MATH 231A Assignment 3

This assignment is due at 11:59 pm on Wednesday, Oct. 30, 2024.

Problem 1 (Exercise 10.10): This exercise generalizes the computation of the homology of spheres, and introduces several important constructions.

The *cone* on a space X is the quotient space $CX = X \times I / X \times \{0\}$, where I is the unit interval $[0, 1]$. The cone is a pointed space, with basepoint $*$ given by the "cone point", i.e. the image of $X \times \{0\}$. (By convention, the cone on the empty space $\varnothing$ is a single point, the cone point.) Regard X as the subspace of CX of all points of the form $(x, 1)$.

Define the *suspension* of a space X to be $SX = CX/X$. Make SX a pointed space by declaring the image of $X \subseteq CX$ to be the basepoint in SX. (By convention, the quotient $W/\varnothing$ is the disjoint union of W with a single point, which is declared to be the basepoint. So $S\varnothing = */\varnothing$ is the discrete two-point space, with the new point as basepoint.)

The quotient map induces a map of pairs $f : (CX, X) \to (SX, *)$. For any $a, b \in I$ with $a \le b$, let $C_a^b X$ denote the image of $X \times [a, b]$ in CX. Thus $C_0^1 X = CX$, $C_0^0 X = *$, and $C_1^1 X = X$.

Let $p : CX \to CX$ send (x, t) to $(x, 3t)$ for $t \le 1/3$ and to $(x, 1)$ if $t \ge 1/3$.

1. Show that CX is contractible.
2. Show that p defines a homotopy equivalence of pairs $\left(C_0^{2/3} X, C_{1/3}^{2/3} X\right) \to (CX, X)$.
3. Show that the evident map $e : \left(C_0^{2/3} X, C_{1/3}^{2/3} X\right) \to \left(SX, C_{1/3}^{1} X/X\right)$ is an excision.
4. Show that p defines a homotopy equivalence of pairs $\left(SX, C_{1/3}^{1} X/X\right) \to (SX, *)$.
5. Conclude from the commutativity of that f induces an isomorphism in homology.
6. Show that there is a natural isomorphism between augmented and reduced homology groups, $\widetilde{H}_{n-1}(X) \to \overline{H}_n(SX)$, for any n.

Problem 2 (Exercise 11.8) Use the Mayer-Vietoris sequence to compute the homology groups of the projective plane P, the Klein bottle K, and the torus T. (The projective plane is obtained by sewing a disk onto a Möbius band along their boundaries. The Klein bottle is obtained either by sewing two Möbius bands together, or by sewing the two boundary components of a cylinder together in a funny way. A torus is obtained by sewing the boundary components of a cylinder together in a less funny way. In each case, it's a good idea to give yourself a hem: glue open "collars" together.)

Problem 3: Read Lectures 12-13 in Miller's notes. You do not need to write anything for this problem.

Problem 4 (Exercise 14.10): Provide the Euclidean space $\mathbb{R}^n$ with the structure of a CW complex.

Problem 5 (Exercise 14.11): Provide each closed oriented connected surface with the structure of a CW complex with just a single 0-cell, several 1-cells, and a single 2-cell. (How about unoriented closed surfaces?)