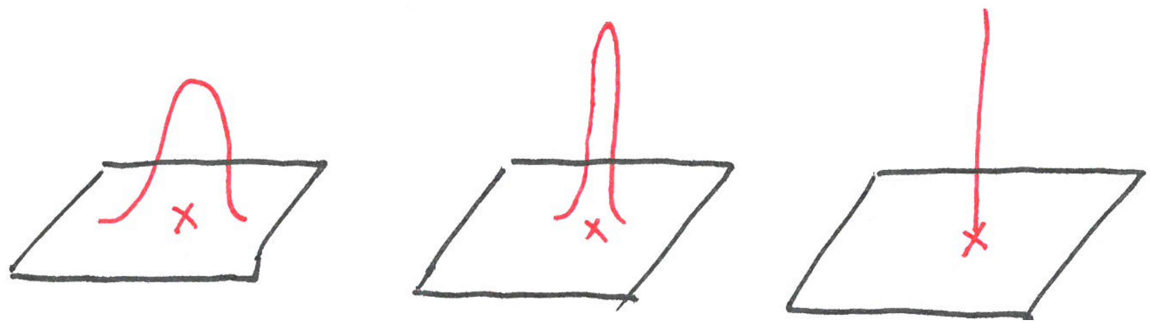
Then the quotient space Y/S^1 becomes a 2-dimensional orbifold. The $U(1)$-bundle

$$\begin{array}{ccc} S^1 & \longrightarrow & Y^3 \\ & & \downarrow \\ & & Y/S^1 \end{array}$$

is an orbifold $U(1)$-bundle.

35.3. Bubbling

Let us take a $PU(2)$-bundle. In the orbifold setting, it is not hard to prove Uhlenbeck's compactness theorem if we can show the limit of a sequence of equivariant connections is still equivariant.



Recall that a bubble at a smooth point will drop the Chern-Weil integral:

$$k = \frac{1}{8\pi^2} \int_X \mathrm{tr}(F_A \wedge F_A)$$

by 1. For 4-dimensional orbifolds, we can still have bubbling. The change of k (under a bubbling) will depend on the stabilizer:

- If $\mathrm{Stab}(x) = \{e\}$, then $k \mapsto k - 1$.
- If $\mathrm{Stab}(x) = \mathbb{Z}_2$, then $k \mapsto k - \frac{1}{2}$.
- If $\mathrm{Stab}(x) = V_4$, then $k \mapsto k - \frac{1}{4}$.

Note that the Chern-Weil integral is happening downstairs, while over an orbifold chart, we get twice (four times) the integral.

Recall that the moduli space of 1-instanton on S^4 is acted on by the conformal group $SO(5,1)$ on $\mathbb{R}^4$. Away from the standard instanton $[A_{std}]$ which is $SO(4)$ invariant, the moduli is parametrized by a center of mass $x \in S^4$ and a scale $r \in (0, \infty)$, so

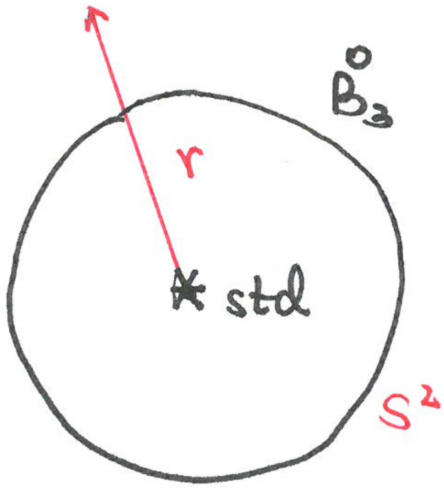
$$\mathcal{M}_1(S^4) \cong \mathring{B}_5.$$

For orbifolds, this picture is generalized as:

$$\mathbb{R}^4/\mathbb{Z}_2 \to \mathcal{M}_1(S^4/\mathbb{Z}_2) \cong \mathring{B}_3,\ x \in \mathrm{Fix}(\mathbb{Z}_2)$$
$$\mathbb{R}^4/V_4 \to \mathcal{M}_1(S^4/V_4) \cong \mathring{B}_2,\ x \in \mathrm{Fix}(V_4)$$
$$\mathbb{R}^4/V_8 \to \mathcal{M}_1(S^4/V_8) \cong \mathring{B}_1,\ x \in \mathrm{Fix}(V_8).$$

In each case, the center of the mass x lies in the fixed point set of $\Gamma = \mathbb{Z}_2, V_4, V_8$. The moduli space is then parametrized by

$$\{(x, r) : x \in \mathrm{Fix}(\Gamma), r \in (0, \infty)\} \bigcup \text{the standard instanton},$$



so it is the interior of some balls. Consider the moduli space of framed instantons on S^4. Either we consider the gauge group that fixes the fiber at the north pole $N \in S^4$, or we work with a pair

$$(A, \mathfrak{f})$$

where $\mathfrak{f}$ is a framing of $\operatorname{ad} P$ at N. In the second case, we use the full gauge group. We have

$$\begin{aligned} \mathbb{R}^4 \to \mathcal{M}_1^o(S^4) &\cong \mathring{B}_5 \times SO(3) && \dim = 8 \\ \mathbb{R}^4/\mathbb{Z}_2 \to \mathcal{M}_1^o(S^4/\mathbb{Z}_2) &\cong \mathring{B}_3 \times O(2) && \dim = 4 \\ \mathbb{R}^4/V_4 \to \mathcal{M}_1^o(S^4/V_4) &\cong \mathring{B}_2 \times V_4 && \dim = 2 \\ \mathbb{R}^4/V_8 \to \mathcal{M}_1^o(S^4/V_8) &\cong \mathring{B}_1 \times (??) && \dim = 1. \end{aligned}$$

(We need Γ to act on $\mathbb{R}^4 \times SO(3)$ freely.)

The group attached is the commutant of Γ in $SO(3)$. These numbers fit into the dimension formula:

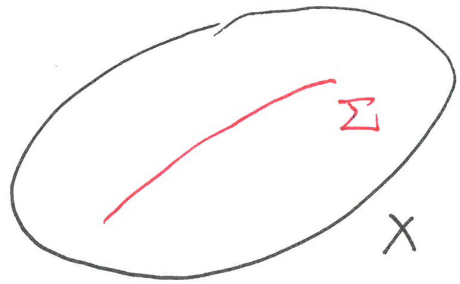
$$\dim(\mathcal{M}_1^0(S^4/\Gamma)) = 8k_\Gamma$$

where k_Γ is the drop of the Chern-Weil integral for a bubbling (at a point with isotropy group Γ). Readers can compare this with the monotonicity condition in symplectic topology, by which we get rid of the Novikov ring.

The number $8k_\Gamma$ also reflects the right dimension jump when we glue an instanton at an orbifold point.

$$\mathcal{M}_{k-k_\Gamma}(X) \mapsto \mathcal{M}_k(X).$$

Uhlenbeck's compactification now involves more strata. A bubble is now more energetically effective.



Suppose $\Gamma = \mathbb{Z}_2$ and the singular locus is an embedded surface $\Sigma \subset X$. Then

$$\begin{aligned}
\overline{\mathcal{M}_k(X^4,\Sigma^2)}^{Uhl} &= \mathcal{M}_k(X^4,\Sigma^2) && \dim = \dim_k \\
&\bigcup \mathcal{M}_{k-1/2}(X,\Sigma)\times\Sigma && \dim = \dim_k -2\ (-4+2) \\
&\bigcup \mathcal{M}_{k-1}(X,\Sigma)\times(X\backslash\Sigma) && \dim = \dim_k -4\ (-8+4) \\
&\bigcup \mathcal{M}_{k-1}(X,\Sigma)\times\operatorname{Sym}^2\Sigma && \dim = \dim_k -4\ (-8+4) \\
&\bigcup \mathcal{M}_{k-3/2}(X,\Sigma)\times\operatorname{Sym}^3\Sigma && \dim = \dim_k -6\ (-12+6).
\end{aligned}$$

Lecture 36. Classification of $SO(3)$ bundles over orbifolds

At some point in the class, Tom started saying:

" I do not know if any of you suffered from English in your high school." He stopped and looked at the audience. I was not a native speaker in English and pointed at myself with a pen.

Tom:" Well, I guess it means different things for different people..."

Tom continued with something I did not understand, and he started to recite a poem:

"This is the way the world ends. Not with a bang but a whimper."

I was totally confused at that time. Two days later, it was when I started typing the lecture note that I realized it was a famous poem by T.S. Eliot, and Tom really meant:

"This is the way **the class** ends. Not with a bang but a whimper."

36.1. Classification of $SO(3)$ bundles over orbifolds

Let X^4 be an closed oriented 4-manifold and $\Sigma \subset X$ be a smoothly embedded 2-surface, not necessarily orientable. We will view $X_\Sigma = (X, \Sigma)$ as an orbifold. The goal of this lecture is to find the topological classification of $SO(3)$-bundles over (X, Σ):

$$PU(2) \to \hat{P} \to (X, \omega).$$

When $\Sigma = \varnothing$, $(X, \varnothing)$ is a smooth manifold. In this case, we have the Dold-Whitney theorem:

Theorem 36.1.1. *A principal $SO(3)$-bundle $P \to X$ over a 4-complex X is classified by*

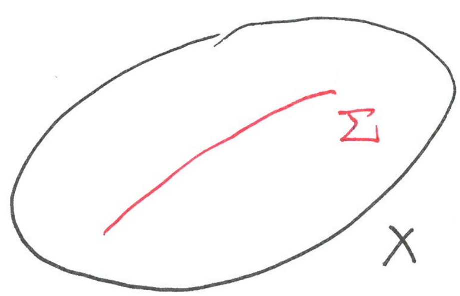
$$p_1(P) \in H^4(X, \mathbb{Z}),\ w_2(P) \in H^2(X, \mathbb{Z}_2).$$

They are subject to a single relation:

$$\mathcal{P}(w_2(P)) \equiv p_1(P) \mod 4.$$

where $\mathcal{P} : H^2(X.\mathbb{Z}_2) \to H^2(X, \mathbb{Z}_4)$ is the Pontryagin square.

As for orbifolds, we need to understand connections on bundles more concretely.



Choose an orbifold connection $\hat{A}$ in $\hat{P}$, let

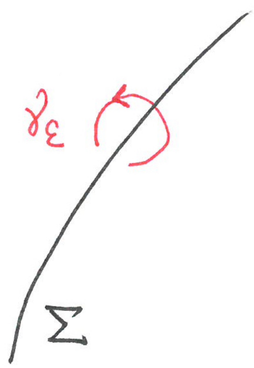
$$A = \hat{A}|_{X_\Sigma \setminus \Sigma = X \setminus \Sigma} \text{ in } P = \hat{P}|_{X_\Sigma \setminus \Sigma}.$$

The holonomy of A at a linking circle of Σ is an element in $SO(3)$ of order 2. Locally, we have a quotient:

$$\mathbb{R}^4 \times SO(3)/\mathbb{Z}_2$$

The isotropy group $\mathbb{Z}_2$ acts on non-trivially at the fiber of a singular point. The limiting holonomy is

$$\sim_{conj} \begin{pmatrix} -1 & & \\ & -1 & \\ & & 1 \end{pmatrix} \in SO(3).$$



Choose an orbifold chart about $x \in \Sigma$. Locally, it is

$$D^2 \times D^2/\mathbb{Z}_2 = (D^2/\mathbb{Z}_2) \times D^2.$$

Removing the zero section, $(D^2 \setminus \{0\}/\mathbb{Z}_2) \times D^2$ is a chart for $X - \Sigma$. Using the polar coordinate, let

$$\eta = d\theta$$

be the angular 1-form on the first factor such that

$$\int_{S^1} \eta = 2\pi.$$

Let E be the associated $\mathbb{R}^3$-bundle to $\hat{P}$ (defined using orbifold charts). By the $\mathbb{Z}_2$ action along Σ, we have

$$E|_\Sigma = L \oplus \lambda$$

on which $[1] \in \mathbb{Z}_2$ acts on by $(-I_2, 1)$. $\lambda \to \Sigma$ is a real line bundle. Let

$$\xi = \begin{pmatrix} 0 & -\frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then $\exp(2\pi\xi) = -I_2$. The sum $A + \eta \otimes \xi$ is a connection on

$$P|_{D^2\backslash\{0\}\times D^2}.$$

Moreover, the linking holonomy is

$$\lim_{\epsilon} hol_{\gamma_\epsilon}(A + \eta \otimes \xi) = \mathrm{Id}\,.$$

Using $A+\eta\otimes\xi$, we get a trivialization of P on $(D^2\backslash\{0\}\times D^2)/\mathbb{Z}_2$ and this trivialization extends to $(D^2 \times D^2)/\mathbb{Z}_2$.

To globalize this picture, we need to make ξ a section of λ. Assuming both λ and Σ are orientable:

- ξ is a non-vanishing section of $\lambda \to \Sigma$. Up to homotopy, this is the same as an orientation of λ.
- η is a connection 1-form in the normal circle bundle of $\mathbb{S}(N) \to \Sigma$. By choosing an orientation of $N \to \Sigma$, we assume

  $$\int_{S^1} \eta = 2\pi.$$

  Thus, up to homotopy, η is determined by an orientation of $N \to \Sigma$.

Therefore, $\xi \otimes \lambda$ is well-defined globally (in a tubular neighborhood of Σ) if both λ and Σ are orientable.

Since it is only necessary to define the product $\xi \otimes \lambda$, we only need:

$$o(\Sigma) \otimes \lambda \to \Sigma$$

is trivial. Here, $o(\Sigma) \to \Sigma$ is the orientation line bundle over Σ. Moreover, an orientation of

$$o(\Sigma) \otimes \lambda$$

determines a homotopy class of $\xi \otimes \lambda$.

From now on, assume $o(\Sigma) \otimes \lambda$ is orientable and an orientation τ is fixed. Using the connection $A + \eta \otimes \xi$, we obtain an $SO(3)$-bundle

$$P_\tau \to X$$

that extends $P \to X\backslash\Sigma$. Moreover,

$$P_\tau \times_{SO(3)} \mathbb{R}^3|_\Sigma = \lambda \oplus K_\tau$$

where K_τ is a rank 2 vector bundle with $o(K_\tau) \cong \lambda$. $A + \eta \otimes \xi$ is a connection in P_τ which respects this splitting on Σ. We have

$$\begin{aligned} p_1(P_\Sigma) &\in H^4(X, \mathbb{Z}), \\ e(K_\tau) &\in H^2(\Sigma, o(K_\tau)) = H^2(\Sigma, \lambda) = H^2(\Sigma, o(\Sigma)) \cong \mathbb{Z}. \end{aligned}$$

This allows us to define

$$k_\tau = \langle -\frac{1}{4} p_1(P_\tau), [X] \rangle$$
$$l_\tau = \langle -\frac{1}{2} e(K_\tau), [\Sigma] \rangle.$$

By changing τ into the opposite orientation $\bar{\tau} = -\tau$, these numbers are changed by

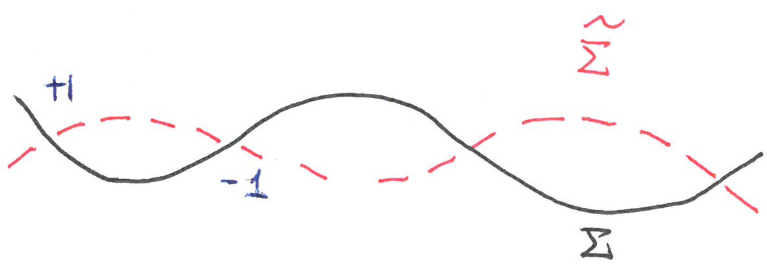
$$k_{\bar{\tau}} = k_\tau + l_\tau - \frac{1}{4}\Sigma \cdot \Sigma.$$
$$l_{\bar{\tau}} = \frac{1}{2}\Sigma \cdot \Sigma - l_\tau.$$

The triple $(k_\tau, l_\tau, w_2(P_\tau))$ classifies the orbifold bundle P (compare with the Dold-Whitney Theorem).

Let us explain the self-intersection number $\Sigma \cdot \Sigma$. When Σ is orientable, perturb the embedding $\Sigma \hookrightarrow X$ slightly into $\tilde{\Sigma}$ and count the intersection points in

$$\tilde{\Sigma} \cap \Sigma$$

with signs.



This counting is dependent of orientations. In fact, only a local orientation is needed to define the sign, so the intersection number can be defined for non-orientable surfaces.

Alternatively, the Euler class of the normal bundle lives in

$$e(N_\Sigma) \in H^2(\Sigma, o(\Sigma)) \cong \mathbb{Z}.$$

Define $\Sigma \cdot \Sigma = \langle e(N_\Sigma), [\Sigma, o(\Sigma)] \rangle$.

When Σ is orientable, $[\Sigma]$ defines a element (up to sign) in $H_2(X, \mathbb{Z})$ and

$$\Sigma \cdot \Sigma = q_X([\Sigma], [\Sigma]).$$

When Σ is not orientable, however, the fundamental class $[\Sigma] \in H_2(\Sigma, o(\Sigma))$ does not map into in $H_2(X, \mathbb{Z})$, so the self intersection number can not be defined in this way.

Example 36.1.2. $\mathbb{RP}^2_\pm \subset S^4$. Let S^4 be the unit sphere in traceless symmetric 3×3 matrices ($\dim_{\mathbb{R}} = 5$):

$$\begin{pmatrix} @ & * & * \\ @ & * & * \\ @ & @ & * \end{pmatrix}.$$

Then $SO(3)$ acts on S^4 by matrix conjugation. Let

$$\begin{aligned} \mathbb{RP}^2_+ &= \{\text{matrices with a positive eigenspace of dimension } 2\}, \\ \mathbb{RP}^2_- &= \{\text{matrices with a negative eigenspace of dimension } 2\} \end{aligned}$$

Hence, if $A \in \mathbb{RP}^2_+$, then A is conjugate to

$$\lambda \begin{pmatrix} 1 & & \\ & 1 & \\ & & -2 \end{pmatrix}$$

where λ is fixed by the length of A. The stabilizer of A is $O(2) \subset SO(3)$, so the orbit is

$$SO(3)/O(2) \cong \mathbb{RP}^2.$$

There is another way to think about $\mathbb{RP}^2_\pm$. A theorem by Kuiper, Arnold and others states that

$$\mathbb{CP}^2/\text{complex conjugation} \cong S^4.$$

The action is induced from the conjugation of complex numbers. Hence, the real projective space $\mathbb{RP}^2_- \subset \mathbb{CP}^2$ is invariant under the action. $\mathbb{RP}^2_- \subset \mathbb{CP}^2$ is Lagrangian. Hence, $T^*\mathbb{RP}^2_- \cong N(\mathbb{RP}^2_+)$ and

$$1 = \chi(\mathbb{RP}^2) = \langle e(T\mathbb{RP}^2_-), [\mathbb{RP}^2_-, o(\mathbb{RP}^2_-)]\rangle = -\langle e(T^*\mathbb{RP}^2_-), [\mathbb{RP}^2_-, o(\mathbb{RP}^2_-)]\rangle.$$

Since $\mathbb{RP}^2_-$ is fixed, after passing to the quotient, the self-intersection number will be doubled:

$$\mathbb{RP}^2_- \cdot \mathbb{RP}^2_- = -2.$$

On the other hand, a real conic with no real roots gives an $S^2 \subset \mathbb{CP}^2$ invariant under conjugation. For a conic, the intersection number is

$$S^2 \cdot S^2 = +4.$$

Passing to the quotient, the self-intersection of $\mathbb{RP}^2_+ = S^2/\mathbb{Z}_2$ is divided by 2:

$$\mathbb{RP}^2_+ \cdot \mathbb{RP}^2_+ = 2.$$

Lecture 37. **(Special Lecture 1) Gluing via Excision**

Tom decided to continue his lecture in this summer. The first lecture is about Gluing via Excision.

37.1. Excision of the Index

Let us recall the easy proof of the excision theorem from Lecture 22. We start with the following setup:

(1) $X_1 = U_1^L \cup U_1^R$ and $X_2 = U_2^L \cup U_2^R$.
(2) $W_1 = U_1^L \cap U_1^R$ is diffeomorphic to $W_2 = U_2^L \cap U_2^R$.
(3) We have elliptic differential operators:

$$D_1 : \Gamma(E_1) \to \Gamma(F_1) \text{ on } X_1$$
$$D_2 : \Gamma(E_2) \to \Gamma(F_2) \text{ on } X_2.$$

(4) There exist bundle isomorphisms

$$\phi_E : E_1|_{W_1} \to E_2|_{W_2},\ \phi_F : F_1|_{W_1} \to F_2|_{W_2}$$

that cover the diffeomorphism in (2) and intertwine D_1 and D_2:

$$D_2 = \phi_F \circ D_1 \circ \phi_E^{-1} \text{ on } W_2.$$

Under these assumptions, we can

(1) Define $X_3 = U_1^L \cup U_2^R$ and $X_4 = U_2^L \cup U_1^R$.
(2) Obtain $E_3 \to X_3$ by gluing $E_1|_{U_1^L}$ and $E_2|_{U_2^R}$ using ϕ_E.
(3) Obtain $E_4 \to X_4$ by gluing $E_2|_{U_1^R}$ and $E_1|_{U_2^L}$ using ϕ_E. Similarly, define $F_3 \to X_3$ and $F_4 \to X_4$ using ϕ_F.
(4) Define elliptic differential operators:

$$D_3 : \Gamma(E_3) \to \Gamma(F_3) \text{ on } X_3$$
$$D_4 : \Gamma(E_4) \to \Gamma(F_4) \text{ on } X_4.$$

Then the conclusion is

$$\operatorname{Ind} D_1 + \operatorname{Ind} D_2 = \operatorname{Ind} D_3 + \operatorname{Ind} D_4.$$

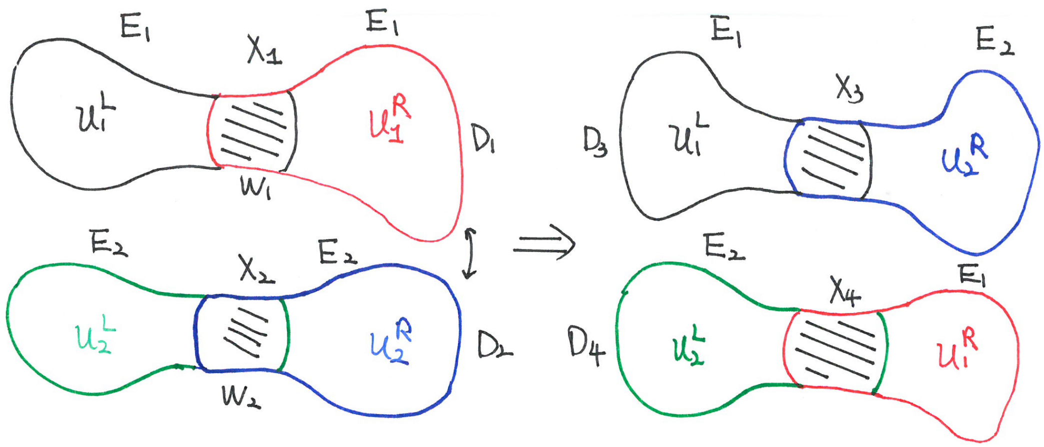
The easy proof of the excision theorem relies on a partition of unity on (ϕ_L, ϕ_R) on X_i $(i = 1, 2)$ such that

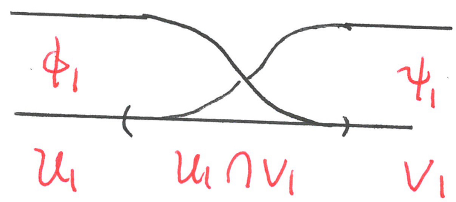
$$\phi_L^2 + \psi_R^2 = 1.$$

As functions on X_i, supp $\phi_L \subset U_i^L$ and supp $\phi_R \subset U_i^R$. Moreover,

$$\phi_L \equiv 1 \text{ on } X_i - U_i^R$$
$$\phi_R \equiv 1 \text{ on } X_i - U_i^L.$$

These functions are not constant only on W_i. Since $W_1 \cong W_2$, the same symbol ϕ_L (and ϕ_R) is used to denote functions on X_i for $1 \leqslant i \leqslant 4$ by abuse of notations.





Consider bundle maps:

$$\Gamma(E_1) \oplus \Gamma(E_2) \xrightarrow{\alpha} \Gamma(E_3) \oplus \Gamma(E_4)$$
$$\Gamma(F_1) \oplus \Gamma(F_2) \xleftarrow{\beta} \Gamma(F_3) \oplus \Gamma(F_4)$$

where

$$\alpha = \begin{pmatrix} \phi_L & -\phi_R \\ \phi_R & \phi_L \end{pmatrix}, \beta = \begin{pmatrix} \phi_L & \phi_R \\ -\phi_R & \phi_L \end{pmatrix}. \tag{49}$$

Formally, $\alpha \circ \beta = \beta \circ \alpha = \mathrm{Id}$. By direct computation,

$$\beta \circ \begin{pmatrix} D_3 & 0 \\ 0 & D_4 \end{pmatrix} \circ \alpha = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix} + \text{ some error terms.}$$

Error terms involve only multiplication operators by derivatives of ϕ_L and ϕ_R, which are compact, so

$$\mathrm{Ind}\, \beta \circ \begin{pmatrix} D_3 & 0 \\ 0 & D_4 \end{pmatrix} \circ \alpha = \mathrm{Ind} \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix}.$$

When manifolds change their sizes, we can try to make norms of error terms small. For instance, let

$$W_1 \cong W_2 = [-T, T] \times Y,$$

where Y is a closed 3-manifold. Let X_1 and X_2 be closed 4-manifolds that contain a long cylinder. Fixing a partition of unity $\{\phi_L^1, \phi_R^1\}$ with $(\phi_L^1)^2 + (\phi_R^1)^2 = 1$, let

$$\phi_L^T(t) = \phi_L^1(\frac{t}{T}),\ \phi_R^T(t) = \phi_R^1(\frac{t}{T})$$

for $t \in [-T, T]$. Then

$$\|\nabla\phi_L^T\|_\infty, \|\nabla\phi_R^T\|_\infty < \frac{C}{T}$$

Hence, the norm of error terms decays as $1/T$ as $T \to \infty$ and they do not affect the index. In particular, if D_1 and D_2 are surjective with uniformly (in T) bounded one-sided inverses, then the same is true of D_1' and D_2'.

37.2. Gluing

For the gluing problem of non-linear operators, the minus sign in definitions (49) of α and β would be annoying. Moreover, it will be necessary to work with an actual partition of unity

$$\phi_L^T + \phi_R^T = 1.$$

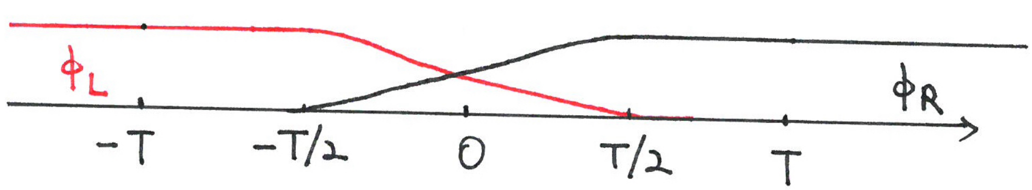
When the parameter T is clear from the context, we will write ϕ_L and ϕ_R for short.

In what follows, we will frequently work with a cylinder of length $2T$:

$$[-T, T] \times Y.$$

Cut-off functions will be applied to a shorter cylinder: $[-T/2, T/2] \times Y$; so

$$\text{supp } d\phi_L, \text{supp } d\phi_R \subset [-T/2, T/2] \times Y.$$



To make it concrete, let us describe the gluing problem first. Consider 4-manifolds with cylindrical ends:

$$X_1^* = X_1 \bigcup [0, \infty) \times Y$$
$$X_2^* = (-\infty, 0] \times Y \bigcup X_2$$

where X_1 and X_2 are compact 4-manifolds with boundary:

$$\partial X_1 = Y, \partial X_2 = -Y.$$

Suppose A_i is an ASD connection on the principle G-bundle $P_i \to X_i$ $(i = 1, 2)$. When restricted to ends, P_1 and P_2 are canonically trivialized:

$$P_1|_{[0,\infty)\times Y} = G \times [0, \infty) \times Y,\ P_2|_{(-\infty,0]\times Y} = G \times (\infty, 0] \times Y.$$

Over the cylinder, we have

$$A_i = A_0 + a_i^\infty$$

where A_0 is a translation invariant ASD connection on $\mathbb{R} \times Y$ in the temporal gauge. Therefore,

$$A_0 = \frac{d}{dt} + B_0$$

for a flat G-connection B_0 on Y. We require B_0 to be irreducible as a critical point of the Chern-Simons functional. Moreover, we assume a_1^∞ and a_2^∞ decay exponentially along the end, in any reasonable Sobolev norms:

$$a_1^\infty \xrightarrow{exp} 0, \text{ as } t \to \infty$$
$$a_2^\infty \xrightarrow{exp} 0, \text{ as } t \to -\infty.$$

The goal is to find some corrected connections:

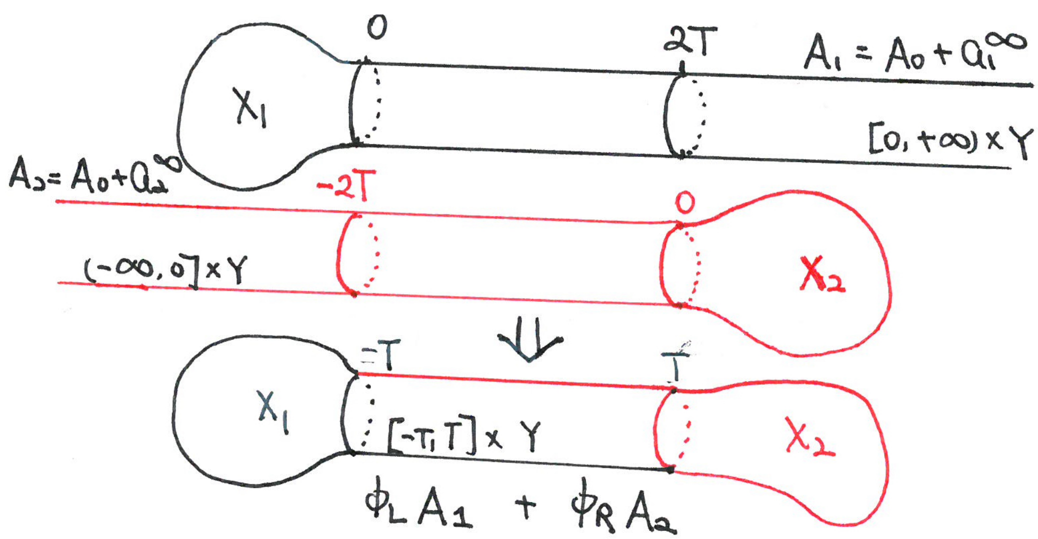
$$A_i^T = A_i + b_i^T$$

with $b_i^T \in \Omega^1(X_i^*, \operatorname{ad} P_i)$ such that the patched connection

$$\phi_L A_1^T + \phi_R A_2^T$$

solves the ASD equation on

$$X_{12}^T = X_1 \cup ([-T, T] \times Y) \cup X_2.$$



For brevity, we will also write $A_i^T = A_0 + a_i^T$ on the cylinder. Hence,

$$a_i^T = a_i^\infty + b_i^T \text{ on } [0, \infty) \times Y \text{ or } (-\infty, 0] \times Y.$$

To make it analogous to excision, we wish to write

$$(-\phi_R) A_2^T + \phi_L A_1^T$$

as a connection on $\mathbb{R} \times Y$. But it does not make any sense. If instead, we try

$$\phi_R A_2^T + \phi_L A_1^T$$

the matrix α will become

$$\begin{pmatrix} \phi_L & \phi_R \\ \phi_R & \phi_L \end{pmatrix},$$

which is not invertible in general. To reconcile this problem, Tom suggested introducing the third cylindrical manifold:

$$X_3^* = \mathbb{R} \times Y$$

and choose $A_3 = A_0$ as the background ASD connection. We wish to find a correction

$$A_3^T = A_3 + a_3^T = A_0 + a_3^T$$

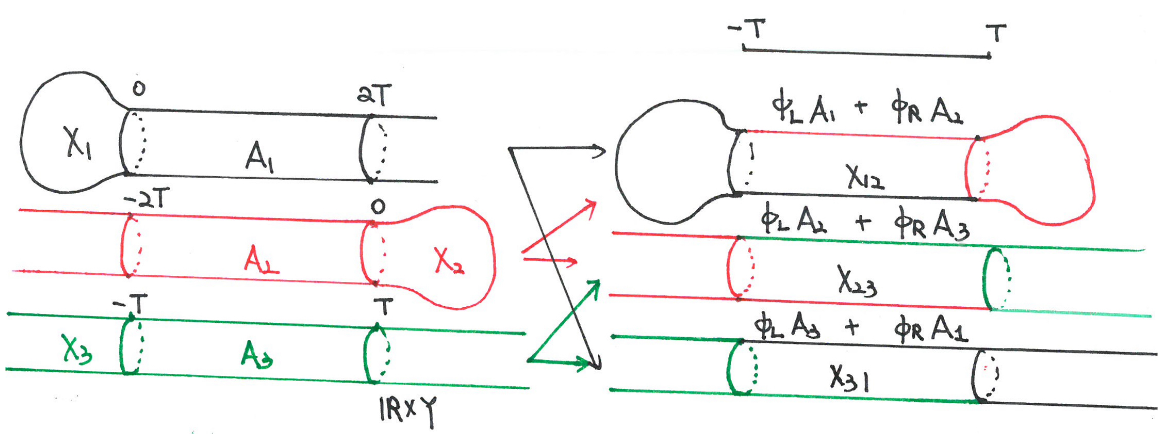
such that patched connections:

$$\begin{aligned} &\phi_L A_1^T + \phi_R A_2^T \text{ on } X_{12}^T = X_1 \cup [-T, T] \times Y \cup X_2, \\ &\phi_L A_2^T + \phi_R A_3^T \text{ on } X_{23}^T = \mathbb{R} \times Y, \\ &\phi_L A_3^T + \phi_R A_1^T \text{ on } X_{31}^T = \mathbb{R} \times Y. \end{aligned}$$

solve ASD equations. The α-matrix we use will be

$$\alpha = \begin{pmatrix} \phi_L & \phi_R & 0 \\ 0 & \phi_L & \phi_R \\ \phi_R & 0 & \phi_L \end{pmatrix},$$

with determinant $\det(\alpha) = \phi_L^3 + \phi_R^3 \geqslant \frac{1}{4}$.



To produce an elliptic system, we require corrections to satisfy the gauge fixing condition:

$$\begin{aligned} d_{A_1}^* b_1^T &= 0 \text{ on } X_1^* \\ d_{A_2}^* b_2^T &= 0 \text{ on } X_2^* \\ d_{A_3}^* b_3^T &= 0 \text{ on } X_3^*. \end{aligned} \tag{50}$$

Finally, we deal with the β-matrix. One choice of β is the inverse of α:

$$\beta = \alpha^{-1} = \frac{1}{\phi_L^3 + \phi_R^3} \begin{pmatrix} \phi_L^2 & -\phi_R\phi_L & \phi_R^2 \\ \phi_R^2 & \phi_L^2 & -\phi_R\phi_L \\ -\phi_R\phi_L & \phi_R^2 & \phi_L^2 \end{pmatrix}.$$

For simplicity, we will use

$$\beta = \begin{pmatrix} \beta_L & 0 & \beta_R \\ \beta_R & \beta_L & 0 \\ 0 & \beta_R & \beta_L \end{pmatrix}$$

where $\beta_R \equiv 1$ on supp ϕ_R and $\beta_L \equiv 1$ on supp ϕ_L. Again, cut-off functions β_L and β_R depend on the length parameter T, but we omit the superscript T from our notations. In particular,

$$\beta_R\phi_R = \phi_R,\ \beta_L\phi_L = \phi_L.$$

This β-matrix will allow us to transfer sections back

$$(X_{12}, X_{23}, X_{31}) \to (X_1^*, X_2^*, X_3^*).$$

The system of equations will be invariant if we permute $(1, 2, 3)$ cyclically:

$$(1, 2, 3) \to (2, 3, 1),$$

so we will focus on the first equation. We compute the self-dual part of the curvature of

$$\phi_L A_1^T + \phi_R A_2^T$$

on the long cylinder $[-T, T] \times Y \subset X_{12}^T$:

$$\begin{aligned} F_{A_{12}}^+ &= d^+(\phi_L a_1^T + \phi_R a_2^T) + \frac{1}{2}[(\phi_L a_1^T + \phi_R a_2^T) \wedge (\phi_L a_1^T + \phi_R a_2^T)]^+ \\ &= \phi_L(d^+ a_1^T + \frac{1}{2}[a_1^T \wedge a_1^T]^+) + \phi_R(d^+ a_2^T + \frac{1}{2}[a_2^T \wedge a_2^T]^+) \\ &\quad + \phi_L(\phi_L - 1)\frac{1}{2}[a_1^T \wedge a_1^T] + \phi_R(\phi_R - 1)\frac{1}{2}[a_2^T \wedge a_2^T] \\ &\quad + (d\phi_L \wedge a_1^T)^+ + (d\phi_R \wedge a_2^T)^+ + \phi_L\phi_R[a_1^T \wedge a_2^T]^+ \\ &= \phi_L(d^+ a_1^T + \frac{1}{2}[a_1^T \wedge a_1^T]^+) + \phi_R(d^+ a_2^T + \frac{1}{2}[a_2^T \wedge a_2^T]^+) \\ &\quad - \frac{1}{2}\phi_L\phi_R[(a_1^T - a_2^T) \wedge (a_1^T - a_2^T)]^+ + (d\phi_L \wedge a_1^T)^+ + (d\phi_R \wedge a_2^T)^+. \end{aligned}$$

We use β-matrix to go back to X_1^*. We need to compute

$$\beta_L F^+_{A_{12}} + \beta_R F^+_{A_{31}}.$$

The result is on the half cylinder $[0,\infty)\times Y \subset X_1^*$ is

$$\begin{aligned}
&(d^+a_1^T + \frac{1}{2}[a_1^T\wedge a_1^T]^+) + \beta_L\phi_R(d^+a_2^T + \frac{1}{2}[a_2^T\wedge a_2^T]^+) + \beta_R\phi_L(d^+a_3^T + \frac{1}{2}[a_3^T\wedge a_3^T]^+)\\
&+\beta_L(d\phi_L\wedge da_1^T)^+ + \beta_R(d\phi_R\wedge da_1^T)^+ + \beta_L(d\phi_R\wedge da_2^T)^+ + \beta_R(d\phi_L\wedge da_3^T)^+\\
&-\frac{1}{2}\beta_L\phi_R\phi_L[(a_1^T-a_2^T)\wedge(a_1^T-a_2^T)]^+ - \frac{1}{2}\beta_R\phi_R\phi_L[(a_1^T-a_3^T)\wedge(a_1^T-a_3^T)]^+.
\end{aligned}$$

The first line can be cast into a simpler form on the whole manifold X_1^*:

$$F^+_{A_1^T} + \beta_L\phi_R F^+_{A_2^T} + \beta_R\phi_L F^+_{A_3^T}$$

These coefficients are precisely entries in the first row of $\beta\cdot\alpha$:

$$\beta\cdot\alpha = \begin{pmatrix} 1 & \beta_L\phi_R & \beta_R\phi_L\\ \beta_R\phi_L & 1 & \beta_L\phi_R\\ \beta_L\phi_R & \beta_R\phi_L & 1\end{pmatrix}.$$

Since $F^+_{A_1}\equiv 0$, we have

$$F^+_{A_1^T} = d^+_{A_1}b_1^T + \frac{1}{2}[b_1^T\wedge b_1^T]^+.$$

Write $b = (b_1^T, b_2^T, b_3^T)^t \in \Omega^1(X_1,\operatorname{ad}P_1)\oplus\ \Omega^1(X_2,\operatorname{ad}P_2)\oplus\ \Omega^1(X_3,\operatorname{ad}P_3)$. It is important to know from this setup that the system of equations to be solved can be cast into a standard form:

$$\beta\cdot\alpha\begin{pmatrix} d^+_{A_1}b_1^T\\ d^+_{A_2}b_2^T\\ d^+_{A_3}b_3^T\end{pmatrix} = L^T(b) + \mu^T(b,b) + E^T(A_1,A_2,A_3).$$

where L^T is a bounded linear operator, μ^T is a bounded bilinear form and E^T is an error term that involves only the reference connections A_1, A_2 and A_3. Use $\mathcal{H}$ to denote any Hilbert space of interest. The point is that as $T\to\infty$, we can make

$$\|L^T\|_{\mathcal{H}\to\mathcal{H}}\to 0 \text{ and } \|E^T\|_{\mathcal{H}}\to 0. \tag{51}$$

For simplicity, we first assume $D_{A_i} = d^+_{A_i}\oplus d^*_{A_i}$ is an invertible operator for $1\leqslant i\leqslant 3$. Hence, the dimension of moduli spaces to be glued is zero:

$$\dim\mathcal{M}_i = 0 \text{ for } 1\leqslant i\leqslant 3.$$

Together with the gauge fixing condition (50), we obtain a system of equations:

$$D(b) = (\beta\cdot\alpha)^{-1}(L^T(b) + \mu^T(b,b) + E^T(A_1,A_2,A_3)). \tag{52}$$

where

$$D = \begin{pmatrix} D_{A_1} & 0 & 0\\ 0 & D_{A_2} & 0\\ 0 & 0 & D_{A_3}\end{pmatrix}.$$

Given (51), this equation has a unique solution with $\|b\|_{\mathcal{H}} \ll 1$ when $T \gg 1$. This will resolve the gluing problem.

In general, if D_{A_i}'s are surjective with non-trivial kernels, we write

$$\ker D_{A_i} = H^1_{A_i}.$$

Let $\Pi_i : \Omega^1(X_i, \operatorname{ad} P_i) \to H^1_{A_i}$ be the L^2-orthogonal projection onto the kernel. Then we would require

$$\Pi_i(b_i^T) = v_i$$

for $1 \leqslant i \leqslant 3$ and for some fixed small vector $v_i \in H^1_{A_i}$. Together with (52), we obtain

$$\begin{cases} D(b) = (\beta \cdot \alpha)^{-1}(L^T(b) + \mu^T(b,b) + E^T(A_1, A_2, A_3)) \\ \Pi(b) = v. \end{cases}$$

This system still has a unique solution when (51) holds and $\|v\|_{\mathcal{H}} \ll 1$.

37.3. How to make error terms small?

Finally, let us explain how to realize (51). The linear part L^T comes from two parts:

$$\beta_L(d\phi_L \wedge \cdot)^+$$
$$\beta_L\phi_R\phi_L[a_1^\infty \wedge \cdot]^+.$$

The first part is made small by making the derivative of ϕ_L small. For the second part, we exploit the fact that a_1^∞ has exponential decay as $t \to \infty$.

As for E^T, we need to deal with

$$\beta_L\phi_R\phi_L[a_1^\infty \wedge a_1^\infty]^+.$$

Either we use the exponential decay of a_1^∞, or we make use of the estimate:

$$|\phi_R^T(t)| \leqslant \frac{C}{T} \cdot t.$$

on $[0,\infty) \times Y \subset X_1$. Here, t is the coordinate function on the cylinder $[0,\infty) \times Y$. Hence,

$$\|\beta_L\phi_R\phi_L[a_1^\infty \wedge a_1^\infty]^+\|_{\mathcal{H}} \leqslant |\frac{C}{T} \cdot te^{-\lambda t}| \leqslant \frac{C_1}{T}.$$

37.4. What if $\beta = \alpha^{-1}$?

In this section, we list the computational result if we take $\beta = \alpha^{-1}$, just for completeness. We will assume (instead of $\phi_L + \phi_R = 1$) that

$$\phi_L^3 + \phi_R^3 = 1.$$

This is allowed, because cut-off functions are applied on a cylinder $[-T,T] \times Y$ and there is a preferred reference connection A_0. The patched connection is

$$\phi_L(A_1^T - A_0) + \phi_R(A_2^T - A_0) + A_0.$$

Therefore, the β-matrix takes a simpler form:

$$\beta = \begin{pmatrix} \phi_L^2 & -\phi_R\phi_L & \phi_R^2 \\ \phi_R^2 & \phi_L^2 & -\phi_R\phi_L \\ -\phi_R\phi_L & \phi_R^2 & \phi_L^2 \end{pmatrix}.$$

We will focus on the first entry of

$$\begin{pmatrix} \phi_L^2 & -\phi_R\phi_L & \phi_R^2 \\ \phi_R^2 & \phi_L^2 & -\phi_R\phi_L \\ -\phi_R\phi_L & \phi_R^2 & \phi_L^2 \end{pmatrix}\begin{pmatrix} F^+_{A_{12}} \\ F^+_{A_{23}} \\ F^+_{A_{31}} \end{pmatrix}$$

and its expression is

$$\begin{aligned} F^+_{A_1^T} &+ \big((\phi_L d\phi_R - \phi_R d\phi_L) \wedge (\phi_L a_2^T - \phi_R a_3^T)\big)^+ \\ &+ \frac{1}{2}(\phi_L^4 + \phi_R^4 - 1)[a_1^T \wedge a_1^T]^+ + \frac{1}{2}\phi_L^2\phi_R(\phi_R - \phi_L)[a_2^T \wedge a_2^T]^+ \\ &+ \frac{1}{2}\phi_L\phi_R^2(\phi_L - \phi_R)[a_3^T \wedge a_3^T]^+ \\ &+ \phi_L\phi_R(\phi_L^2[a_1^T \wedge a_2^T]^+ - \phi_R\phi_L[a_2^T \wedge a_3^T]^+ + \phi_R^2[a_3^T \wedge a_1^T]^+) \end{aligned}$$

By replacing all superscripts T by ∞, we obtain an expression for the error term $E^T(A_1, A_2, A_3)$:

$$\begin{aligned} &\big((\phi_L d\phi_R - \phi_R d\phi_L) \wedge (\phi_L a_2^\infty - \phi_R a_3^\infty)\big)^+ \\ &\quad+ \frac{1}{2}(\phi_L^4 + \phi_R^4 - 1)[a_1^\infty \wedge a_1^\infty]^+ + \frac{1}{2}\phi_L^2\phi_R(\phi_R - \phi_L)[a_2^\infty \wedge a_2^\infty]^+ \\ &\quad+ \frac{1}{2}\phi_L\phi_R^2(\phi_L - \phi_R)[a_3^\infty \wedge a_3^\infty]^+ \\ &\quad+ \phi_L\phi_R(\phi_L^2[a_1^\infty \wedge a_2^\infty]^+ - \phi_R\phi_L[a_2^\infty \wedge a_3^\infty]^+ + \phi_R^2[a_3^\infty \wedge a_1^\infty]^+). \end{aligned}$$

The first line will give us the leading term of the error. The expression of the quadratic non-linear term μ^T is obtained in a similar manner. Lastly, the linear correction $L^T(b)$ has an expression of

$$\big((\phi_L d\phi_R - \phi_R d\phi_L) \wedge (\phi_L b_2^T - \phi_R b_3^T)\big)^+ + \text{ other terms.}$$

Lecture 38. (Special Lecture 2) Floer's Exact Triangle

38.1. The setup

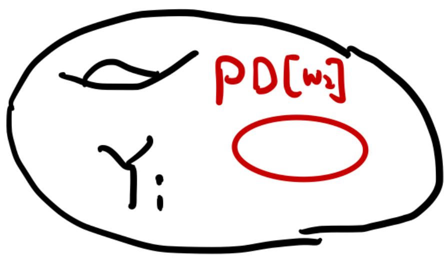
Let $(Y_i, P_i, w_i), 0 \leqslant i \leqslant 2$ be closed oriented 3-manifolds with admissible $SO(3)$-principal bundles. Here, $w_i = w_2(P_i)$ is the second Stiefel-Whitney class of $P_1 \to Y_i$ and "admissible" means there exists a closed embedded surface $\Sigma_i \subset Y_i$ with $g(\Sigma) \geqslant 1$ such that the evaluation

$$\langle w_2(P_i), [\Sigma_i] \rangle \neq 0 \in \mathbb{Z}_2.$$

In this case, the instanton Floer homology

$$I(Y_i, \omega_i)$$

is well-defined. We work with determinant-1 gauge group: the subgroup of $SO(3)$-gauge transformations that lifts to $SU(2)$.



We require (Y_i, w_i)'s to be produced in a special way. Let $(Y, \partial Y = \mathbb{T}^2)$ be a 3-manifold whose boundary is connected and homeomorphic to a 2-torus. Let

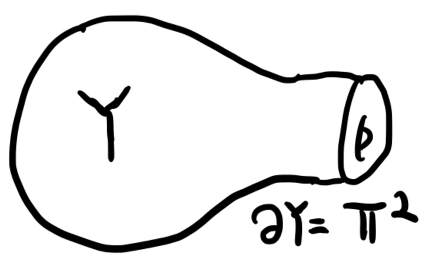
$$\gamma_0,\ \gamma_1,\ \gamma_2$$

be oriented simple closed curves on ∂Y such that

$$\gamma_i \cdot \gamma_{i+1} = -1 \text{ for } 0 \leqslant i \leqslant 2.$$

Alternatively, we require

$$\gamma_0 \cdot \gamma_1 = -1, \gamma_0 + \gamma_1 + \gamma_2 = 0 \in \pi_1(\mathbb{T}^2).$$



From the second description, it is clear that such a triple $(\gamma_0, \gamma_1, \gamma_2)$ exists. Let

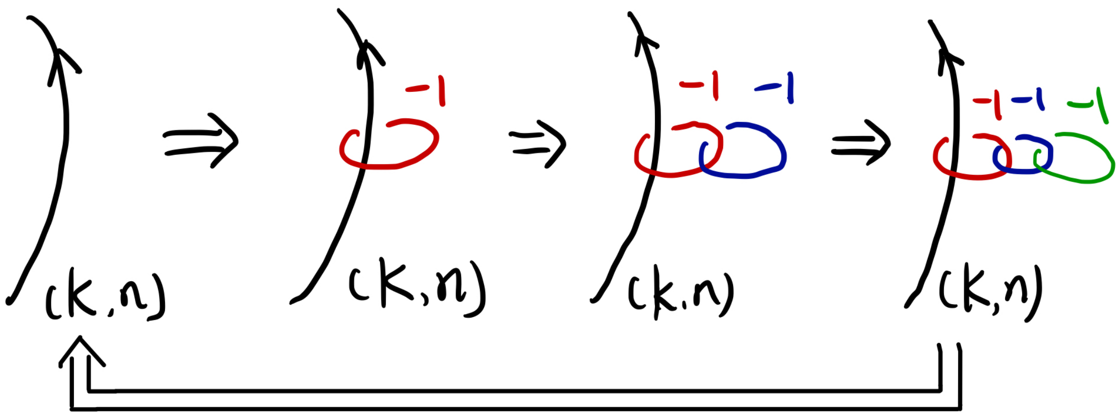
$$Y_i = Y \coprod_{[\gamma_i]=[\partial D^2]} (D^2 \times S^1).$$

To glue Y and $D^2 \times S^1$ along the common boundary, we need to specify a homeomorphism $\phi : \partial Y \cong \partial D^2 \times S^1$; we require the image of $[\partial D^2]$ in ∂Y to be $[\gamma_i]$ (up to homotopy). The homeomorphism type of the resulting closed manifold Y_i is determined by this image $\phi_*[\partial D^2]$ (not by ϕ).

The Poincaré dual of w_i is taken to be

$$PD[w_i] = [pt \times S^1] \subset D^2 \times S^1 \subset Y_i.$$

A different way to think about the triple (Y_0, Y_1, Y_2) is by surgery diagrams. Suppose the surgery diagram of Y_0 is given by (K, n) where $K \subset S^3$ stands for a knot and $n \in \mathbb{Z}$ denotes the framing. We obtain Y_1 by doing (-1)-surgery along a meridian of K:



Iterating the process to the meridian, we obtain Y_2 and Y_3. It is important to know that $Y_3 \cong Y_0$. To see this, we need a relation from Kirby Calculus:

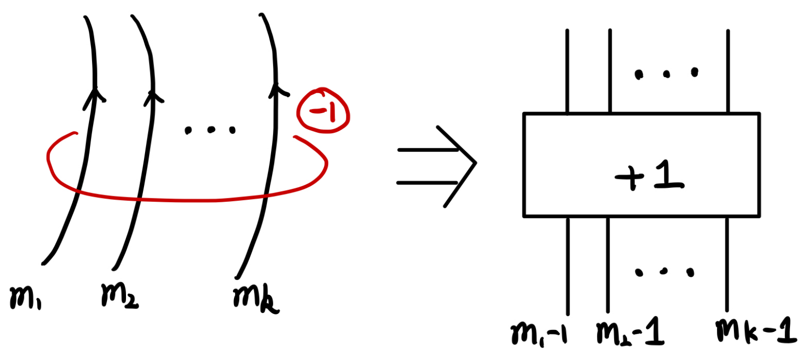
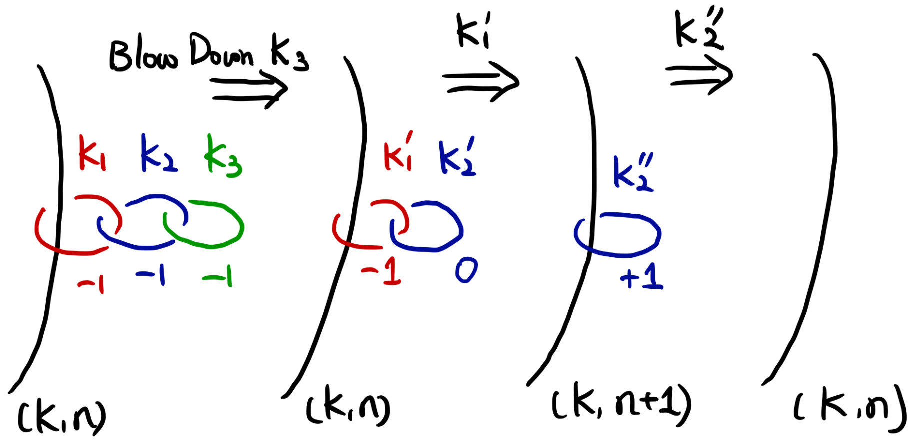


FIGURE 2. An Important Relation

Applying the relation in the following order, we come back to Y_0 from Y_s:

(1) Blow down along K_3.
(2) Blow down along K_1'.

(3) Blow down along K_2''.



In particular, there is a cobordism

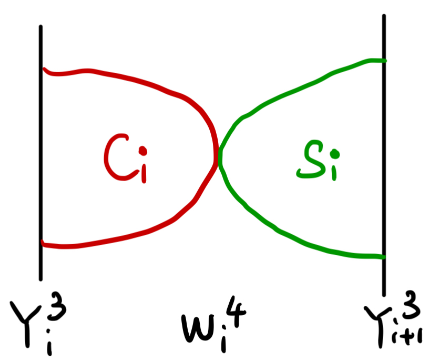
$$W_i^4 : Y_i^3 \to Y_{i+1}^3.$$

by adding a 2-handle. Let the core and co-core of W_i^4 be

$$C_i \text{ and } S_i$$

respectively, and $\zeta_i = C_i \cup S_i$ be their union. Then

$$\zeta_i \cap Y_i = \gamma_i,\ \zeta_i \cap Y_{i+1} = \gamma_{i+1}.$$



The Poincaré dual of ζ_i determines a class in $H^2(W_i, \mathbb{Z}_2)$ whose restriction at boundaries are w_i and w_{i+1}. Formally, we have a cobordism map

$$(W_i, \zeta_i) : (Y_i^3, \gamma_i) \to (Y_{i+1}^3, \gamma_{i+1}).$$

which induces a morphism between Instanton Floer Homology groups:

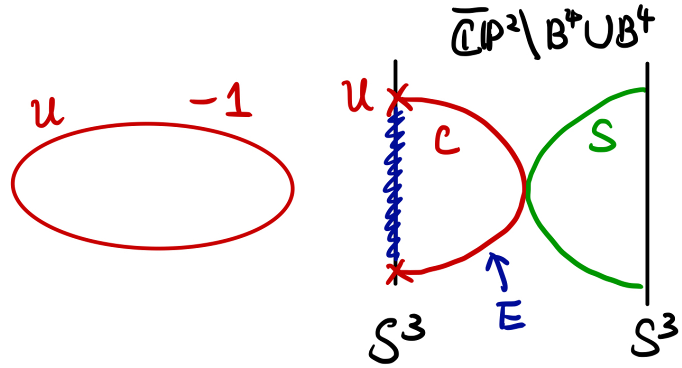
$$I_*(W_i, \zeta_i) : I_*(Y_i, \gamma_i) \to I_*(Y_{i+1}, \gamma_{i+1}).$$

38.2. Why Blow Down?

Tom digressed a bit to explain the sign convention in (??). In S^3, doing (-1)-surgery along a unknot U gives us cobordism:

$$W^4 = \overline{\mathbb{CP}}^2 \backslash B_1^4 \coprod B_2^4 : S_1^3 \to S_2^3.$$

The unknot U bounds a disk $D_1 \subset S_1^3$. Together with the core disk C_1, we obtain a closed 2-sphere of self-intersection (-1) inside W.



Generalize this picture a little bit. Let us do (n_1, n_2)-surgery along a Hopf-link, then we obtain a cobordism map:

$$W^4 : S^3 \to Y^3.$$

and two spheres of self intersections n_1 and n_2 inside W:

$$C_1, C_2.$$

Moreover, $C_1 \cap C_2 = 1$, since we start off with a Hopf link. We arrange that they intersect at a single point:

$$p = C_1 \cap C_2 \in W^4.$$

By blowing up at $p \in W^4$, we obtain a different cobordism

$$W \# \overline{\mathbb{CP}}^2 : S^3 \to Y^3.$$

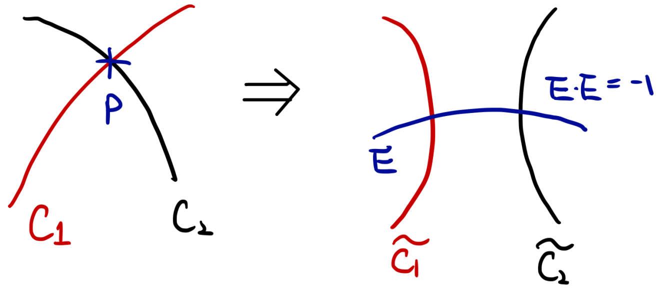
It is tempting to figure out the surgery diagram associated to this cobordism. Pretend that C_1 and C_2 are complex curves within a complex surface intersecting at p. Blowing up at p will produce an exceptional divisor E and the proper transformation $\tilde{C}_i$ of C_i. Moreover,

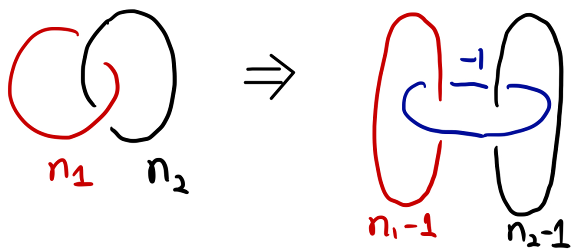
$$E \cdot E = -1, E \cdot \tilde{C}_i = 1, i = 1, 2.$$

It is important to figure out the self-intersection $\tilde{C}_i \cdot \tilde{C}_i$. Algebraic Geometry tells us

$$\begin{aligned} C_i \cdot C_i &= (\tilde{C}_i + E)^2 = \tilde{C}_i \cdot \tilde{C}_i + 2\tilde{C}_i \cdot E + E \cdot E \\ &= \tilde{C}_i \cdot \tilde{C}_i + 1. \end{aligned}$$

So the self intersection number of $\tilde{C}_i$ is $n_i - 1$ and the surgery diagram is





38.3. Floer's Exact Triangle

Theorem 38.3.1 (Floer 1989)**.** *The digram*

$$\begin{tikzcd} & I_*(Y_1, w_1) \arrow[dr, "I_*(W_1;\zeta_1)"] & \\ I_*(Y_0, w_0) \arrow[ur, "I_*(W_0;\zeta_0)"] & & I_*(Y_2, w_2) \arrow[ll, "I_*(W_2;\zeta_2)"] \end{tikzcd}$$

is an exact triangle.

The first step is to show consecutive compositions are zeros. If we compose cobordisms:

$$W_i : Y_i \to Y_{i+1} \text{ and } W_{i+1} : Y_{i+1} \to Y_{i+2},$$

then we obtain a cobordism

$$Z_i = W_{i+1} \circ W_i : Y_i \to Y_{i+2}.$$

The co-core of W_i and the core of W_{i+1} form a sphere E_i of self-intersection (-1):

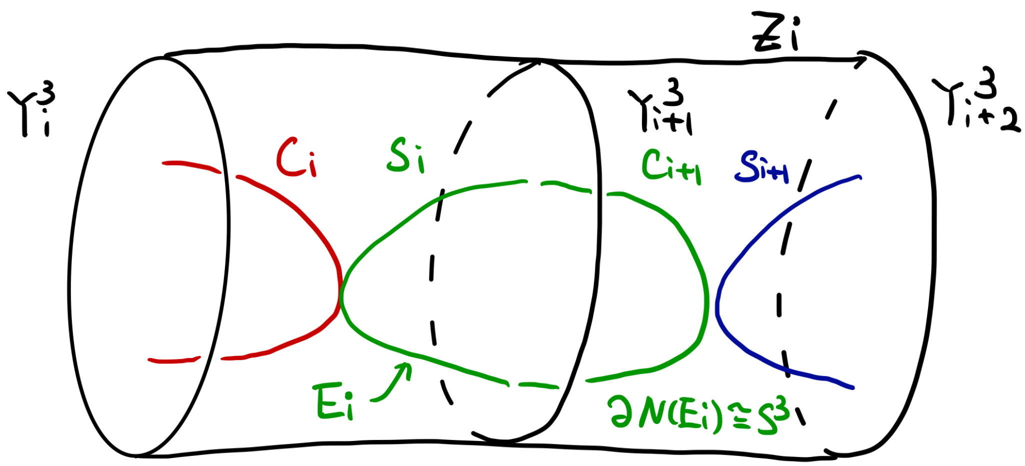
$$E_i = S_i \cup C_{i+1},\ \ E_i \cdot E_i = -1.$$

This means the boundary of a tubular neighborhood $N(E_i)$ of E_i is homeomorphic to S^3:

$$\partial N(E_i) \cong S^3.$$

Moreover, $\langle w_2, E_i\rangle \neq 0 \in \mathbb{Z}_2$. Take critical points $\alpha \in \mathrm{Crit}(\mathcal{CS}, Y_i)$ and $\beta \in \mathrm{Crit}(\mathcal{CS}, Y_{i+2})$. The composition

$$I_*(W_{i+1}; \zeta_{i+1}) \circ I_*(W_i; \zeta_i) = I_*(Z_i; \zeta_{i+1} \cup \zeta_i).$$



is defined by counting the zero dimensional moduli space

$$\mathcal{M}_k(Z_i, \alpha, \beta)$$

for some instanton number $k = \langle c_2(P_i' \to Z_i), [Z_i, \partial Z_i] \rangle$. This number $k \in \mathbb{Z}$ is fixed by requiring

$$\dim \mathcal{M}_k(Z_i, \alpha, \beta) = 0.$$

The idea is to choose a special metric on Z_i such that the moduli space is empty. So on the chain level the cobordism map $I_*(Z_i)$ is forced to be zero.

The special metric is constructed by stretching the neck along $\partial N(E_i) = S^3$. Let

$$Z_i^T = Z_i \backslash N(E_i) \coprod [-T, T] \times S^3 \coprod N(E_i)$$

with fixed metrics on $Z_i \backslash N(E_i)$ and $N(E_i)$. When $T \gg 0$, an ASD connection on Z_i is close to the flat connection over the cylinder $[-T, T] \times S^3$.

Lemma 38.3.2. *Since $\partial N(E_i) = S^3$, one can fill in a 4-ball to obtain a new cobordism between Y_i and Y_{i+2}. This cobordism is simply*

$$\overline{W_{i+2}} : Y_{i+2} \to Y_i.$$

and $Z_i = \overline{W_{i+2}} \# \overline{\mathbb{CP}}^2$.

In particular,

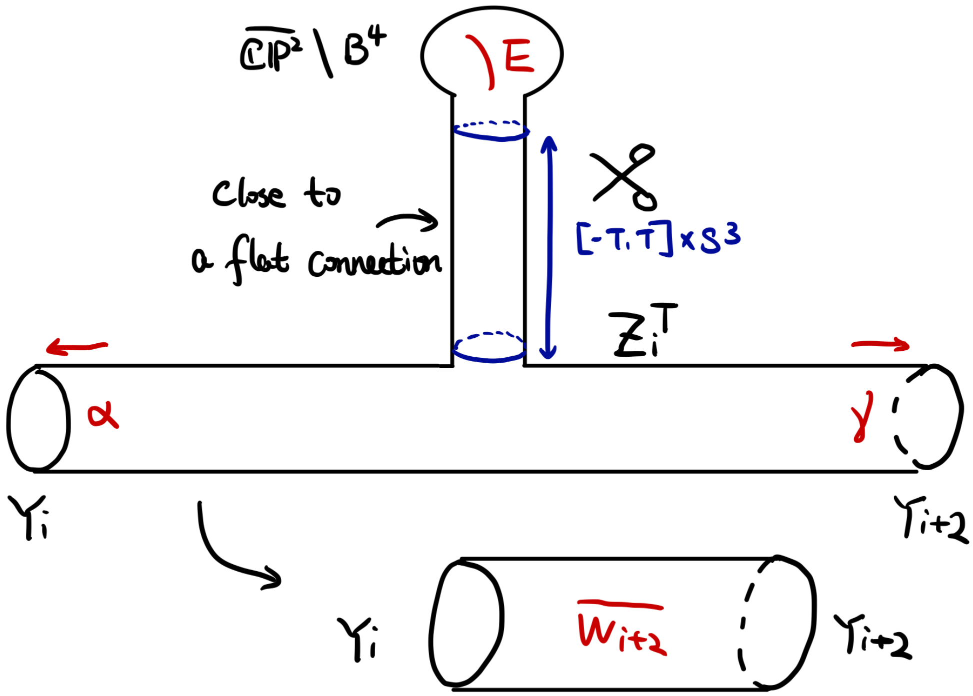
$$b^+(Z_i) = b^+(\overline{W_{i+2}}) \text{ and } b^-(Z_i) = b^-(\overline{W_{i+2}}).$$

and $0 = \dim \mathcal{M}_k(Z_i^T, \alpha, \beta) = \dim \mathcal{M}_k(\overline{W_{i+2}}, \alpha, \beta)$. As $T \to \infty$, a gluing theorem states that an ASD connection A in $\mathcal{M}_k(Z_i^T, \alpha, \beta)$ breaks up into

$$A \rightsquigarrow (A', A_0)$$

with $A' \in \mathcal{M}_k(\overline{W_{i+2}}, \alpha, \beta)$ and an ASD connection A_0 on $\overline{\mathbb{CP}}^2$. Moreover,

$$\mathcal{E}(A) = \mathcal{E}(A') + \mathcal{E}(A_0).$$



Since $\mathcal{E}(A)$ and $\mathcal{E}(A')$ are determined by the instanton number k, we must have $\mathcal{E}(A_0) = 0$. To draw a contradiction, it suffices to show an ASD connection on $\overline{\mathbb{CP}}^2$ with $\langle w_2, E_i\rangle \neq 0$ must have non-zero energy.

On $\overline{\mathbb{CP}}^2$, lift the $SO(3)$-bundle P to a $U(2)$-bundle $P' = P'_{U(2)}$. $w_2 \neq 0$ implies $c_1(P')$ is odd. The energy formula says

$$\mathcal{E}(A) = c_2 - \frac{1}{2}c_1^2 \neq 0.$$

38.4. The Second Step

The second step is to show the triangle is exact. The first step provides a concrete chain homotopy between $C_*(W_{i+1}) \circ C_*(W_i)$ and 0:

$$\begin{aligned} K_i : C_*(Y_i, w_i) &\to C_*(Y_{i+2}, w_{i+2}) \\ \alpha &\mapsto \sum_\gamma \# \coprod_{T\in[0,\infty)} \mathcal{M}_k(Z_i^T, \alpha, \gamma)\gamma. \end{aligned}$$

where the instanton number $k(\alpha,\gamma)$ is chosen (for each (α,γ)) such that

$$\dim \mathcal{M}_k(Z_i^T, \alpha, \gamma) = -1.$$

Since the moduli space depends on an extra parameter T, the union has formal dimension 0 and there is a reasonable signed count.

To show K_i is a chain homotopy, we need to prove:

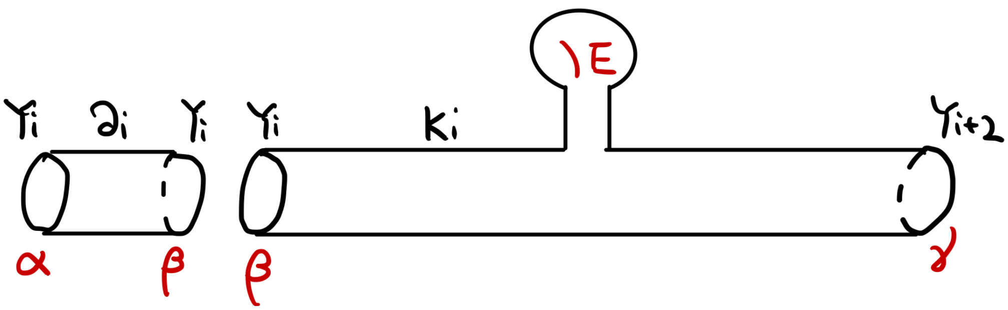
$$\partial_{i+2} \circ K_i + K_i \circ \partial_i = C_*(W_{i+1}) \circ C_*(W_i) - 0. \tag{53}$$

The idea is to compactify the moduli space

$$\coprod_{T \in [0,\infty)} \mathcal{M}_k(Z_i^T, \alpha, \beta)$$

with $\dim \mathcal{M}_k(Z_i^T, \alpha, \beta) = 0$. The formal dimension of this parametrized moduli space is 1 and its compactification is a disjoint union of 1-manifolds. Each term in 53 appears as a boundary term in the compactification:

(1) The boundary of $T \in [0, \infty]$ will give $C_*(W_{i+1}) \circ C_*(W_i)$ ($T = 0$) and 0 ($T = \infty$).
(2) The energy can slide off along ends $(-\infty, 0] \times Y_i$ and $Y_{i+2} \times [0, \infty)$, which contribute to $K_i \circ \partial_i$ and $\partial_{i+2} \circ K_i$.



Therefore, we arrive at a diagram, where arrows are indexed by domains.

$$\begin{tikzcd}
& C_*(Y_1, w_1) \arrow[dr, "C_*(W_1)"] \arrow[dl, bend right, "K_1"'] & \\
C_*(Y_0, w_0) \arrow[ur, "C_*(W_0)"] \arrow[rr, bend right, "K_0"'] & & C_*(Y_2, w_2). \arrow[ll, "C_*(W_2)"] \arrow[ul, bend right, "K_2"']
\end{tikzcd}$$

Note that $K_1 W_0 + W_2 K_0 : C_*(Y_0) \to C_*(Y_0)$ is a chain map (at least) over $\mathbb{Z}_2$:

$$\begin{aligned}
\partial_0(K_1 W_0 + W_2 K_0) &= K_1 \partial_0 W_0 + W_2 W_1 W_0 + W_2 \partial_2 K_0 \\
&= (K_1 W_0 + W_2 K_0)\partial_0 + 2 W_2 W_1 W_0 \\
&= (K_1 W_0 + W_2 K_0)\partial_0 \mod 2.
\end{aligned}$$

Theorem 38.4.1 (Distinguished Triangulation Detection Lemma, Ozvasth-Szabo, Kontsevich; See also Seidel's book)**.** *If $K_1 W_0 + W_2 K_0$ is* ***an isomorphism on homology****, or it is chain-homotopy equivalent to a chain isomorphism , then*

$$(C_*(Y_0), \partial_0) \cong \mathrm{Cone}(C_*(Y_1) \xrightarrow{W_1} C_*(Y_2))$$

is a quasi-isomorphism.

Recall that the mapping cone of a chain map defined by

$$(C_{12}, \partial_{12}) = (C_*(Y_1) \oplus C_*(Y_2), \begin{pmatrix} \partial_1 & 0 \\ W_1 & \partial_2 \end{pmatrix})$$

The morphism in the detection theorem is constructed by

$$f : (C(Y_0), \partial_0) \to (C_{12}, \partial_{12}), \alpha \mapsto (W_0(\alpha), K_0(\alpha)).$$

whose inverse is given by

$$g : (C_{12}, \partial_{12}) \to (C(Y_0), \partial_0), (\beta, \gamma) \mapsto K_1(\beta) + W_2(\gamma).$$

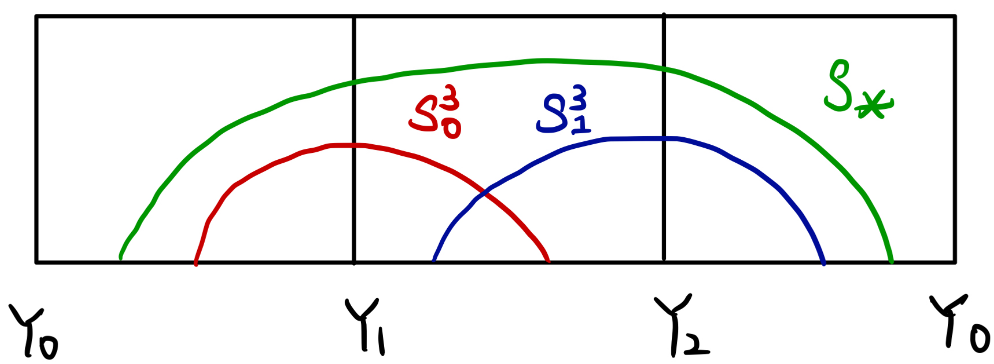
so $g \circ f = K_1W_0 + W_2K_0$. The condition on $g \circ f$ is certainly necessary for the theorem to hold.

Let us now explain why $K_1W_0 + W_2K_0$ is chain-homotopy equivalent to the identity map from the geometry. Consider the composition

$$U = W_2 \circ W_1 \circ W_0 : Y_0 \to Y_0,$$

in which we have five separating submanifolds:

(1) Y_1.
(2) Y_2.
(3) $S^3_0 = \partial N(E_0)$, where $E_0 \subset Z_0$ is the first exceptional curve.
(4) $S^3_1 = \partial N(E_1)$, where $E_1 \subset Z_1$ is the second exceptional curve.
(5) $S_* = S^1 \times S^2 = \partial N(E_1 \cup E_2)$.



Starting with a fixed metric g_0 on U, we can stretch the metric g_0 in five different ways. If two submanifolds are disjoint, then they can be stretched at the same time. There are five such pairs:

$$(H_1, H_2) = (Y_1, S^3_1), (S^3_1, S_*), (S_*, S^3_0), (S^3_0, Y_2), (Y_2, Y_1).$$

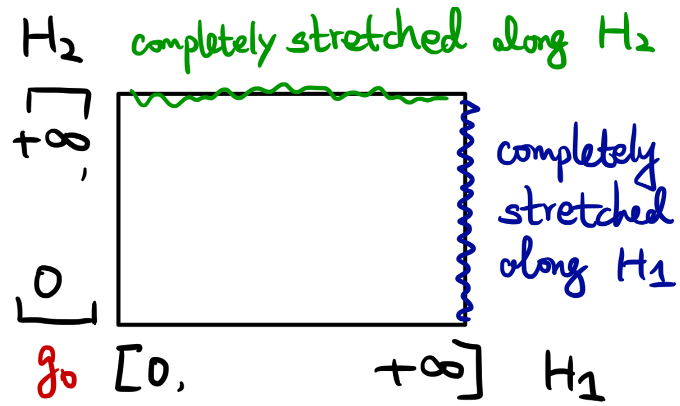
For each pair (H_1, H_2), we form a family of metrics parametrized by a square:

$$(T_1, T_2) \in [0, \infty] \times [0, \infty].$$

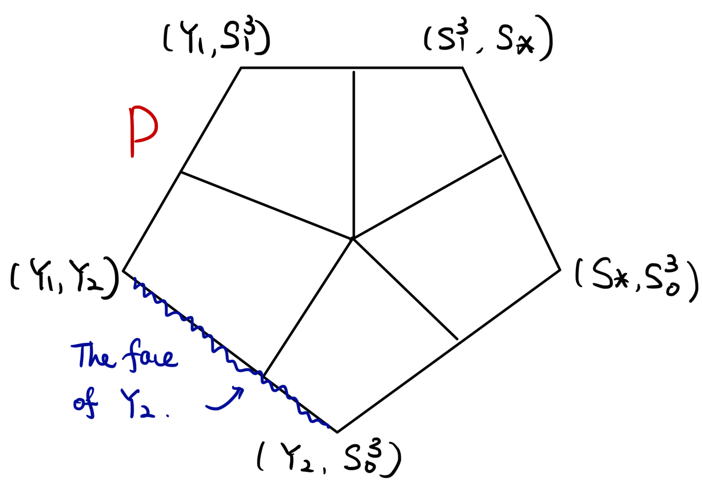
where T_i denotes the length of the cylinder along H_i:

$$[0, T_i] \times H_i.$$

When $(T_1, T_2) = (0, 0)$, the metric is not stretched and it is just g_0. When $(T_1, T_2) = (0, \infty)$, the metric is completely stretched along H_2, but not stretched along H_1 at all. When $(T_1, T_2) = (\infty, \infty)$, it means the metric is completely stretched along both H_1 and H_2.



Since we have five pairs, the five squares are combined and form a closed pentagon, denoted by P:



To construct a chain homotopy between $K_1W_0+W_2K_0$ and the identity map, define

$$J : C(Y_0) \to C(Y_0)$$

$$\alpha \mapsto \sum_{\gamma} \#\left(\coprod_{p \in P} M_k(U, g_p, \alpha, \beta)\right)\beta$$

The instanton number $k(\alpha,\gamma)$ is chosen so that the formal dimension of each individual moduli space is -2.

$$\dim M_k(U_p,\alpha,\beta) = -2.$$

Hence, the parametrized moduli space

$$\coprod_{p\in P} M_k(U,g_p,\alpha,\beta)$$

has formal dimension 0. To prove

$$\partial_0 J + J\partial_0 = K_1W_0 + W_2K_0 + \mathrm{Id},$$

consider the compactification of the 1-dimensional moduli space:

$$\coprod_{p\in P} M_{k'}(U,g_p,\alpha,\gamma)$$

(where $k'(\alpha,\gamma)$ is chosen so that each individual moduli space has dimension (-1)). The boundary terms have two origins:

(1) The boundary ∂P of the parameter space P.
(2) The energy can slide off along ends $(-\infty,0]\times Y_0$ and $Y_0\times[0,\infty)$, which contribute to $J\circ\partial_0$ and $\partial_0\circ J$.

In conclusion, we have

$$\partial_0 J + J\partial_0 = \text{ contributions from } \partial P.$$

There are five components of ∂P. We already know some boundary contributions:

(1) The face of Y_1: $(Y_2,Y_1)-(Y_1,S^3_1)$. The metric is completely stretched along Y_1, so we get an 1-parameter family of metrics on
$$Z_1 : Y_1 \to Y_0.$$
In one direction, the metric is completely stretched along $Y_2\subset Z_1$; in the other direction, it is stretched along $S^3_1\subset Z_1$. The counting (by definitions) is
$$K_1W_0.$$
(2) The face of Y_2: $(S^3_0,Y_2)-(Y_2,Y_1)$. The metric is completely stretched along Y_2, so we get an 1-parameter family of metrics on
$$Z_0 : Y_0 \to Y_2.$$
In one direction, the metric is completely stretched along $Y_1\subset Z_0$; in the other direction, it is stretched along $S^3_0\subset Z_0$. The counting (by definitions) is
$$W_2K_0.$$

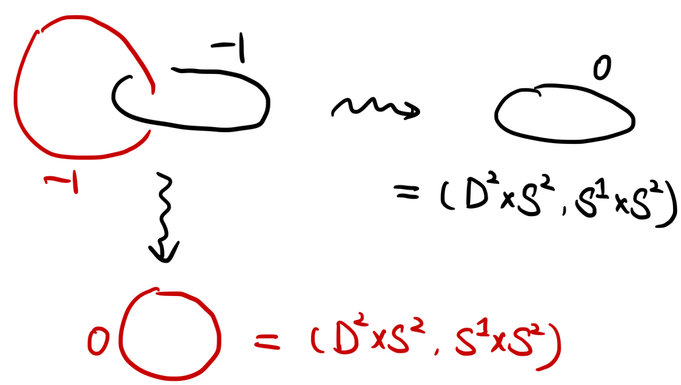
(3) The face of S^3_0: $(S_*, S^3_0) - (S^3_0, Y_2)$. The metric is completely stretched along S^3_0. By the argument from the first step, we do not have any contribution from this part because ASD connections on $\overline{\mathbb{CP}}^2$ with $w_2 \neq 0$ must have non-zero energy.
(4) The face of S^3_1: $(Y_1, S^3_1) - (S^3_1, S_*)$. It is similar to the face of S^3_1. No contributions.
(5) The face of S_*: $(S^3_1, S_*) - (S_*, S^3_0)$. The metric is stretched completely along S_*.

The remaining task is to show the face of $S_* = S^1 \times S^2$ will contribute to the identity morphism on $C_*(Y_0)$.

Lemma 38.4.2. *Recall that S_* is the boundary of $U_1 := N(E_0 \cup E_1)$. Write $U = U_1 \coprod_{S_*}(U - U_1)$. Then*

$$U_1 = (S^2 \times D^2) \# \overline{\mathbb{CP}}^2.$$
$$[0,1] \times Y = (U - U_1) \coprod_{S_*} S^1 \times D^3.$$

We focus on U_1 first. E_1 and E_2 are two (-1)-spheres intersecting positively at a single point. In terms of the surgery diagram, there are two ways to blow down (-1)-spheres to obtain $S^2 \times D^2$:



Doing 0-surgery along the unknot produce a 0-sphere in the resulting 4-manifold, which, as a consequence, has to be a trivial disk bundle $D^2 \times S^2$ over S^2. So (D^4, S^3) is changed into $(D^2 \times S^2, S^1 \times S^2)$ by this surgery.

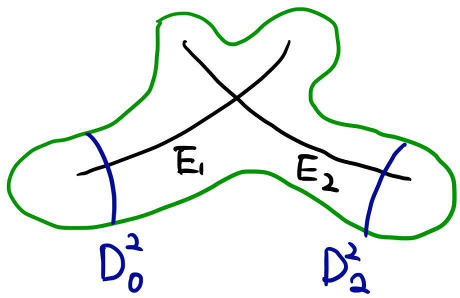
In particular, the intersection form of U_1 is

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \text{(represented by } E_0 \text{ and } E_1\text{)}.$$
$$\sim \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \text{(represented by } E_0 \text{ and } E_0 + E_1\text{)}.$$

The Poincaré dual of $w_2(P)$ on U_1 is the union:

$$E_0 \cup E_1 \cup D_0 \cup D_2$$

where $D_0 = U_1 \cap C_0$ (the core disk) and $D_2 = U_1 \cap S_2$ (the co-core disk). It has even evaluations on both E_1 and E_2, so $w_2(P) = 0 \in H^2(U_1, \mathbb{Z}_2)$. In particular, $w_2(P) = 0 \in H^2(S_*, \mathbb{Z}_2)$.



Another way to think about this is to take the intersection:

$$(E_0 \cup E_1 \cup D_0 \cup D_2) \cap S_* = (D_0 \cap S_*) \cup (D_2 \cap S_*).$$

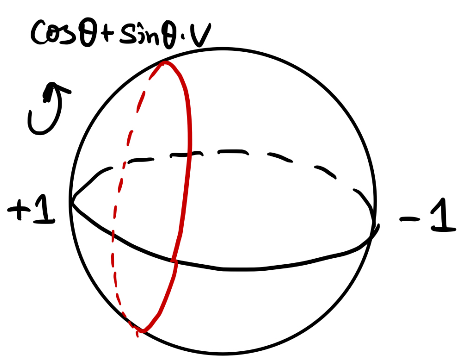
One may think of $(D_0 \cap S_*)$ as in $(D^2 \times S^2, S^1 \times S^2)$ (blow down the other (-1)-sphere). Then D_0 is a fiber disk and $D_0 \cap S_*$ generates $\pi_1(S_*) \cong \mathbb{Z}$. Since we have two copies of generators, $w_2(P) = 0 \in H^2(S^1 \times S^2, \mathbb{Z}_2)$.

Hence, we may think of P as the trivial $SU(2)$-bundle (instead of $SO(3)$) over $S^1 \times S^2$. The moduli space of flat connections is

$$\{\mathbb{Z} \to SU(2)\}/\text{conj} \cong [0, \pi].$$

$$\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \leftrightarrow \theta.$$

The representation $\rho : \mathbb{Z} \to SU(2)$ is determined by $\rho(1) = \cos\theta + v \sin\theta$ with $v \in \operatorname{Im}\mathbb{H}$ and $|v| = 1$. Conjugations by $SU(2)$ will rotate v in $\operatorname{Im}\mathbb{H}$, so the quotient space is parametrized by $\theta \in [0, \pi]$.



38.5. The Final Step.

(Tom didn't explain enough details on this part).

Consider the 4-manifold with cylindrical ends:

$$U_1 \coprod_{S^*} S_* \times [0, \infty).$$

Fix a flat connection $\theta_0 \in (0, \infty) \subset \mathcal{M}(S^1 \times S^2)$. We choose the instant number k of P such that the moduli space $\mathcal{M}_{k,\theta_0}(U_1, S_*)$ of ASD connections on U with θ_0 as the limit at ∞ has dimension (-1).

Lemma 38.5.1. *Given a family of metrics on U_1 parametrized by*

$$\mathbb{R} = (-\infty, 0] \cup [0, \infty).$$

When $T \in [0, \infty)$, the metric is stretched along S_1^3. When $T \in (-\infty, 0]$, the metric is stretched along S_0^3. Then the sign count of

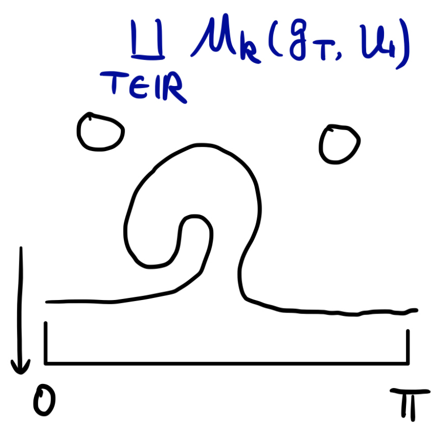
$$\coprod_{T \in \mathbb{R}} \mathcal{M}_{k,\theta_0}(U_1, S_*, g_T)$$

equals 1*.*

The reason is that as $T \to \infty$, $\mathcal{M}_k(U_1, S_*, g_T)$ contains a unique point with $\theta(T) \to \pi$. As $T \to -\infty$, $\mathcal{M}_k(U_1, S_*, g_T)$ contains a unique point with $\theta(T) \to 0$. Hence, the map

$$\theta : \coprod_{T \in \mathbb{R}} \mathcal{M}_k(U_1, S_*, g_T) \to [0, \pi]$$

has degree 1.



Finally, we need to compare the gluing picture

$$U_1 \coprod_{S^*} S_* \times [0, \infty) \;+\; (-\infty, 0] \times S_* \coprod_{S^*} (U - U_1).$$

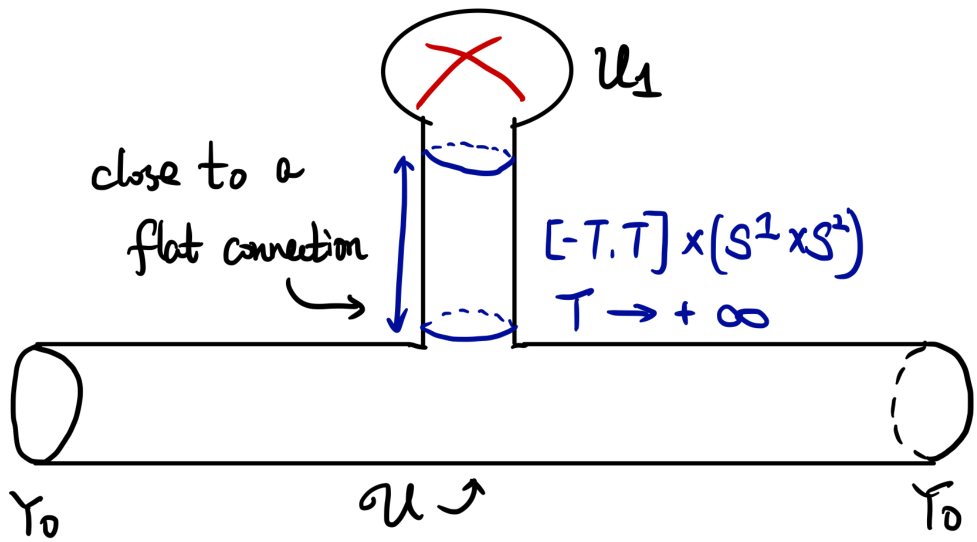
with

$$S^1 \times D^3 \coprod_{S*} S_* \times [0, \infty) \ + \ (-\infty, 0] \times S_* \coprod_{S*} (U - U_1).$$

The latter one will give us the identity cobordism:

$$[0, 1] \times Y_0 : Y_0 \to Y_1,$$

which induces the identity map on Floer Homology.

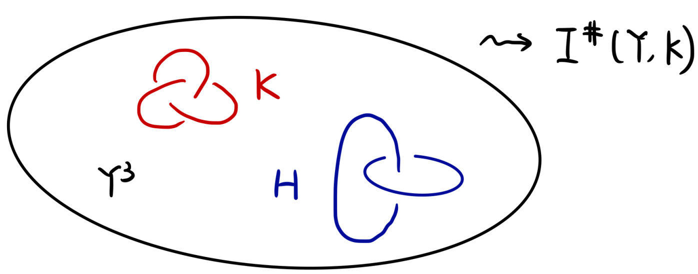


Lecture 39. (Special Lecture 3) Khovanov Homology detects the Unknot

39.1. Instanton Knot Floer Homology

Let Y^3 be a closed homology 3-sphere and $K : S^1 \to Y^3$ be a smoothly embedded knot in Y^3. By adding a disjoint Hopf link $H \subset Y$, we define the knot Floer homology by counting singular instanton:

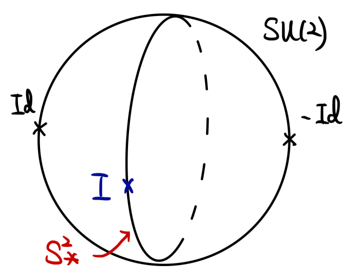
$$I_*^{\#}(Y, K) = I_*(Y, K \coprod H).$$



Consider the singular $SO(3)$-representation variety of $(Y = S^3, H)$ with $w_2 \neq 0$. The complement $S^3 \backslash H$ deformation retracts to $\mathbb{T}^2$, so

$$\pi_1(S^3 \backslash H) \cong \mathbb{Z} \oplus \mathbb{Z}.$$

For a singular π_1-representation, the meridian of each link component is sent to an element of order 2 in $SO(3)$. Working instead with the 2-fold cover $SU(2)$, they have order 4. These elements form a 2-sphere (the equator) $S_*^2 \subset SU(2)$.



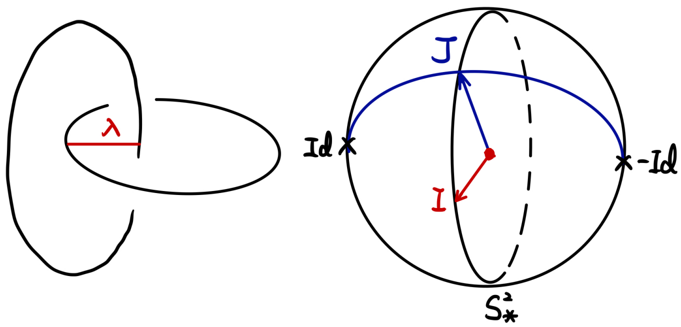
so

$$\begin{aligned}\mathcal{R}^I(S^3, H) &= \{(\alpha, \beta) \in SU(2) \times SU(2) : [\alpha, \beta] = \mathrm{Id}, \alpha \sim \beta \sim \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\}/\sim \\ &= \{(I, I), (I, -I)\} = S^2 \coprod S^2/\text{conjugations} \\ &= 2 \text{ points}.\end{aligned}$$

Each singular representation is reducible. The conjugation action of $SU(2)$ on these 2-spheres are not free. Stabilizers are not trivial.

Let λ be an arc connecting different components of the Hopf link H and let $w_2 = PD[\lambda] \neq 0 \in H^2(S^3, H)$. In this case, the singular representation variety is

$$\mathcal{R}^I_w(S^3, H) = \{(\alpha, \beta) \in SU(2) \times SU(2) : [\alpha, \beta] = -1, \alpha \sim \beta \sim \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\} \sim$$
$$= \{(I, J)\} = 1 \text{ point.}$$



Indeed, α takes values in the equator S^2 of $SU(2)$, while β takes value in the intersection:

$$\alpha^\perp \cap S^2.$$

So the singular representation variety is the unit sphere bundle $\mathbb{S}(TS^2)$ of the tangent space of S^2, and $SU(2)/Z(SU(2))$ acts freely and transitively on this space.

In general, define $(Y, K)^\# = (Y, K)\#(S^3, H)$ where we take the connected sum of 3-manifolds and disjoint union of links. Define

$$\mathcal{R}^\#(Y, K) = \mathcal{R}^I_w((Y, k)^\#).$$

Equivalently, this is the based representation variety $\mathcal{R}^0(Y, K)$.

Example 39.1.1. Let $U \subset S^3$ be the unknot. Then

$$\mathcal{R}^\#(S^3, U) = S^2.$$

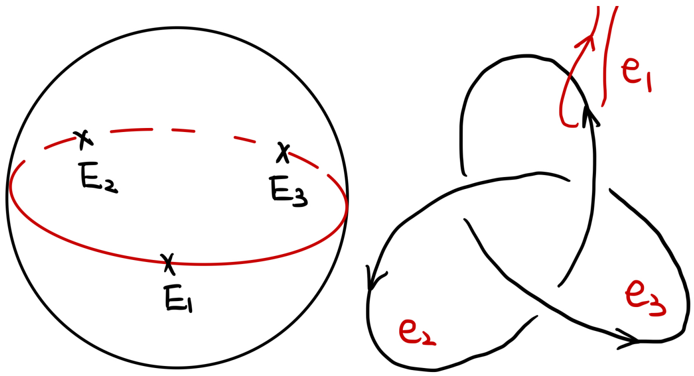
In general, let $U_k = \coprod_k U$ be the n-component unlink. Then

$$\mathcal{R}^\#(S^3, U) = \underbrace{S^2 \times \cdots \times S^2}_{k \text{ copies}}.$$

Example 39.1.2. Let $K \subset S^3$ be the trefoil. By the Wirtinger presentation, each arc of a knot diagram gives a generator of π_1 of the knot complement and each crossing gives a relation. In particular, we get

$$\pi_1(S^3 \backslash K) = \langle e_1, e_2, e_3 \big| \, e_i = e_{i+1} e_{i+2} e_{i+1}^{-1} \rangle.$$

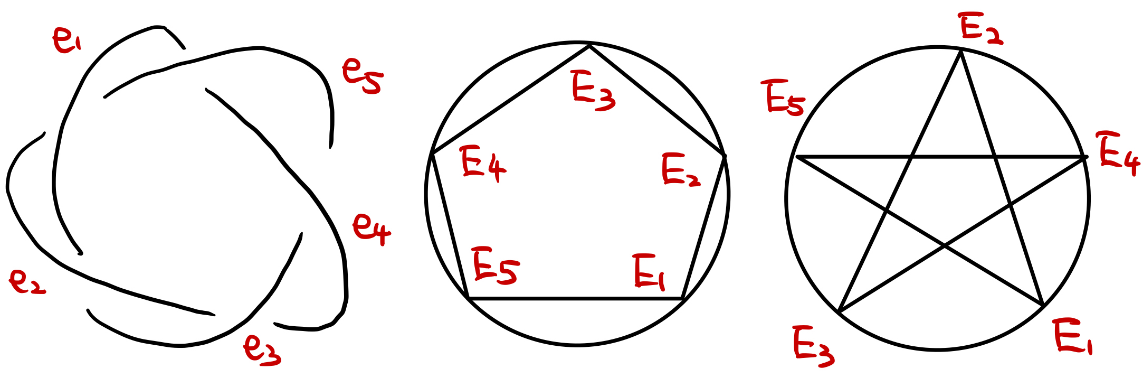
Let $E_i = \rho(e_i) \subset S^2 \subset SU(2)$ be the image of e_i under a singular representation, then $E_i^2 = \text{Id}$. These relations imply that rotating E_{i+2} along E_{i+1} by 180° yields



E_i. Hence, E_i's must distribute evenly along a great circle in $S^2_* \subset SU(2)$. When $E_1 = E_2 = E_3 \in S^2_*$, representations are reducible and contribute to a copy of S^2 in $\mathcal{R}^\#(S, K)$, which happens for any other knots. When $E_i's$ are distinct, we obtain a copy of $SO(3)$:

$$\mathcal{R}^\#(S^3, K) = SO(3) \coprod S^2.$$

In general, for the $(2, 2k+1)$-torus knot $T(2, 2k+1)$, there are $2k+1$ points distributed evenly on a great circle of S^2, and there are k possible orders of them. For instance, when $k = 2$:



Hence, $\mathcal{R}^\#(S, T(2, 2k+1)) = \coprod_k SO(3) \coprod S^2$.

Compute the singular homology of these representation varieties:

$$\begin{aligned} H_*(\mathcal{R}^\#(S, U)) &= \mathbb{Z} \oplus \mathbb{Z} \\ H_*(\mathcal{R}^\#(S, U_k) &= (\mathbb{Z} \oplus \mathbb{Z})^{\otimes k} \\ H_*(\mathcal{R}^\#(S, K)) &= \underbrace{(\mathbb{Z} \oplus \mathbb{Z})}_{S^2} \oplus \underbrace{(\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z})}_{\mathbb{RP}^3}. \end{aligned}$$

All these groups agree with Khovanov homology of links, suggesting a relations between Khovanov homology and Instanton Floer homology.

Let us briefly describe the original instanton knot Floer homology first defined by Floer. The complement of a knot $K \subset S^3$ is a homology $S^1 \times D^2$. Removing a tubular neighborhood of K, a Seifert surface S_K of K defines a canonical curve in $\partial N(K)$—the longitude:

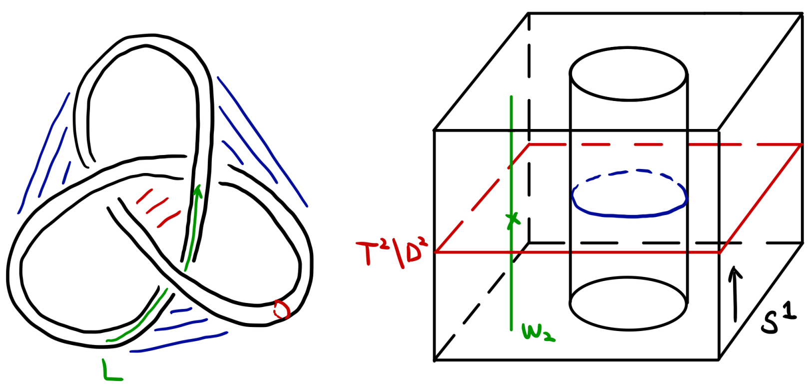
$$L = \partial S_K \subset N(K).$$

On the other hand, $(\mathbb{T}^2 \backslash D^2) \times S^1$ is a 3-manifold with a torus boundary. By identifying

$$L \mapsto [\partial D^2] \times \{pt\}, m \mapsto \{pt\} \times S^1,$$

we glue $S^3 \backslash N(K)$ and $(\mathbb{T}^2 \backslash D^2) \times S^1$ along their common boundary and obtain a homology $\mathbb{T}^3$:

$$\mathbb{T}^3_K := S^3 \backslash N(K) \coprod (\mathbb{T}^2 \backslash D^2) \times S^1.$$



Let $PD(w_2) = \{pt\} \times S^1 \subset (\mathbb{T}^2 \backslash D^2) \times S^1$ and let $\Sigma_K = S_K \cup (\mathbb{T}^2 \times D^2) \times \{pt\}$. Then $\langle w_2, [\Sigma] \rangle \neq 0$. The $SO(3)$-bundle is admissible. The original instanton knot Floer homology is defined as

$$I^\natural(S^3, K) := I_*(\mathbb{T}^3_K, w_2 = PD[m]).$$

In general, the boundary ∂S_K of Seifert surfaces of a link can take different homology classes in

$$\underbrace{\mathbb{T}^2 \times \mathbb{T}^2 \times \cdots \times \mathbb{T}^2}_{n \text{ copies}}$$

where $n = \#$ of link components. One may verify that

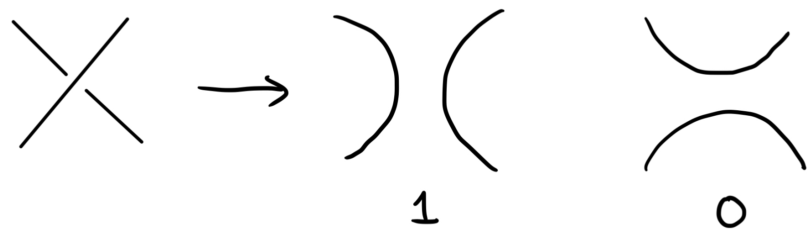
$$\mathcal{R}(\mathbb{T}^3_k, w_2 = PD[m]) = \mathcal{R}^\#(S, K),$$

so they define the same Floer homology groups.

39.2. Khovanov Homology

Caution: Kronheimer-Mrowka and Khovanov use different conventions for 1 and 0 resolutions of knots.

For each knot diagram $D(K)$, there are two ways to resolve a crossing:

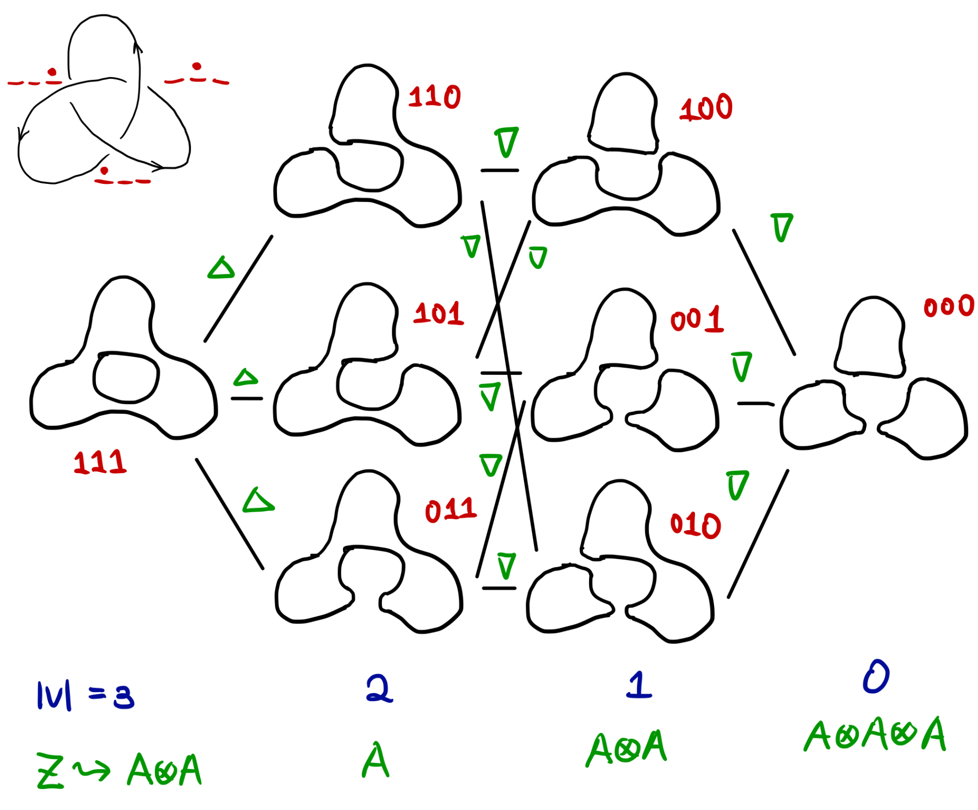


If the knot diagram $D(K)$ has n crossings, there are 2^n different ways to resolve all crossings, each of which corresponds to a vertex of the cube:

$$\{0,1\}^n \subset [0,1]^n.$$

Each resolved digram is a disjoint union of circles in $\mathbb{R}^2$.

Example 39.2.1. Starting with the knot diagram of the trefoil as shown above, we arrive at a cube of resolutions:



For each edge of $[0,1]^n$, we associated an arrow from 1 to 0 in the resolution.

We apply an $(1+1)$-dimensional TQFT to the cube of resolutions. Let

$$A = \mathbb{Z}[x]/(x^2) = H^*(S^2, \mathbb{Z}).$$

The ring structure of $H^*(S^2, \mathbb{Z})$ defines a multiplication $\Delta : A \otimes A \to A$. The co-multiplication $\nabla : A \to A \otimes A$ is defined by

$$\begin{aligned} 1 &\mapsto 1 \otimes x + x \otimes 1 \\ x &\mapsto x \otimes x. \end{aligned}$$

Consider the diagonal embedding $\iota : S^2 \hookrightarrow S^2 \times S^2$. ∇ is also realized as the composition:

$$\begin{aligned} A = H^*(S^2, \mathbb{Z}) &\xrightarrow{PD} H_{2-*}(S^2, \mathbb{Z}) \xrightarrow{\iota_*} H_{2-*}(S^2 \times S^2, \mathbb{Z}) \\ &\xrightarrow{PD} H^{2+*}(S^2 \times S^2, \mathbb{Z}) = A \otimes A. \end{aligned}$$

Define the unit and co-unit as:

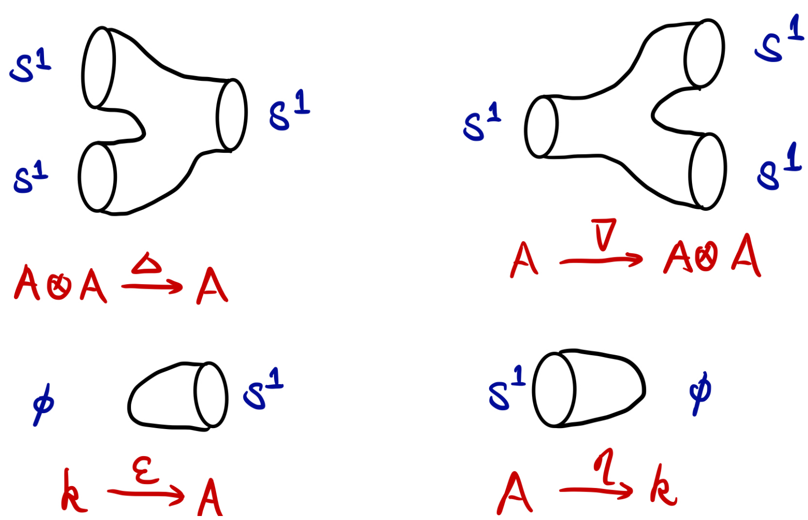
$$\begin{aligned} \epsilon &: \mathbb{Z} \to A, 1 \mapsto 1 \\ \eta &: A \to \mathbb{Z}, 1 \mapsto 0, x \mapsto 1. \end{aligned}$$

Then $(\Delta, \nabla, \epsilon, \eta)$ makes A a Frobenius algebra. In general, we can take $A = H^*(X, \mathbb{Z})$ for any oriented smooth manifold X. The ring structure, Poincaré duality and the diagonal embedding defines a Frobenius algebra structure on A. However, Khovanov homology to be defined will depend on the particular knot diagram unless $X = S^2$.

We need a functor:

$$Z : \mathrm{Cob}_2 \to \mathbb{Z}\text{-Mod}$$

For the definition, let $Z(S^1) = A$ and $Z(\coprod_k S^1) = A^{\otimes k}$. For each elementary cobordism, define:



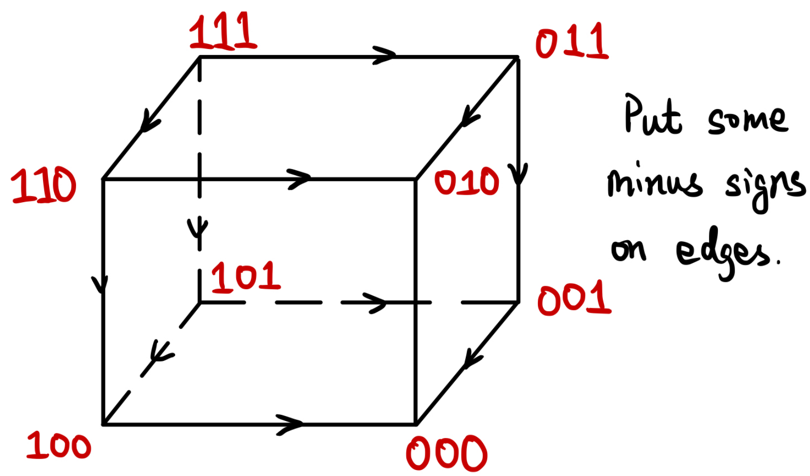
Apply the functor Z to the cube of resolutions. Each vertex $v \in \{0,1\}^n$, the resolved diagram $v(D(K))$ is a union of circles, and $Z(v(D(K))$ is a tensor product of A. Each edge is either splitting a circle or merging two circles. Faces of $[0,1]^n$ imply the resulting cube of morphisms are commutative.

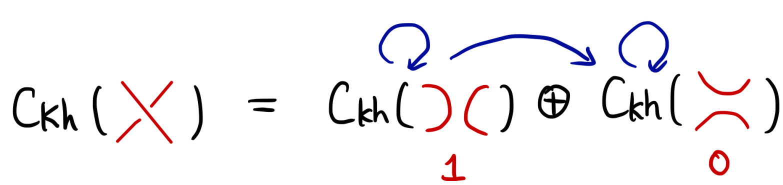
$$\begin{aligned} \text{each vertex} &\mapsto A^{\otimes n},\ n = \text{the number of circles} \\ \text{each edge} &\mapsto \ \text{either } \Delta \text{ or } \nabla. \end{aligned}$$

We form a graded chain complex by taking the direct sum

$$C_{Kh}(D(K)) = \bigoplus_{v \in \{0,1\}^n} Z(v(D(K))).$$

where the grading is given by $|v| := \sum_{1 \in v} 1$. The differential ∂_{Kh} is the direct sum of all edge morphisms, which decreases the grading by 1. We need to assign some minus signs on edge morphisms to assure $\partial_{Kh}^2 = 0$. The chain homotopy type of $(C_{Kh}(D(K)), \partial_{Kh})$ is independent of the digram $D(K)$, and so an invariant of $K \subset S^3$.





Take a crossing of $D(K)$. Then 0- and 1-resolution result in two links K_0 and K_1 respectively. There is a short sequence of chain complexes:

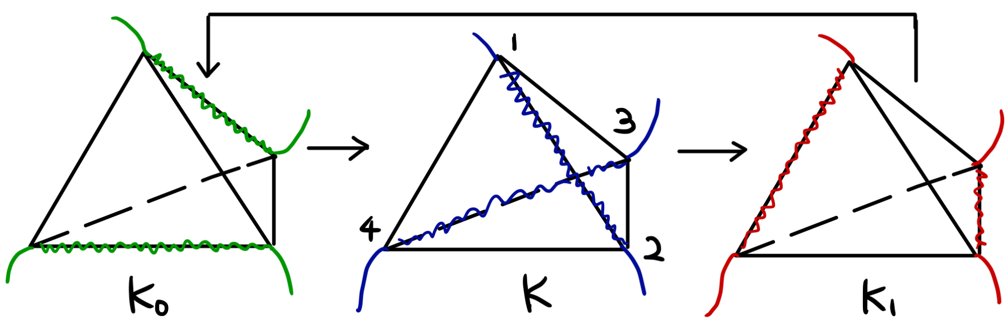
$$0 \to C_{Kh}(D(K_0)) \to C_{Kh}(D(K)) \to C_{Kh}(D(K_1)) \to 0$$

which induces an exact triangle:

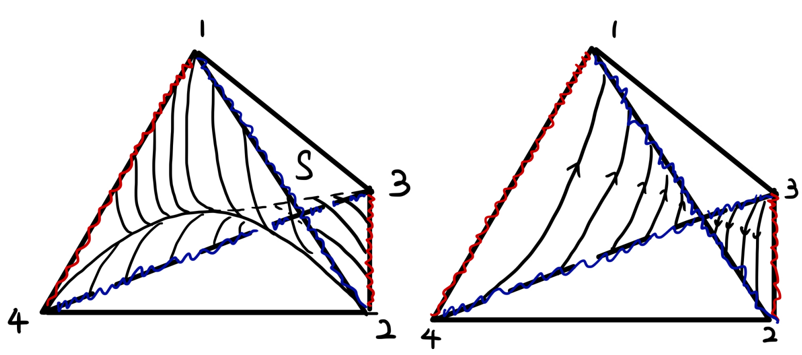
$$
\begin{tikzcd}
& \mathrm{Kh}(K) \arrow[dr] & \\
\mathrm{Kh}(K_0) \arrow[ur] & & \mathrm{Kh}(K_1). \arrow[ll, "\delta"]
\end{tikzcd}
$$

39.3. Exact Triangles

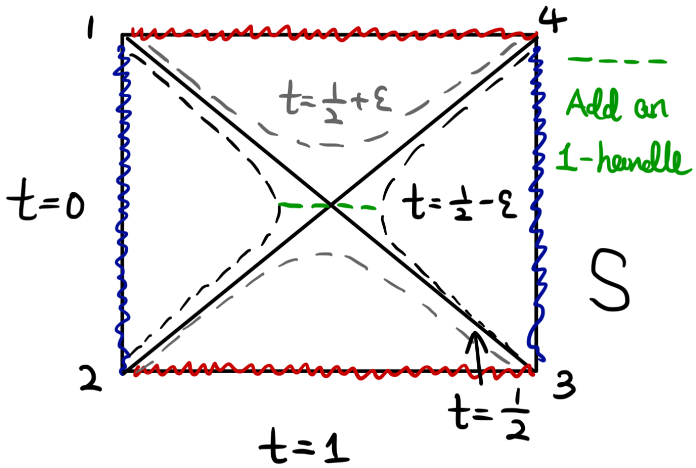
The same exact triangle holds for instanton knot Floer homology, which comes with natural cobordisms of links. Imagine that the resolution of a crossing happens within a tetrahedron T:



The cobordism from K to K_1 happens within $T \times [0, 1]$ by attaching a handle. Consider the surface $S \subset T$:



The cobordism can be visualized as:

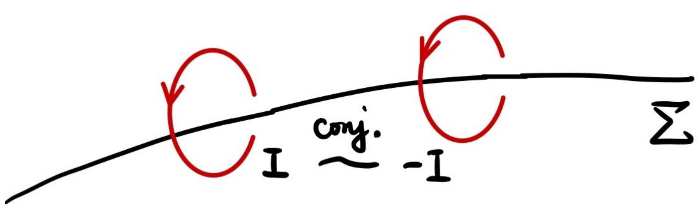


Recall that for each cobordism $(X, \Sigma) : (Y_-, K_-) \to (Y_+, K_+)$, there is a natural morphism

$$I^{\#}(X, \Sigma) : I^{\#}(Y_-, K_-) \to I^{\#}(Y_+, K_-)$$

defined by counting singular ASD connections on (X, Σ).

The surface Σ might be **non-orientable**, but the construction still works. In this case, the normal bundle of Σ is not orientable, and the linking holonomy is only well-define up to $\pm\,\mathrm{Id}$. Since g is conjugate to $-g$ for $g \in S^2_* \subset SU(2)$ (i.e. g is an element of order 4), a singular connection still makes sense.



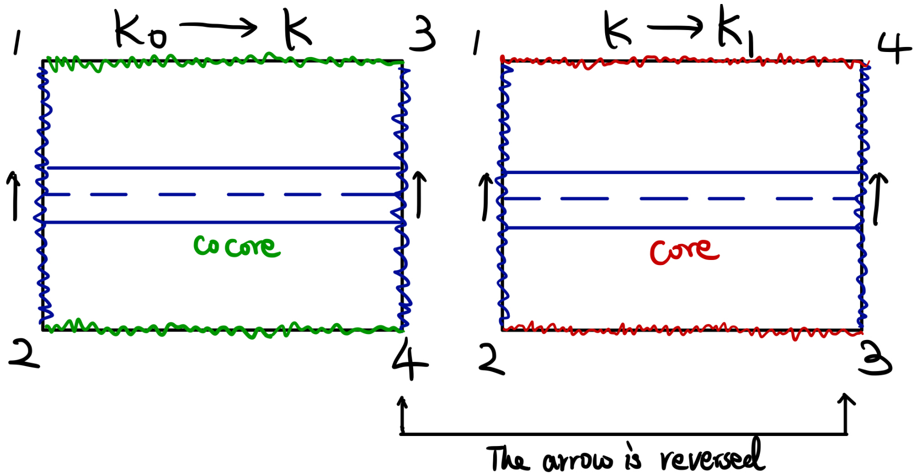
In what follows, we will work with the digram (See Special Lecture 2):

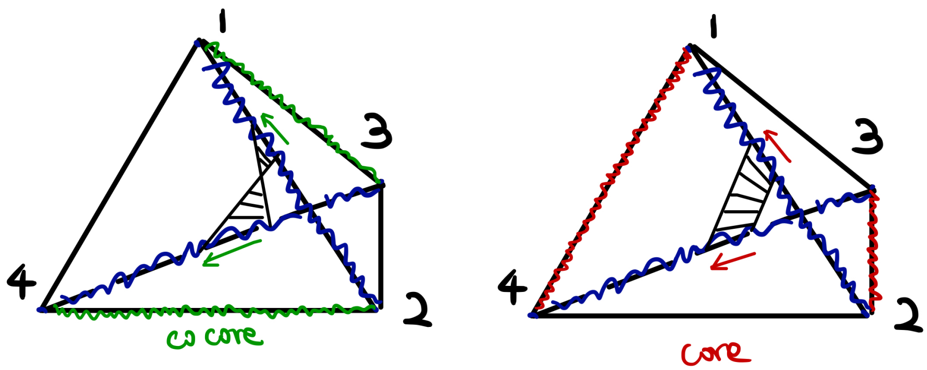
$$\begin{tikzcd}
& C_*(S^3, K_1) \arrow[dr, "C_*(W_1)"] \arrow[dl, bend right, "K_1"'] & \\
C_*(S^3, K_0) \arrow[ur, "C_*(W_0)"] \arrow[rr, bend right, "K_0"'] & & C_*(S^3, K_2). \arrow[ll, "C_*(W_2)"] \arrow[ul, bend right, "K_2"']
\end{tikzcd}$$

where knots are indexed in a different way. It remains to verify it is an exact triangle on the level of homology. The proof invokes the detection lemma (See Special Lecture 2). The subtlety is that each cobordism $W_i = (S^3 \times I, \Sigma_i)$ comes with an orientable surface Σ_i, but their compositions

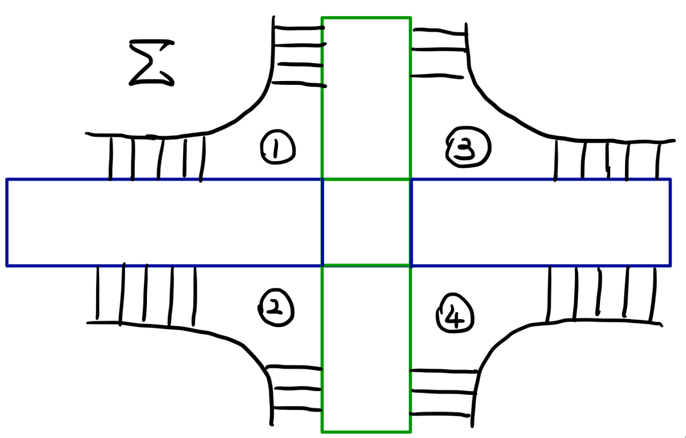
$$\Sigma_{i+1} \circ \Sigma_i$$

become non-orientable. To see this, consider the co-core of the 1-handle in Σ_i and the core of the 1-handle in Σ_{i+1}. They form a Möbius band in $\Sigma_{i+1} \circ \Sigma_i$:

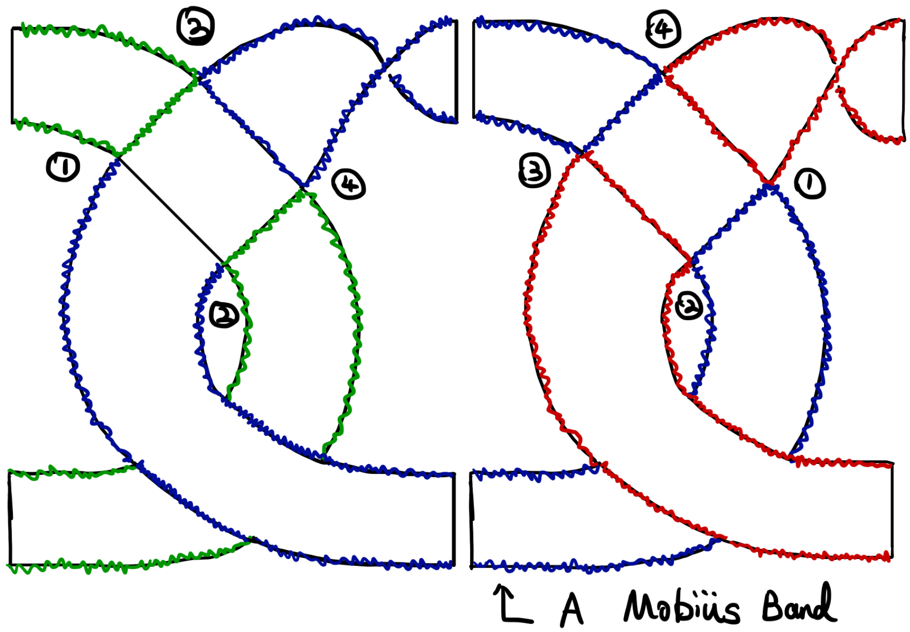




One way to visualize the composition $\Sigma_{i+1} \circ \Sigma_i$ is to think of each Σ as



The composition then becomes



39.4. Some Non-Orientable Surfaces in 4-Manifolds

When X is an oriented 4-manifold, the self-intersection number $\Sigma \cdot \Sigma$ makes sense no matter if Σ is orientable (see Lecture 36).

Example 39.4.1. $\mathbb{RP}^2_\pm \subset S^4$. Let S^4 be the unit sphere of traceless symmetric 3×3 matrices ($\dim_{\mathbb{R}} = 5$):

$$\begin{pmatrix} x & y & z \\ -y & u & v \\ -z & -v & -x-u \end{pmatrix}.$$

The group $SO(3)$ acts on S^4 by matrix conjugation. Let

$$\mathbb{RP}^2_+ = \{\text{matrices with a positive eigenspace of dimension } 2 \subset S^4\},$$
$$\mathbb{RP}^2_- = \{\text{matrices with a negative eigenspace of dimension } 2 \subset S^4\}.$$

That is to say, any matrix $A \in \mathbb{RP}^2_+$ is conjugate to

$$\lambda \begin{pmatrix} 1 & & \\ & 1 & \\ & & -2 \end{pmatrix}$$

where λ is a positive constant fixed by the length of A. The stabilizer of A is $O(2) \subset SO(3)$, so the orbit is

$$SO(3)/O(2) \cong \mathbb{RP}^2.$$

It is easier to compute the self intersection of $\mathbb{RP}^2_\pm$ from a different description. A theorem by Kuiper, Arnold and others states that

$$\mathbb{CP}^2/\text{complex conjugation} \cong S^4.$$

A real conic with no real roots is an $S^2 \subset \mathbb{CP}^2$ invariant under conjugation. For instance, we take

$$\mathbb{CP}^1 = \{x^2 + y^2 + z^2 = 0\} \subset \mathbb{CP}^2.$$

For a conic, the intersection number is $S^2 \cdot S^2 = +4$. Passing to the quotient, the self-intersection of $\mathbb{RP}^2_+ = S^2/\mathbb{Z}_2$ is divided by 2, so

$$\mathbb{RP}^2_+ \cdot \mathbb{RP}^2_+ = 2.$$

On the other hand, the fixed point set of the action is a real projective space $\mathbb{RP}^2_- \subset \mathbb{CP}^2$. $\mathbb{RP}^2_- \subset \mathbb{CP}^2$ is a Lagrangian submanifold. Hence, $T^*\mathbb{RP}^2_- \cong N(\mathbb{RP}^2_+)$ and

$$\begin{aligned} 1 = \chi(\mathbb{RP}^2) = \langle e(T\mathbb{RP}^2_-), [\mathbb{RP}^2_-, o(\mathbb{RP}^2_-)]\rangle &= -\langle e(T^*\mathbb{RP}^2_-), [\mathbb{RP}^2_-, o(\mathbb{RP}^2_-)]\rangle \\ &= -\langle e(N(\mathbb{RP}^2_-)), [\mathbb{RP}^2_-, o(\mathbb{RP}^2_-)]\rangle. \end{aligned}$$

Since $\mathbb{RP}^2_-$ is fixed, after passing to the quotient, the self-intersection number will be doubled:

$$\mathbb{RP}^2_- \cdot \mathbb{RP}^2_- = -2.$$

Signs of self-intersection numbers depend only on the orientation of S^4.

Reversing the construction, we take the double branched cover of S^4 along $\mathbb{RP}^2_\pm$:

$$\begin{aligned} (S^4, \mathbb{RP}^2_+) &\mapsto \overline{\mathbb{CP}}^2 \\ (S^4, \mathbb{RP}^2_-) &\mapsto \mathbb{CP}^2. \end{aligned}$$

In Special Lecture 2, the composition of W_0 and W_1 is

$$W_1 \circ W_0 = \overline{W_2} \# \overline{\mathbb{CP}}^2.$$

In our case, it becomes

$$W_1 \circ W_0 = (S^3 \times I, \Sigma_1) \circ (S^3 \times I, \Sigma_0) = \overline{(S^3 \times I, \Sigma_2)} \# (S^4, \mathbb{RP}^2_+).$$

where we take connected sums of both 4-manifolds and embedded 2-surfaces. By using the same technique as Special Lecture 2, one can prove

$$\begin{tikzcd} & I^\#(S^3, K) \arrow[dr] & \\ I^\#(S^3, K_0) \arrow[ur] & & I^\#(S^3, K_1). \arrow[ll, "\delta"] \end{tikzcd}$$

is indeed an exact triangle. Equivalently, we have

$$C^\#(S^3, K) \xrightarrow[\cong]{\text{chain homotopy equiva.}} \mathrm{Cone}(C^\#(S^3, K_1) \xrightarrow{f_i} C^\#(S^3, K_0)). \tag{54}$$

39.5. A Spectral Sequence

By resolving another crossing of the knot digram $D(K)$, we can iterate the chain homotopy equivalence (54):

$$C^{\#}(S^3, K) \xrightarrow{\cong} \mathrm{Cone}(C^{\#}(S^3, K_1) \xrightarrow{f_1} C^{\#}(S^3, K_0))$$

$$\xrightarrow{\cong} \mathrm{Cone}\left(\begin{tikzcd} C^{\#}(K_{11}) \arrow[r, "f_{11\to 01}"] \arrow[d, "f_{11\to 10}"] \arrow[dr, "f_{11\to 00}"] & C^{\#}(K_{01}) \arrow[d, "f_{01\to 00}"] \\ C^{\#}(K_{10}) \arrow[r, "f_{10\to 00}"] & C^{\#}(K_{00}) \end{tikzcd}\right).$$

We replaced $C^{\#}(S^3, K_i)$ for $i = 0, 1$ by

$$\mathrm{Cone}\left(\begin{tikzcd} C^{\#}(K_{i1}) \arrow[d, "f_{i1\to i0}"] \\ C^{\#}(K_{i0}) \end{tikzcd}\right) = C^{\#}(K_{i1}) \oplus C^{\#}(K_{i0}) \circlearrowright \begin{pmatrix} \partial_{i1} & 0 \\ f_{i1\to i0} & \partial_{i0} \end{pmatrix}.$$

The horizontal map f_1 in (54) is replaced by the sum of three maps:

$$f_{11\to 01},\ f_{11\to 00},\ f_{10\to 00}.$$

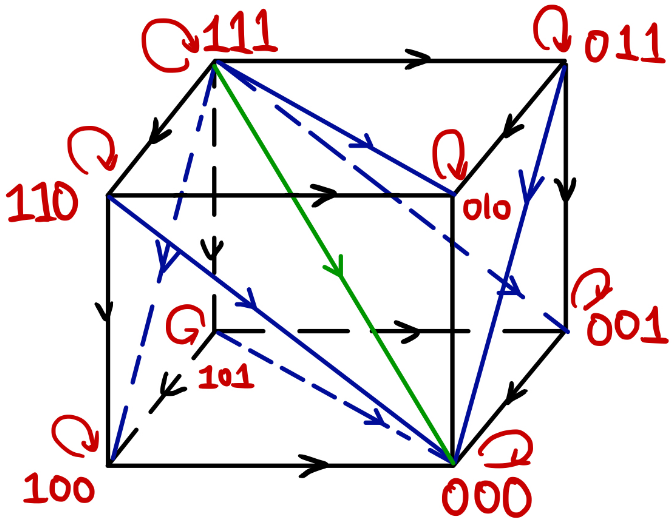
The diagonal map $f_{11\to 00}$ is not a chain map. It is a "canonical" chain homotopy between two compositions of chain maps:

$$f_{01\to 00} \circ f_{11\to 01} \text{ and } f_{10\to 00} \circ f_{11\to 10}.$$

Indeed, compare the right-bottom entry of the multiplication:

$$\begin{pmatrix} \partial_{11} & 0 \\ f_{11\to 10} & \partial_{10} \end{pmatrix} \begin{pmatrix} f_{11\to 01} & 0 \\ \boldsymbol{f_{11\to 00}} & f_{10\to 00} \end{pmatrix} = \begin{pmatrix} f_{11\to 01} & 0 \\ \boldsymbol{f_{11\to 00}} & f_{10\to 00} \end{pmatrix} \begin{pmatrix} \partial_{01} & 0 \\ f_{01\to 00} & \partial_{00} \end{pmatrix}$$

Keep resolving more crossings and keep iteration. We end up with a cube of maps:



At each vertex v of $[0,1]^n$, it is a chain complex computing the singular instanton knot homology of a unlink. For each edge, it is associated with a chain map. A filtration on this large chain complex is given by the norm of v:

$$|v| := \sum_{1 \in v} 1.$$

Hence, we obtain a spectral sequence abutting to $I^{\#}(S^3, K)$. Moreover,

(1) E_1-page. At each vertex $v \in \{0,1\}^n$, the resolved digram is a unlink. (E_1, d_1) computes the homology at each vertex. Hence, it is $A^{\otimes n}$ where n is the number of link components.
(2) E_2-page. (E_2, d_2) consists of chain maps along edges of $[0,1]^n$. They agree with Khovanov differentials. Hence, the E_2-page computes the Khovanov homology of (S^3, K).

In particular, we obtain $\dim \mathrm{Kh}(K) \geqslant \dim I^{\#}(S^3, K)$. To finish the proof of unknot detection, we also need:

(1) Compare $I^{\#}(S^3, K)$ with $I^{\natural}(S^3, K)$, the original version of knot Floer homology defined by Floer.
(2) Using Sutured manifolds technique, show that $I^{\natural}(S^3, K)$ detects the unknot.

Note that

$$\dim \mathrm{Kh}(U) = \min_K \dim \mathrm{Kh}(K).$$

If $\dim \mathrm{Kh}(K)$ is minimal, both $\dim I^{\#}(S^3, K)$ and $\dim I^{\natural}(S^3, K)$ are minimal as well, so K is the unknot.

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