MATH 231A Assignment 2

This assignment is due at 11:59 pm on Wednesday, Oct. 16, 2024.

Problem 1 (Exercise 5.15):

1. Let A_* be a chain complex. It is *acyclic* if $H_*(A_*) = 0$, and *contractible* if it is chain-homotopy-equivalent to the trivial chain complex. Prove that a chain complex is contractible if and only if it is acyclic and for every n the inclusion $Z_n A \hookrightarrow A_n$ is a split monomorphism of abelian groups.

2. Give an example of an acyclic chain complex that is not contractible.

Problem 2 (Exercise 8.8): Let $0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$ be a short exact sequence. Establish bijections among the following three sets.

1. The set of homomorphisms $\sigma : C \to B$ such that $p\sigma = 1_C$.

2. The set of homomorphisms $\pi : B \to A$ such that $\pi i = 1_A$.

3. The set of homomorphisms $\alpha : A \oplus C \to B$ such that $\alpha(a, 0) = ia$ for all $a \in A$ and $p\alpha(a, c) = c$ for all $(a, c) \in A \oplus C$.

Show that any homomorphism as in (3) is an isomorphism. (Any one of these is a *splitting* of the short exact sequence, and the sequence is then said to be *split*.)

Problem 3 (Exercise 9.8): Suppose that

$$\begin{tikzcd}
\cdots \arrow[r] & A_n \arrow[r] \arrow[d] & B_n \arrow[r] \arrow[d] & C_n \arrow[r] \arrow[d, "\cong"] & A_{n-1} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & A'_n \arrow[r] & B'_n \arrow[r] & C'_n \arrow[r] & A'_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

is a "ladder": a map of long exact sequences. So both rows are exact and each square commutes. Suppose also that every third vertical map is an isomorphism, as indicated. Prove that these data determine a long exact sequence

$$\cdots \to A_n \to A'_n \oplus B_n \to B'_n \to A_{n-1} \to \cdots$$

Problem 4 (Exercise 9.9, "3×3 lemma"): Let

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A' \arrow[r] \arrow[d] & B' \arrow[r] \arrow[d] & C' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A'' \arrow[r] \arrow[d] & B'' \arrow[r] \arrow[d] & C'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

be a commutative diagram of abelian groups. Assume that all three columns are exact, that all but one of the rows is exact, and that the compositions in the remaining row are trivial. Prove that the remaining row is also exact. (Hint: view each row as a chain complex.)

Problem 5 (Exercise 9.11, long exact homology sequence of a triple): Let (C, B, A) be a "triple," so C is a space, B is a subspace of C, and A is a subspace of B. Show that there are natural transformations $\partial : H_n(C, B) \to H_{n-1}(B, A)$ such that

$$\cdots \to H_n(B, A) \xrightarrow{i_*} H_n(C, A) \xrightarrow{j_*} H_n(C, B) \xrightarrow{\partial} H_{n-1}(B, A) \to \cdots$$

is exact, where $i : (B, A) \to (C, A)$ and $j : (C, A) \to (C, B)$ are the inclusions of pairs.