

## MATH 231A ASSIGNMENT 3

This assignment is due at 11:59 pm on Wednesday, Oct. 30, 2024.

**Problem 1 (Exercise 10.10):** This exercise generalizes the computation of the homology of spheres, and introduces several important constructions.

The *cone* on a space  $X$  is the quotient space  $CX = X \times I / X \times \{0\}$ , where  $I$  is the unit interval  $[0, 1]$ . The cone is a pointed space, with basepoint  $*$  given by the “cone point”, i.e. the image of  $X \times \{0\}$ . (By convention, the cone on the empty space  $\emptyset$  is a single point, the cone point.) Regard  $X$  as the subspace of  $CX$  of all points of the form  $(x, 1)$ .

Define the *suspension* of a space  $X$  to be  $SX = CX/X$ . Make  $SX$  a pointed space by declaring the image of  $X \subseteq CX$  to be the basepoint in  $SX$ . (By convention, the quotient  $W/\emptyset$  is the disjoint union of  $W$  with a single point, which is declared to be the basepoint. So  $S\emptyset = */\emptyset$  is the discrete two-point space, with the new point as basepoint.)

The quotient map induces a map of pairs  $f : (CX, X) \rightarrow (SX, *)$ . For any  $a, b \in I$  with  $a \leq b$ , let  $C_a^b X$  denote the image of  $X \times [a, b]$  in  $CX$ . Thus  $C_0^1 X = CX$ ,  $C_0^0 X = *$ , and  $C_1^1 X = X$ .

Let  $p : CX \rightarrow CX$  send  $(x, t)$  to  $(x, 3t)$  for  $t \leq 1/3$  and to  $(x, 1)$  if  $t \geq 1/3$ .

1. Show that  $CX$  is contractible.
2. Show that  $p$  defines a homotopy equivalence of pairs  $(C_0^{2/3} X, C_{1/3}^{2/3} X) \rightarrow (CX, X)$ .
3. Show that the evident map  $e : (C_0^{2/3} X, C_{1/3}^{2/3} X) \rightarrow (SX, C_{1/3}^1 X/X)$  is an excision.
4. Show that  $p$  defines a homotopy equivalence of pairs  $(SX, C_{1/3}^1 X/X) \rightarrow (SX, *)$ .
5. Conclude from the commutativity of that  $f$  induces an isomorphism in homology.
6. Show that there is a natural isomorphism between augmented and reduced homology groups,  $\tilde{H}_{n-1}(X) \rightarrow \overline{H}_n(SX)$ , for any  $n$ .

**Problem 2 (Exercise 11.8)** Use the Mayer-Vietoris sequence to compute the homology groups of the projective plane  $P$ , the Klein bottle  $K$ , and the torus  $T$ . (The projective plane is obtained by sewing a disk onto a Möbius band along their boundaries. The Klein bottle is obtained either by sewing two Möbius bands together, or by sewing the two boundary components of a cylinder together in a funny way. A torus is obtained by sewing the boundary components of a cylinder together in a less funny way. In each case, it's a good idea to give yourself a hem: glue open “collars” together.)

**Problem 3:** Read Lectures 12-13 in Miller's notes. You do not need to write anything for this problem.

**Problem 4 (Exercise 14.10):** Provide the Euclidean space  $\mathbb{R}^n$  with the structure of a CW complex.

**Problem 5 (Exercise 14.11):** Provide each compact oriented connected surface with the structure of a CW complex with just a single 0-cell and a single 1-cell and a single 2-cell. (How about unoriented surfaces?)