Class 13 ASD equation

Instanton theory is another gauge theory

Instead of SW equ (3d, 4d). we consider

flat conn $F_A = 0$ for 3d and ASD equ $F_A^+ = 0$ for 4d

both are special cases of Yang-Mills eqn.

Similarly, we will consider configuration space, gauge group

0, 1, 2-dim (compact) moduli space to define Floer homology.

We will use similar construction for sutured instanton homology

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For closed 3-mfd Y and cobordism $W: Y_0 \to Y_1$ or closed 4-manifold X,

we consider principal $SO(3)$ bundle P

(originally, Floer consider $SU(2)$ bundle on integral homology sphere Y)

Later, Donaldson, KM also consider $SO(3)$, $U(2)$, $SO(3)-SU(2)$ mixed theories

Lem. isomorphism class of $SO(3)$ bundle P over Y
(Facts) is determined by second Stiefel-Whitney class $w_2(P) \in H^2(Y; \mathbb{Z}/2)$

similar for W

For closed 4-mfd X, we also consider ponjuyakin class $p_1 \in H^4(X; \mathbb{Z})$

i.e. $SO(3)$ P determined by (w_2, p)

$SU(2)$ bundles are determined by second Chern class $c_2 \in H^4(X; \mathbb{Z})$

(first Chern class always vanishes) $\langle c_2, [X] \rangle$ called <u>instanton num</u>

Any $SU(2)$ bundle over Y is trivial.

$\mathcal{A}(X,P)$ = space of conns affine over $\Omega^1(X, \mathrm{ad}P) = \Gamma(T^*X \otimes \mathrm{ad}P)$

$\mathrm{ad}P$ = associated vector bundle for adjoint rep $P \times_{\mathrm{ad}} \mathfrak{g}$ over X

$\mathfrak{g}$ = Lie algebra of $G = SO(3)$ (or $SU(2)$)

$G \times \mathfrak{g} \to \mathfrak{g}$

$(g, x) \mapsto g^{-1} x g$

$\mathcal{G}(X,P)$ = space of gauge transformation

= {bundle automorphisms of P that are G-equ}

$\eta : P \to P \quad \eta \in \mathcal{G}(X,P) \iff \tilde{\eta} : P \to G$ equ for conj on G

For $p \in P$ $\quad \eta(p) = pg \qquad \tilde{\eta}(p) = g$

$\eta(pg_0) = pg_0 h \qquad \tilde{\eta}(p \cdot g_0) = h$

G-equ $\longrightarrow$ $\| $

$\eta(p) \cdot g_0$

conj equivariant

$\Rightarrow \eta(p) = pg_0 h g_0^{-1} \qquad \Rightarrow \underbrace{\tilde{\eta}(pg_0)}_{h} = g_0^{-1} \underbrace{\tilde{\eta}(p)}_{g_0 h g_0^{-1}} g_0$

$\tilde{\eta} : P \to G$ equ for conj on $G \iff u : X \to \mathrm{Ad}P = P \times_{\mathrm{Ad}} G$

$\mathrm{Ad} : G \times G \to G$

$(g, h) \mapsto g^{-1} h g$

$x = \pi(p) \quad u(x) = [(p, \tilde{\eta}(p))] = [(p \cdot g, g^{-1} \tilde{\eta}(p) g)]$ fiber iso to G

Conclusion $\quad \mathcal{G}(X,P) = \Gamma(\mathrm{Ad}P)$ Lie group

its Lie algebra is $\Gamma(\mathrm{ad}P)$

action of gauge transformation

$u(A) = A - (d_A u) u^{-1}$

$\mathcal{B}(X,P) = \mathcal{A}(X,P) / \mathcal{G}(X,P)$ action may not free

We can also take $\mathcal{G}_0(X,P) \subset \mathcal{G}(X,P)$ with fixed value at y_0
called based gauge group.

$$\mathcal{B}_0(X,P) = \mathcal{A}(X,P)/\mathcal{G}_0(X,P)$$

We replace X by Y, W.

Chern-Simons functional: Fix conn A_0, for $\mathfrak{g}$-valued 1-form a

usually $\frac{1}{8\pi^2}\int_Y tr(d_{A_0}a\wedge a + \frac{2}{3}a\wedge a\wedge a)$

$$CS(a) = \int_Y tr(a\wedge(F_{A_0} + \tfrac{1}{2}d_{A_0}a + \tfrac{1}{6}[a\wedge a]))$$

Similar to CSD functional, we find formal gradient as follows
(critical pts)

$$D(CS_a)(b) = \lim_{t\to 0}\frac{CS(a+tb) - CS(a)}{t}$$

$$= \int_Y tr(b\wedge(F_{A_0} + \tfrac{1}{2}d_{A_0}a + \tfrac{1}{6}[a\wedge a]) + a\wedge(\tfrac{1}{2}d_{A_0}b) + a\wedge(\tfrac{1}{3}[a\wedge b]))$$

Let $A = A_0 + a$ $\quad F_A = F_{A_0} + d_{A_0}a + \frac{1}{2}[a\wedge a]$

$d_A b = d_{A_0}b + \frac{1}{2}[a\wedge b]$ $\quad d_A a = d_{A_0}a + \frac{1}{2}[a\wedge a]$

$$= \int_Y tr(b\wedge F_A - \tfrac{1}{2}b\wedge d_{A_0}a - \tfrac{1}{3}[a\wedge a] + a\wedge(\tfrac{1}{2}d_{A_0}b) + a\wedge\tfrac{1}{3}[a\wedge b])$$

$$= \int_Y tr(b\wedge F_A + \tfrac{1}{2}(a\wedge d_A b + d_A a\wedge b))$$

$$\int_Y \tfrac{1}{2}d(tr(a\wedge b)) = 0$$

$$= \int_Y tr(b\wedge F_A)$$

b arbitrary $\Rightarrow F_A = 0$ $\quad$ A flat

Let $\langle a,b\rangle_{L_2} = \int_Y tr(a\wedge *_3 b) = \int_Y tr(b\wedge *_3 a)$

$\langle \mathrm{grad}(CS)_A, b\rangle_{L^2} = D(CS_a)(b)$

$\Rightarrow \mathrm{grad}(CS)_A = *_3 F_A$ because $*_3^2 = 1$

$*_3$ 3-dim Hodge star. (depending on the metric)

In general, $*: \Omega^k \to \Omega^{n-k}$ $n = \dim$ of mfd M

$e^1, \dots, e^n$ orthornomal basis of $T^*M = \Omega^1$

$dvol_M = e^1 \wedge \cdots \wedge e^n$

$*\alpha$ satisfies $\alpha \wedge *\alpha = dvol_M$

Ex. $*(e^1 \wedge \cdots \wedge e^k) = e^{k+1} \wedge \cdots \wedge e^n$

4d case $*^2 = 1 : \Omega^2 \to \Omega^2$

For $w \in \Omega^2$ $w^{\pm} = \frac{w \pm * w}{2}$ ± 1-eigenspace of $*$

On the other hand, we compute $CS(uA_0) - CS(A_0)$

Suppose $uA_0 - A_0 = a$ $F^4_A = F^3_{A_0 + ta} + dt \wedge a$

Then

$$CS(a) = \int_Y tr\left(a \wedge \left(F_{A_0} + \tfrac{1}{2} d_{A_0} a + \tfrac{1}{6}[a \wedge a]\right)\right)$$

$$= \int_Y tr \int_0^1 a \wedge \left(F_{A_0} + t d_{A_0} a + \tfrac{1}{2} t^2 [a \wedge a]\right) dt$$

$$= \int_0^1 \int_Y tr(a \wedge F_{A_0 + ta})\, dt$$

$$= \int_{[0,1] \times Y} tr(dt \wedge a \wedge F_{A_0 + ta})$$

$$= \int_{S^1 \times Y} tr(F^4_A \wedge F^4_A)$$

$= 8\pi^2 c_2(\mathbb{P})$ by Chern-Weil formula

$\mathbb{P}$ bundle over $S^1 \times Y$ by gluing uA_0 A_0

$CS : \mathcal{A}(Y,P)/\mathcal{G}(Y,P) \longrightarrow \mathbb{R}/8\pi^2\mathbb{Z}$

Class 14 Instanton Floer homology

Last time, we introduce Chern-Simons functional for fixed A_0

$$CS(a) = \int_Y tr\left(a \wedge \left(F_{A_0} + \frac{1}{2} d_{A_0} a + \frac{1}{6}[a \wedge a]\right)\right)$$

regarded as $B(Y,P) = A(Y,P)/G(Y,P) \to \mathbb{R}/8\pi^2\mathbb{Z}$

on $\mathbb{R}_t \times Y = W$ with product metric

$$dt \wedge dvol_Y = dvol_M$$

$w \in \Omega^2(Y)$ $\quad *_4 w = dt \wedge *_3 w$

$\phi \in \Omega^1(Y)$. $\quad *_4(dt \wedge \phi) = *_3 \phi$

4d conn $\mathbb{A} = A_0 dt + \sum_{i=1}^{3} A_i dy_i$ $\quad y_1, y_2, y_3$ on Y

We can fix a gauge (by transportation along $\mathbb{R}$-factor)

s.t. $A_0 = 0$ (called <u>temporal gauge</u>) $\mathbb{A} = A(t) = (0, A_i(t))$

$$\underbrace{F^4_{\mathbb{A}}}_{\text{on } \mathbb{R}\times Y} = \underbrace{F^3_{A(t)}}_{\text{on } Y} + dt \wedge \frac{\partial A(t)}{\partial t}$$

from local form $F = da + \frac{1}{2}[a \wedge a]$

$da = dt \wedge \frac{\partial a}{\partial t} + dy_i \wedge \frac{\partial a}{\partial y_i}$

2-form $\quad$ 1-form

$$*_4 F_{\mathbb{A}} = *_4 F_{A(t)} + *_4\left(dt \wedge \frac{\partial A(t)}{\partial t}\right)$$

$$= dt \wedge *_3 F_{A(t)} + *_3 \frac{\partial A(t)}{\partial t}$$

If $\frac{\partial A(t)}{\partial t} = - *_3 F_{A(t)}$, then

$$= -\left(dt \wedge \frac{\partial A(t)}{\partial t} + F_{A(t)}\right) = -F_{\mathbb{A}}$$

i.e. negative gradient flow equation
equivalent to the anti-self-dual (ASD) equation
in the temperal gauge

$$F_{A_0+a} = F_{A_0} + d_{A_0} a + \frac{1}{2}[a \wedge a]$$

$F_A^+ = F_A + * F_A$ $\quad F^+_{A_0+a} = F^+_{A_0} + d^+_{A_0} a + \frac{1}{2}[a\wedge a]^+$

linearization $d_A^+ a = 0$

gauge fixing $d_A^* a = 0$

$d_A^+ \oplus d_A^*$ elliptic operator while d_A^+ isn't.

Consider X closed 4-mfd or especially $X = S^1 \times Y$

$\mathcal{M}(P) = \{ \text{conn } A \text{ on } P \mid F_A^+ = 0 \} / \mathcal{G}(P)$

solutions are called <u>instantons</u>

reducible — stabilizer is nontrivial (S^1 or $SU(2)$)

irreducible — stabilizer is trivial ($\{\pm 1\}$) — in $SU(2)$ case

$SO(3) = SU(2)/\{\pm 1\}$

- After Banach completion and small perturbation,
  we have smooth str near irreducible sol
  stratification near reducible sol
  (KM used admissible condition to rule out reducibles)
- dim of moduli space computed by index of Fredholm operator

- Compactness result by Uhlenbeck, in short, some bubbles phenomenon appears over dim 8, compact resut for dim ≤ 7

  in Floer homology, only use dim 0, 1, 2

  in sutured instanton, use cohomology class $\mu(R)$ $\mu(pt)$

  evaluate on dim 2 3 4 5

  for independence of choice

For general 3-mfd, Kronheimer–Mrowka introduces admissible pair (Y, w), w 1-cycle where exists surface $\Sigma \subset Y$ s.t. $w \cdot \Sigma$ is odd (In particular $b_1(Y) > 0$)

Recall that $SO(3)$ bundles P over Y are classified by

$$w_2(P) = PD[w] \in H^2(Y; \mathbb{Z}/2)$$

$CI^w(Y)$: generator critical pts of $CS \Leftrightarrow$ flat conn (over $\mathbb{Z}$)

differential : for two generators x, y

Define $$M(x,y) = \left\{ A(t) \text{ on } \mathbb{R}_t \times Y \;\; \begin{array}{l} \lim_{t \to -\infty} A(t) = x \\ \lim_{t \to +\infty} A(t) = y \end{array} \right\} \Big/ \text{gauge}$$

$\check{M}_1(x,y)$ compactification of 1d part, has $\mathbb{R}$ translation action

$$dx = \sum_y \left(\# \check{M}_1(x,y)/\mathbb{R}\right) y$$

$\langle d^2 x, z\rangle = 0$ from $\check{M}_2(x,z)/\mathbb{R}$

which is an oriented 1-mfd with boundary corresponding $\langle d^2 x, z\rangle$

Moreover, there is a relative grading $gr(x,y) \in \mathbb{Z}/8\mathbb{Z}$

s.t. $gr(x,y) = \dim M(x,y) \mod 8$

Cobordism map for $W: Y_0 \to Y_1$ $U: w_0 \to w_1$

again construct $W^* = W \cup (-\infty, 0] \times Y_0 \cup Y_1 \times [0,\infty)$

Consider ASD equ $F_A^+ = 0$ on W^* with limits

$CI^U(W): CI^{w}(Y_0) \to CI^{w}(Y_1)$ is defined by

counting 0-dim moduli space

It is a chain map from counting boundary of

1-dim moduli space

Since the choices of extra data forms

a contractible space, the naturality result of

$CI^w(Y)$ (hence $I^w(Y) = H(CI^w(Y))$ instanton homology)

follows from the cobordism map for $W = Y \times I$

with different data on $Y \times \{\pm 1\}$

Floer's original construction: $Y = \mathbb{Z}HS^3$

all reducibles are abelian, factor through

$$\begin{tikzcd} \pi_1(Y) \arrow[r] \arrow[dr] & SU(2) \\ & H_1(Y) = \{0\} \arrow[u] \end{tikzcd}$$

Throw away the unique reducible θ for construction

(In equivariant instanton theory, needs to use θ)

Sometimes, consider equivalent setting using vector bundles

- E a $U(2)$ vector bundle over Y
- $\Lambda^2 E \cong$ a fixed line bundle $\mathcal{L}$.
- $c_1(\mathcal{L}) = PD[w]$ evaluates odd on some Σ
- Fix A_0 on $\mathcal{L}$

$\mathcal{A}(E) = \{$ conn A on E s.t. A induces A_0 on $\Lambda^2 E\}$

$\mathcal{G}(E) = \{$ gauge transformation fixing $\mathcal{L}\}$

Another way to understand w:

If A is a flat conn on Y, we have $\rho = \rho_A$ s.t.

$$\pi_1(Y) \xrightarrow{\rho} SO(3)$$

Consider the map induced by inclusion $\pi_1(Y\setminus w) \to \pi_1(Y)$

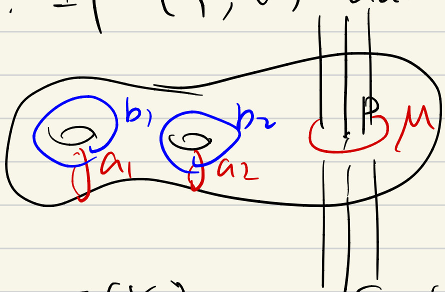
there is no lift, but a lift

$$\begin{tikzcd} & SU(2) \arrow[d] \\ \pi_1(Y) \arrow[r, "\rho"] \arrow[ur, dashed] & SO(3) \end{tikzcd} \qquad \begin{tikzcd} & & SU(2) \arrow[d] \\ \pi_1(Y\setminus w) \arrow[r] \arrow[urr, dashed, "\tilde{\rho}'"] \arrow[rr, bend right, "\rho'"'] & \pi_1(Y) \arrow[r] & SO(3) \end{tikzcd}$$

$w_2(\rho) = PD[w] \iff \tilde{\rho}'([\mu_w]) = -1$ for meridians μ_w of w

Lem. If (Y, w) admissible, then no reducible flat conn on Σ,

Pf



$\mu = [a_1, b_1]\cdot[a_2, b_2] \in \pi_1(\Sigma)$
commutator

$\rho : \pi_1(Y) \to SO(3)$ reducible (i.e. ρ not surjective)

$\rho' : \pi_1(\Sigma\setminus p) \to SO(3)$ red

$\tilde{\rho}' : \pi_1(\Sigma\setminus p) \to SU(2)$ red $\iff \operatorname{Im}\tilde{\rho}'$ abelian

$-1 = \tilde{\rho}'(\mu) = \tilde{\rho}'([a_1, b_1]\cdot[a_2, b_2]) = 1$

Contradiction! $\square$

Class 15 sutured instanton homology.

Last time, we construct $I^w(Y)$ for admissible pair (Y,w) (i.e. $\exists\, \Sigma$ s.t. $w\cdot\Sigma$ odd)

This time, we will construct the μ-action

$$\mu: H_*(Y) \longrightarrow \mathrm{End}(I^w(Y))$$

which is an analogy of cup product for usual homology.

First, consider 4-mfd W (we take $W=\mathbb{R}\times Y$ later) with bundle P. $A^*(P)\subset A(P)$ irreducible conn

$$B^*(P) = A^*(P)/\mathcal{G}(P)$$

universal $SO(3)$ bundle $\mathbb{P} = A^*\times P/\mathcal{G}$ over $B^*\times W$

we have $p_1(\mathbb{P})\in H^4(B^*\times W)$ ponjuyakin class

$$\tilde{\mu}: H_k(W) \longrightarrow H^{4-k}(B^*) \quad \text{slash product}$$
$$\alpha \longmapsto p_1(\mathbb{P})/\alpha$$

Prop: Write $M_k^*(P)\subseteq B^*(P)$ for k-dim part of the ASD solutions. For any $\Sigma_1,\dots,\Sigma_n\subset W$, we can find $V_{\Sigma_i}\subseteq B^*(P)$ codim 2 s.t. for $2n\le k$ the intersection $M_k^*\cap V_{\Sigma_1}\cap\cdots\cap V_{\Sigma_n}$ is transreverse and dual to

$\tilde{\mu}(\Sigma_1)\cup\cdots\cup\tilde{\mu}(\Sigma_n)$ on M_k^* (used in Donaldson polynomial)

In $W = \mathbb{R}\times Y$ case, for $\alpha \in H_k(Y) \cong H_k(\mathbb{R}\times Y)$
find V_α codim $4-k$ whose PD dual represents
$\tilde{\mu}(\alpha) \in H^{4-k}(\mathcal{B}^*(p))$

Define $\mu(\alpha): CI^w(Y) \longrightarrow CI^w(Y)$

$$x \longmapsto \sum_y \#(\underbrace{\check{M}_k(x,y)}_{\text{Compactification}} \cap V_\alpha)\, y$$

Counting $\partial \check{M}_{k+1}(x,y) \cap V_\alpha \rightsquigarrow \mu(\alpha)$ is a chain map

Prop $\mu(\alpha)\mu(\beta) = (-1)^{\deg\alpha \deg\beta}\mu(\beta)\mu(\alpha)$ on $I^w(Y)$
(not hold on $CI^w(Y)$ over $\mathbb{Z}$ similar to cup product) $\square$

Recall in sutured monopole, we find closure (Y,R)
for balanced (M,γ) and define

$$SHM(M,\gamma) = HM(Y|R) = \bigoplus_{\langle c_1(\mathfrak{s}),[R]\rangle = 2g(R)-2} \widecheck{HM}(Y,\mathfrak{s})$$

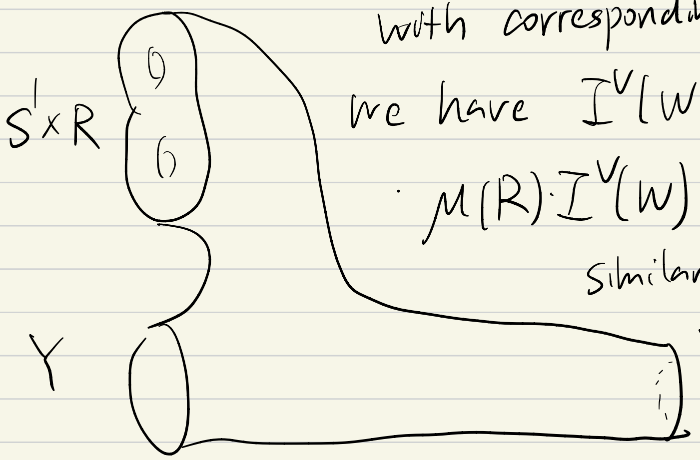
To define sutured instanton $SHI(M,\gamma)$,
we need to find analogy of $c_1(\mathfrak{s})$
KM's idea comes from computation of $Y = S^1\times F$ $w = S^1\times pt$
Prop (Muñoz) The simultanuously generalized eigenspace
of $\mu(F)$ (deg 2) and $\mu(pt)$ (deg 4) (over $\mathbb{C}$) lies in
(*) $\{((\sqrt{-1})^r \cdot 2k, (-1)^r\cdot 2) \mid r\in\mathbb{Z},\ 0\le k\le g(F)-1\}$ when $g(F)>0$,
Moreover, $(2g-2, 2)$-eigenspace is 1-dimensional.

(adjunction formula)

Cor: For closed surface $R \subset Y$ with $w \cdot R$ odd, the eigenvalues of $\mu(R)$, $\mu(pt)$ lies in $(*)$ by replacing $g(F)$ by $g(R)$

Pf: Consider a slice $R \times \{t_0\} \subset Y \times I$, removing its nbhd, we obtain $W: S^1 \times R \sqcup Y \to Y$ with corresponding $V = w \times I \setminus N(R)$

from odd pairing and $w \in H_1(Y; \mathbb{Z}/2)$



We have $I^V(W): I^{S^1}(S^1 \times R) \otimes I^w(Y) \to I^w(Y)$ surj

$$\mu(R) \cdot I^V(W)(a \otimes b) = I^V(W)(\mu(R)a \otimes b)$$

similar for $\mu(pt)$ and $\mu(R)^2 + \mu(pt)$

Hence eigenvalues are subsets $\square$

Def. $I^w(Y|R) = (2g(R)-2, 2)$-eigenspace of $(\mu(R), \mu(pt))$ on $I^w(Y)$ over $\mathbb{C}$

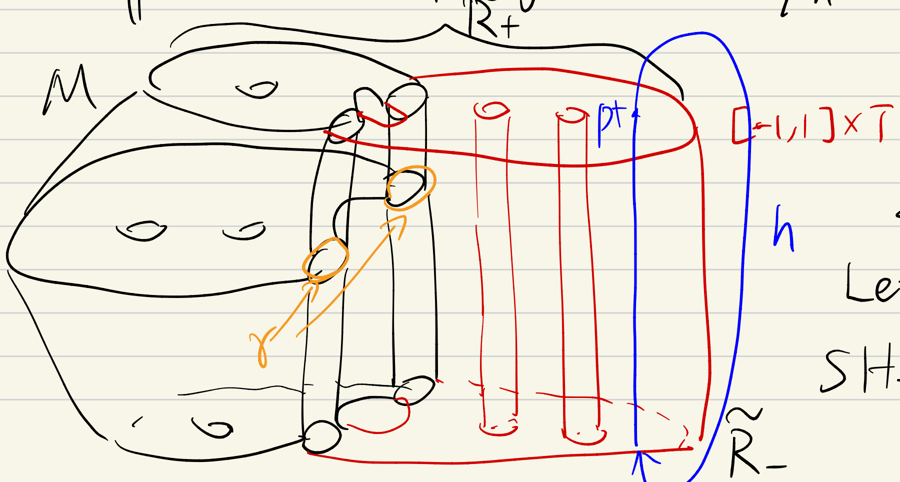
Ex. $I^{S^1}(S^1 \times F | F) \cong \mathbb{C}$ by Muñoz computation

Recall for balanced (M, γ), we construct preclosure $\tilde{M}$

$$\tilde{M} = M \cup_f [-1,1] \times T \qquad f(\partial T) \xrightarrow{\cong} \gamma$$

$$\partial \tilde{M} = \tilde{R}_+ \cup \tilde{R}_- \qquad \tilde{R}_\pm = R_\pm(\gamma) \cup \{\pm 1\} \times T$$

A diffeo $h: \tilde{R}_+ \to \tilde{R}_-$ gives $Y = \tilde{M}/h$, $R =$ image of $R_\pm$



Now furthermore choose $pt \in T$, choose h s.t. $h(\{+1\} \times pt) = \{-1\} \times pt$

Let $w =$ image of $I \times pt$

$$SHI(M, \gamma) = I^w(Y|R)$$

Class 16 More on Floer's excision

- Y closed oriented 3-mfd 1 or 2 components
- $\Sigma_1, \Sigma_2 \subset Y$ connected surfaces of equal genus

If Y is connected, $[\Sigma_1], [\Sigma_2]$ independent

If Y is disconn, Σ_i nonseparating in Y_i

$([\Sigma_i] \neq 0)$

Write $\Sigma = \Sigma_1 \cup \Sigma_2$

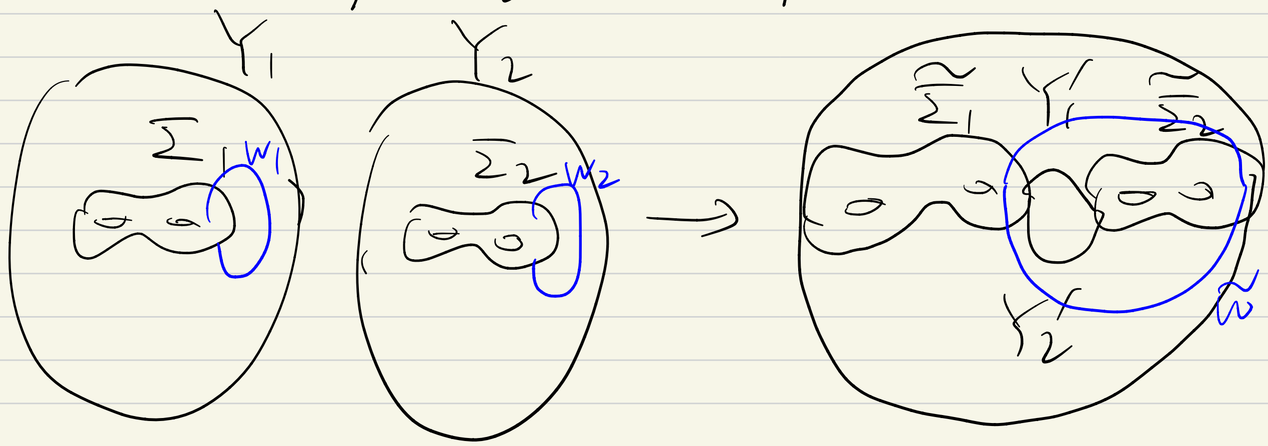
- Fix ori-rev diffeo $h: \Sigma_1 \to \Sigma_2$
- (new) 1-cycles w_1, w_2 s.t. $w_i \cdot \Sigma_i$ equal and odd

Construct $\tilde{Y}$ by gluing $Y' = Y \setminus N(\Sigma)$

along $\Sigma_1 \xrightarrow{h} -\Sigma_2 \quad \Sigma_2 \xrightarrow{h^{-1}} -\Sigma_1$

obtain $\tilde{\Sigma}$ by image. similarly $\tilde{w}$ from $w = w_1 \cup w_2$

Ex



Thm (Floer, Kronheimer–Mrowka)

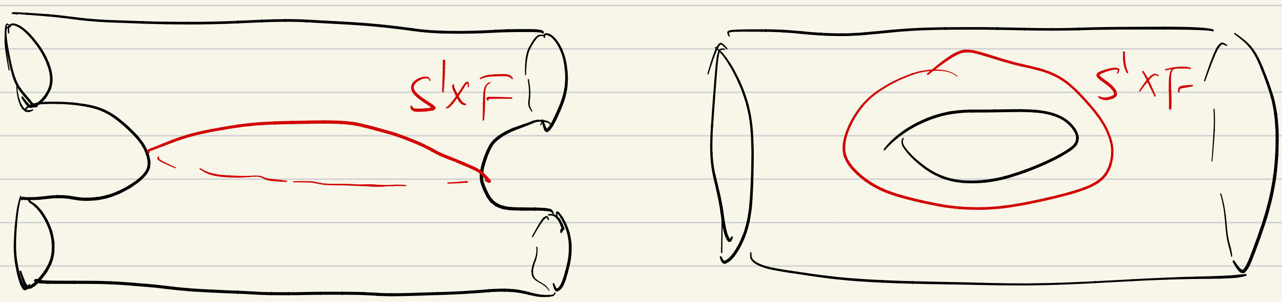
If $g(\Sigma_i) \geq 1$ (≥ 2 in HM), then there exist cobordism maps

$$I^w(Y|\Sigma) \rightleftarrows I^{\tilde{w}}(\tilde{Y}|\tilde{\Sigma}) \text{ that are iso}$$

The reason now we can assume $g(\Sigma_i)=1$ is because of the computation on $I^{S^1}(T^3|T^2)\cong\mathbb{C}$

(In HM, need to use local coefficient)

The proof is similar to HM $\square$



Prop $Y=S^1\times_h F$ mapping tori f fixed a pt

$F=pt\times F$ $\omega=[-1,1]\times pt/h$. $I^{\omega}(Y|F)\cong\mathbb{C}$

Pf: Excision for $S^1\times_h F$ and $S^1\times_{h^{-1}} F$

Cor. $SHI(M,\gamma)$ indep of h $\square$

Prop $Y=S^1\times F$ α any curve on $F=pt\times F$

Then $I^{S^1\cup\alpha}(Y|F)\cong\mathbb{C}$

Pf: By excision along F, we have

$$I^{S^1\cup\alpha}(Y|F)\otimes I^{S^1\cup\alpha}(Y|F)\cong I^{S^1\cup 2\alpha}(Y|F)$$
$$\cong I^{S^1}(Y|F)\cong\mathbb{C}\ \square$$

Cor For $R\subset Y$, curve $\alpha\subset R$, we have

$$I^{\omega\cup\alpha}(Y|R)\cong I^{\omega}(Y|R)$$

another excision to show SHI indep of $g(T)$.

Excision: $\Sigma_i \subset Y_i$ surface ($g \geq 2$ in HM)

$T_i \subset Y_i$ tori $w_i \subset Y_i$ 1-cycles (no w in HM)

$T_i \cap \Sigma_i = C_i$ $|T_i \cap w_i| = 1$

$h: T_1 \to T_2$ $h(C_1) = C_2$ $h(T_1 \cap w_1) = T_2 \cap w_2$

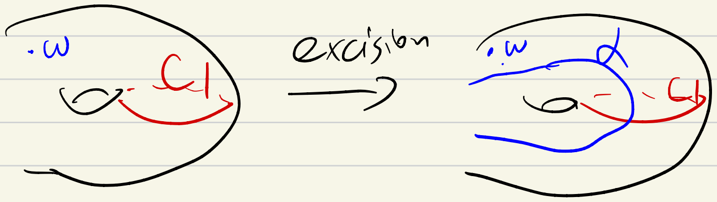
T_i cut open Y_i, Σ_i and reglue by h

Thm $I^w(Y|\Sigma) \cong I^{\tilde{w}}(\tilde{Y}|\tilde{\Sigma})$ by cobordism maps

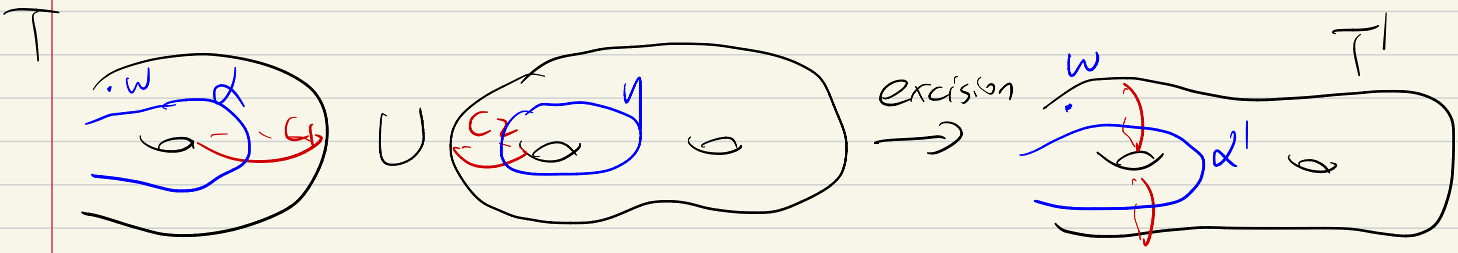
Pf: Similar to the first excision thm, and use the fact that $I^{S^1}(T^3) \cong \mathbb{C} \oplus \mathbb{C}$, in eigenspace of eigenvalues $(0,+2), (0,-2)$ for $(\mu(T^2), \mu(pt))$

Rem. In HM case, there is a version of such excision in local coefficient due to computation of $HM(T^3; \Gamma_\eta)$ local coeff.

Pf of $SHI(M,\gamma)$ indep of $g(T)$ (iso result by Kronheimer-Mrowka (Naturality result by Baldwin-Sivek))



$g(T') = g(T)+1$



$$I^{w\cup\alpha}(Y|R) \otimes I^{\eta}(S^1\times F_2 | F_2) \xrightarrow{\cong} I^{w\cup\alpha}(Y'|R')$$

$g(R') = g(R)+1$

We need to prove $I^{\eta}(S^1\times F_2|F_2) \cong \mathbb{C}$ (no w)

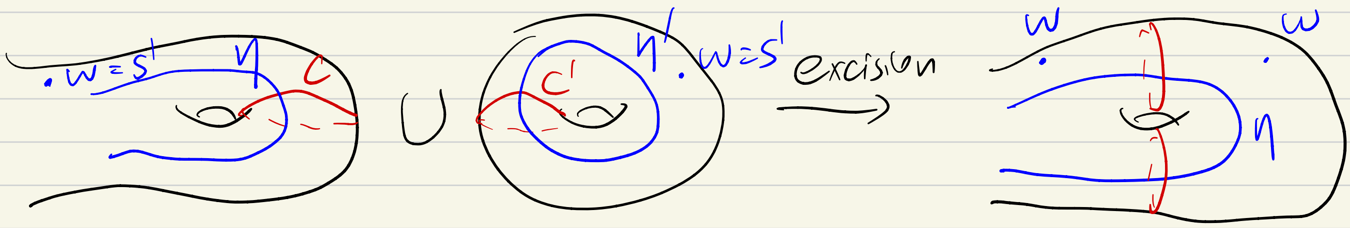
Prop: $Y = S^1\times F$, η a curve on $F = pt\times F$

s.t. $\exists$ another curve c, $|\eta \cap c| = 1$ Then $I^{\eta}(S^1\times F|F) \cong \mathbb{C}$

(I^{η} is well-defined because $|\eta \cdot S^1\times \delta| = 1$)

More generally, $(\lambda, 2)$-eigen of $I^{S^1\cup\eta}(S^1\times F)$ and $I^{\eta}(S^1\times F)$ are isomorphic

Pf: Excision



$$I^{S^1\cup\eta}(S^1\times F|F) \otimes I^{S^1\cup\eta'}(S^1\times T^2|T^2) \xrightarrow{\cong} I^{2S^1\cup\eta}(S^1\times F|F)$$

$$\begin{array}{ccc} I^{S^1\cup\eta}(S^1\times F|F) & I^{S^1\cup\eta'}(S^1\times T^2|T^2) & I^{2S^1\cup\eta}(S^1\times F|F) \\ \text{\rotatebox{90}{$\cong$}} & \text{\rotatebox{90}{$\cong$}} & \text{\rotatebox{90}{$\cong$}} \\ \mathbb{C} & \mathbb{C} & I^{\eta}(S^1\times F|F) \end{array}$$

The general eigen result follows from excision on $+2$-eigenspace of $\mu(pt)$

Class 17 gradings on SHI (and similar for SHM)

Thm (KM) If M is irreducible, then (M,γ) is taut iff $SHI(M,\gamma)\neq 0$.

If M is a homology product, then (M,γ) is product iff $SHI(M,\gamma)\cong \mathbb{C}$ (omit pf)

Sketch (not taut $\Rightarrow SHI=0$).

If $R_\pm(\gamma)$ compressible, then R is compressible

$\Rightarrow \exists R'\subseteq Y \quad [R']=[R] \quad 1\leq g(R')<g(R)$

$\Rightarrow \mu(R)=\mu(R') \quad I^w(Y|R)=0$ because eigenvalues of $\mu(R')$ on $+2$ eigenspace of $\mu(pt)$ are contained in $\{\pm 2,\cdots,\pm 2g(R')-2\}$

Similar proof for $R_\pm(\gamma)$ not norm minimizing

(taut $\Rightarrow SHI\neq 0$)

taut $\Rightarrow$ hierarchy $(M,\gamma)\overset{S_1}{\rightsquigarrow}\cdots\overset{S_n}{\rightsquigarrow}(M_n,\gamma_n)=\coprod(B^3,S^1)$

$(S^1\times F, F, S^1)$ is a closure of (M_n,γ_n)

$SHI(M_n,\gamma_n)\cong\mathbb{C}$ (also used for product $\Rightarrow SHI\cong\mathbb{C}$)

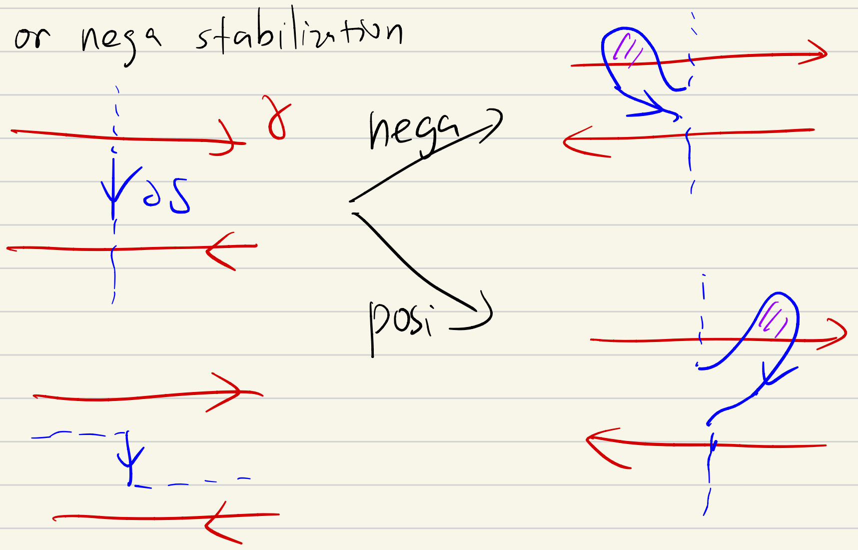
We need to show $SHI(M_{i+1},\gamma_{i+1})$ is a direct summand of $SHI(M_i,\gamma_i)$ for surface decomposition S_i

Indeed we will define $\mathbb{Z}$-gr for $S\subset(M,\gamma)$

$$SHI(M,\gamma)=\bigoplus_{j\in\mathbb{Z}\text{ or }\mathbb{Z}+\frac{1}{2}} SHI(M,\gamma,S,j)$$

$$SHI(M_{i+1},\gamma_{i+1})\cong SHI(M_i,\gamma_i,S_i,\tfrac{1}{2}(\tfrac{1}{2}|\partial S\cap\gamma|-\chi(S))$$

We can assume $\frac{1}{2}|\partial S \cap \gamma| - \chi(S)$ is even, if not, we perturb S by posi or nega stabilization



∂S intersects $R_\pm(\gamma)$ at circles and arcs. $\partial S_\pm$

We find arcs α on T which are linearly indep in homology

Suppose $\widetilde{\partial S_\pm} = \partial S_\pm \cup \{\pm 1\} \times \alpha$ are circles with the same number

Choose h so that $h(\widetilde{\partial S_+}) = \widetilde{\partial S_+}$.

Define $\bar{S}$ = image of $\tilde{S} = S \cup [-1,1] \times \alpha$ in Y

$\chi(\bar{S}) = \chi(\tilde{S}) = \chi(S) - \frac{1}{2}|\partial S \cap \gamma|$ even so h exists

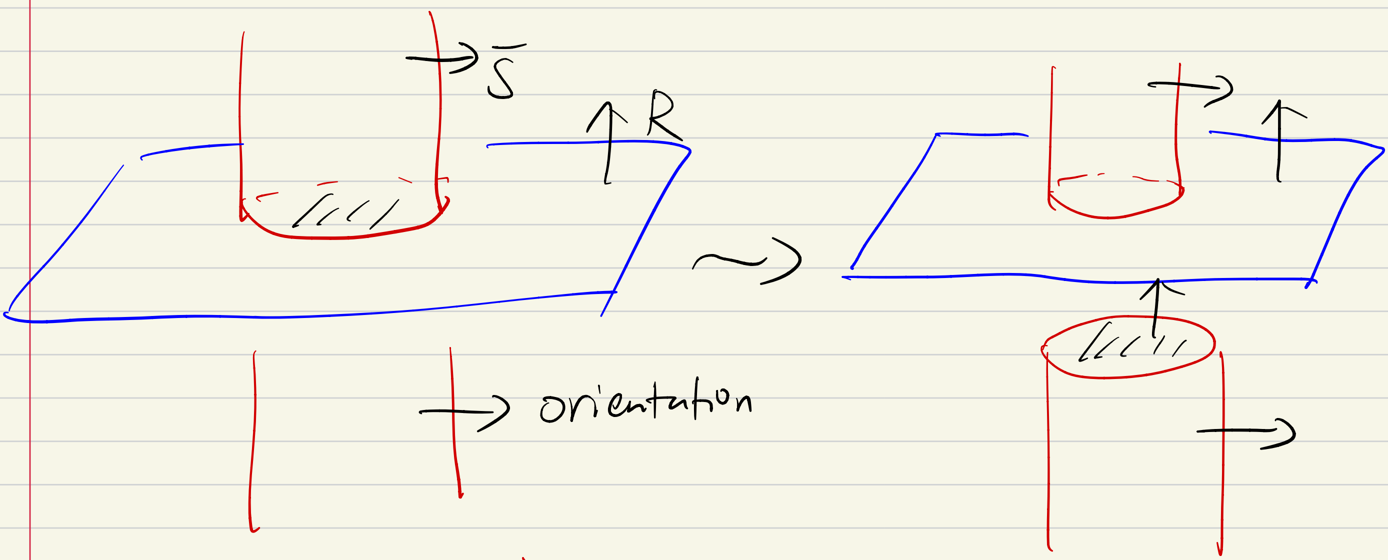
$SHI(M, \gamma, S, j) = (2j, 2g(R)-2, 2)$-eigenspace of

$\mu(\bar{S}), \mu(R), \mu(pt)$ on $I^{w \cup \alpha}(Y)$

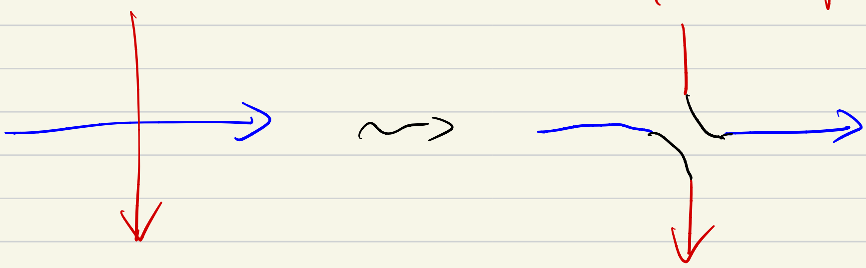
Thm (Zhenkun Li) The grading is independent of all choices (pairing of endpts, arcs, gluing diffeo h)

Rough idea about surface decomposition:

Consider cut and paste operation on $\bar{S}$ and R



1d analog



We write F for the resulting surface

$\chi(F) = \chi(\bar{S}) + \chi(R) \quad [F] = [\bar{S}] + [R]$

(Y, F, w) is a closure of (M', γ') after decomposition

$$\begin{array}{ccc} I^w(Y|F) & = I^w(Y|\bar{S}) \cap & I^w(Y|R) \\ \| & & \cap\!\!| \\ SHI(M', \gamma') & & SHI(M, \gamma) \end{array}$$