MATH 231A Assignment 1

This assignment is due at 11:59 pm on Wednesday, Oct. 2, 2024.

Problem 1 (Exercise 1.8):

1. Let $[n]$ denote the totally ordered set $\{0, 1, \dots, n\}$. Let $\phi : [m] \to [n]$ be an order preserving function (so that if $i \leq j$ then $\phi(i) \leq \phi(j)$). Identifying the elements of $[n]$ with the vertices of the standard simplex Δ^n, ϕ extends to an affine map $\Delta^m \to \Delta^n$ that we also denote by ϕ. Give a formula for this map in terms of barycentric coordinates: If we write $\phi(s_0, \dots, s_m) = (t_0, \dots, t_n)$, what is t_j as a function of $(s_0, \dots, s_m)$?

2. Write $d^j : [n-1] \to [n]$ for the order preserving injection that omits j as a value. Show that an order preserving injection $\phi : [n-k] \to [n]$ is uniquely a composition of the form $d^{j_k} d^{j_{k-1}} \cdots d^{j_1}$, with $0 \leq j_1 < j_2 < \cdots < j_k \leq n$. Do this by describing the integers $j_1, \dots, j_k$ directly in terms of ϕ, and then verify the straightening rule

$$d^i d^j = d^{j+1} d^i \quad \text{for} \quad i \leq j.$$

3. Use the relations among the d_i's to prove that

$$d^2 = 0 : S_n(X) \to S_{n-2}(X).$$

Problem 2 (Example 2.1): Suppose X is a topological space and $\sigma : \Delta^1 \to X$ is a continuous map. Define $\phi : \Delta^1 \to \Delta^1$ by sending $(t, 1-t)$ to $(1-t, t)$. Precomposing σ with ϕ gives another singular simplex $\overline{\sigma}$ which reverses the orientation of σ. Show that it is not true that $\overline{\sigma} = -\sigma$ in $S_1(X)$ but $\overline{\sigma} \equiv -\sigma \bmod B_1(X)$. (This means that there is a 2-chain in X whose boundary is $\overline{\sigma} + \sigma$. If $d_0\sigma = d_1\sigma$, so that $\sigma \in Z_1(X)$, then $\overline{\sigma}$ and $-\sigma$ are homologous cycles, so that $[\overline{\sigma}] = -[\sigma]$ in $H_1(X)$.)

Problem 3 (Exercise 2.3): Construct an isomorphism

$$H_n(X) \oplus H_n(Y) \to H_n(X \coprod Y).$$

Problem 4 (Exercise 3.7): Write $\pi_0(X)$ for the set of path-components of a space X. Construct an isomorphism

$$\mathbb{Z}\pi_0(X) \to H_0(X).$$

Problem 5: Read Lecture 4 in Miller's notes. You do not need to write anything for this problem.