

Class 13 ASD equation

Instanton theory is another gauge theory

Instead of SW eqn (3d, 4d) we consider

flat conn $\bar{F}_A = 0$ for 3d and ASD eqn $\bar{F}_A^\dagger = 0$ for 4d
both are special cases of Yang-Mills eqn.

Similarly, we will consider configuration space, gauge group

0, 1, 2-dim (compact) moduli space to define Floer homology.

We will use similar construction for sutured instanton homology

For closed 3-mfd Y and cobordism $W: Y_0 \rightarrow Y_1$ or closed 4-manifold X ,
we consider principal $SO(3)$ bundle P

(originally, Floer consider $SU(2)$ bundle on integral homology sphere Y)

Later, Donaldson, KM also consider $SO(3)$, $U(2)$, $SO(3)$ - $SU(2)$ mixed theories

Lem. isomorphism class of $SO(3)$ bundle P over Y

(Facts) is determined by second Stiefel-Whitney class $w_2(P) \in H^2(Y; \mathbb{Z}/2)$

similar for W

But for X , we also consider Pontryagin class $p_1 \in H^4(X; \mathbb{Z})$

i.e. $SO(3)$ P determined by (w_2, p)

$SU(2)$ bundles are determined by second Chern class $c_2 \in H^4(X; \mathbb{Z})$

(first Chern class always vanishes) $\langle c_2, [X] \rangle$ called instanton num

Any $SU(2)$ bundle over Y is trivial.

$A(X, P) = \text{space of conns affine over } \Omega^1(X, \text{ad} P) = \Gamma(T^*X \otimes \text{ad} P)$

$\text{ad} P = \text{associated vector bundle for adjoint rep } P \times_{\text{ad}} \mathfrak{g} \text{ over } X$

$\mathfrak{g} = \text{Lie algebra of } G = \text{SO}(3) \text{ (or } \text{SU}(2))$

$$G \times \mathfrak{g} \rightarrow \mathfrak{g}$$

$$(g, x) \mapsto g^{-1} x g$$

$G(X, P) = \text{space of gauge transformation}$

$= \{ \text{bundle automorphisms of } P \text{ that are } G\text{-equiv} \}$

$$\eta: P \rightarrow P \quad \eta \in G(X, P) \iff \tilde{\eta}: P \rightarrow G \text{ equiv for conj on } G$$

$$\text{For } p \in P \quad \eta(p) = p g \quad \tilde{\eta}(p) = g$$

$$\eta(p g) = p g h \quad \tilde{\eta}(p \cdot g) = h$$

$$G\text{-equiv} \xrightarrow{\quad} \eta(p) g$$

$$\Rightarrow \eta(p) = p g h g^{-1} \Rightarrow \tilde{\eta}(p g) \stackrel{\text{conj equivariant}}{=} g^{-1} \tilde{\eta}(p) g$$

$$\tilde{\eta}: P \rightarrow G \text{ equiv for conj on } G \iff u: X \rightarrow \text{Ad} P = P \times_{\text{Ad}} G$$

$$\text{Ad}: G \times G \rightarrow G$$

$$(g, h) \mapsto g^{-1} h g$$

$$x = \pi(p) \quad u(x) = [(p, \tilde{\eta}(p))] = [(p \cdot g, g^{-1} \tilde{\eta}(p) g)] \text{ fiber iso to } G$$

Conclusion $G(X, P) = \Gamma(\text{Ad} P)$ Lie group

its Lie algebra is $\Gamma(\text{ad} P)$

Action of gauge transformation

$$u(A) = A - (d_A u) u^{-1}$$

$$B(X, P) = A(X, P) / G(X, P) \quad \text{action may not free}$$

We can also take $G_0(X, P) \subset G(X, P)$ with fixed value at y_0
called based gauge group.

$$B_0(X, P) = A(X, P) / G_0(X, P)$$

We replace X by Y, W .

Chern-Simons functional: Fix conn A_0 , for $\mathfrak{g}$ -valued 1-form a
usually $\frac{1}{8\pi^2} \int_Y \text{tr}(d_{A_0} a \wedge a + \frac{2}{3} a \wedge a \wedge a)$

$$CS(a) = \int_Y \text{tr} \left(a \wedge \left(\bar{F}_{A_0} + \frac{1}{2} d_{A_0} a + \frac{1}{6} [a \wedge a] \right) \right)$$

Similar to CSD functional, we find formal gradient as follows
(critical pts)

$$D(CS_a)(b) = \lim_{t \rightarrow 0} \frac{CS(a+tb) - CS(a)}{t}$$

$$= \int_Y \text{tr} \left(b \wedge \left(\bar{F}_{A_0} + \frac{1}{2} d_{A_0} a + \frac{1}{6} [a \wedge a] \right) + a \wedge \left(\frac{1}{2} d_{A_0} b \right) + a \wedge \left(\frac{1}{3} [a \wedge b] \right) \right)$$

$$\text{Let } A = A_0 + a \quad \bar{F}_A = \bar{F}_{A_0} + d_{A_0} a + \frac{1}{2} [a \wedge a]$$

$$d_A b = d_{A_0} b + \frac{1}{2} [a \wedge b] \quad d_A a = d_{A_0} a + \frac{1}{2} [a \wedge a]$$

$$\rightarrow = \int_Y \text{tr} \left(b \wedge \bar{F}_A - \frac{1}{2} b \wedge d_{A_0} a - \frac{1}{3} [a \wedge a] \right. \\ \left. + a \wedge \left(\frac{1}{2} d_{A_0} b \right) + a \wedge \frac{1}{3} [a \wedge b] \right)$$

$$= \int_Y \text{tr} \left(b \wedge \bar{F}_A + \frac{1}{2} (a \wedge d_A b + d_A a \wedge b) \right)$$

$$\parallel \\ \int_Y \frac{1}{2} d(\text{tr}(a \wedge b)) = 0$$

$$= \int_Y \text{tr}(b \wedge \bar{F}_A)$$

$$b \text{ arbitrary} \Rightarrow \bar{F}_A = 0 \quad A \text{ flat}$$

$$\text{Let } \langle a, b \rangle_{L_2} = \int_Y \text{tr}(a \wedge *_3 b) = \int_Y \text{tr}(b \wedge *_3 a)$$

$$\langle \text{grad}(CS)_A, b \rangle_{L^2} = D(CS_A)(b)$$

$$\Rightarrow \text{grad}(CS)_A = *_{\bar{3}} \bar{F}_A \quad \text{because } *_{\bar{3}}^2 = 1$$

$*_{\bar{3}}$ 3-dim Hodge star. (depending on the metric)

In general, $*$: $\Omega^k \rightarrow \Omega^{n-k}$ $n = \dim$ of mfd M

$e^1, \dots, e^n$ orthonormal basis of $T^*M = \Omega^1$

$$d\text{vol}_M = e^1 \wedge \dots \wedge e^n$$

$*\alpha$ satisfies $\alpha \wedge *\alpha = d\text{vol}_M$

$$\text{Ex. } *(e^1 \wedge \dots \wedge e^k) = e^{k+1} \wedge \dots \wedge e^n$$

$$4\text{d case } *^2 = 1: \Omega^2 \rightarrow \Omega^2$$

$$\text{For } w \in \Omega^2 \quad w^\pm = \frac{w \pm *w}{2} \quad \pm 1\text{-eigenspace of } *$$

on $\mathbb{R}_t \times Y = W$ with product metric

$$dt \wedge d\text{vol}_Y = d\text{vol}_M$$

$$w \in \Omega^2(Y) \quad *_4 w = dt \wedge *_3 w$$

$$\phi \in \Omega^1(Y) \quad *_4(dt \wedge \phi) = *_3 \phi$$

$$4\text{d conn } A = A_0 dt + \sum_{i=1}^3 A_i dy_i \quad y_1, y_2, y_3 \text{ on } Y$$

We can fix a gauge (by transportation along $\mathbb{R}$ -factor)

s.t. $A_0 = 0$ (called temporal gauge) $A = A(t) = (0, A_i(t))$

$$F_A^4 = F_{A(t)}^3 + dt \wedge \frac{\partial A(t)}{\partial t}$$

on $\mathbb{R} \times Y$

on Y

from local form $F = da + \frac{1}{2}[a, a]$
 $da = dt \wedge \frac{\partial a}{\partial t} + dy_i \wedge \frac{\partial a}{\partial y_i}$

$$\begin{aligned}
 \star_4 \bar{F}_A &= \star_4 \bar{F}_{A(t)} + \star_4 \left(dt \wedge \frac{\partial A(t)}{\partial t} \right) \\
 &= dt \wedge \star_3 \bar{F}_{A(t)} + \star \frac{\partial A(t)}{\partial t}
 \end{aligned}$$

$\swarrow$ 2-form $\swarrow$ 1-form

If $\frac{\partial A(t)}{\partial t} = -\star_3 \bar{F}_{A(t)}$, then

$$\Rightarrow = - \left(dt \wedge \frac{\partial A(t)}{\partial t} + \bar{F}_A \right) = -\bar{F}_A$$

i.e. negative gradient flow equation

equivalent to the anti-self-dual (ASD) equation
in the temporal gauge

On the other hand, we compute $CS(uA_0) - CS(A_0)$

Suppose $uA_0 - A_0 = a$ $\bar{F}_A^4 = \bar{F}_{A_0+t a}^3 + dt \wedge a$

Then

$$CS(a) = \int_Y \text{tr} \left(a \wedge \left(\bar{F}_{A_0} + \frac{1}{2} d_{A_0} a + \frac{1}{6} [a, a] \right) \right)$$

$$= \int_Y \text{tr} \int_0^1 a \wedge \left(\bar{F}_{A_0} + t d_{A_0} a + \frac{1}{2} t^2 [a, a] \right) dt$$

$$= \int_0^1 \int_Y \text{tr} (a \wedge \bar{F}_{A_0+t a}) dt$$

$$= \int_{[0,1] \times Y} \text{tr} (dt \wedge a \wedge \bar{F}_{A_0+t a})$$

$$= \int_{S^1 \times Y} \text{tr} (\bar{F}_A^4 \wedge \bar{F}_A^4)$$

$$= 8\pi^2 C_2(\mathbb{P}) \quad \text{by Chern-Weil formula}$$

$\mathbb{P}$ bundle over $S^1 \times Y$ by gluing uA_0, A_0

$$CS: A(Y, P)/G(Y, P) \rightarrow \mathbb{R}/8\pi^2 \mathbb{Z}$$

Class 13 instanton Floer homology

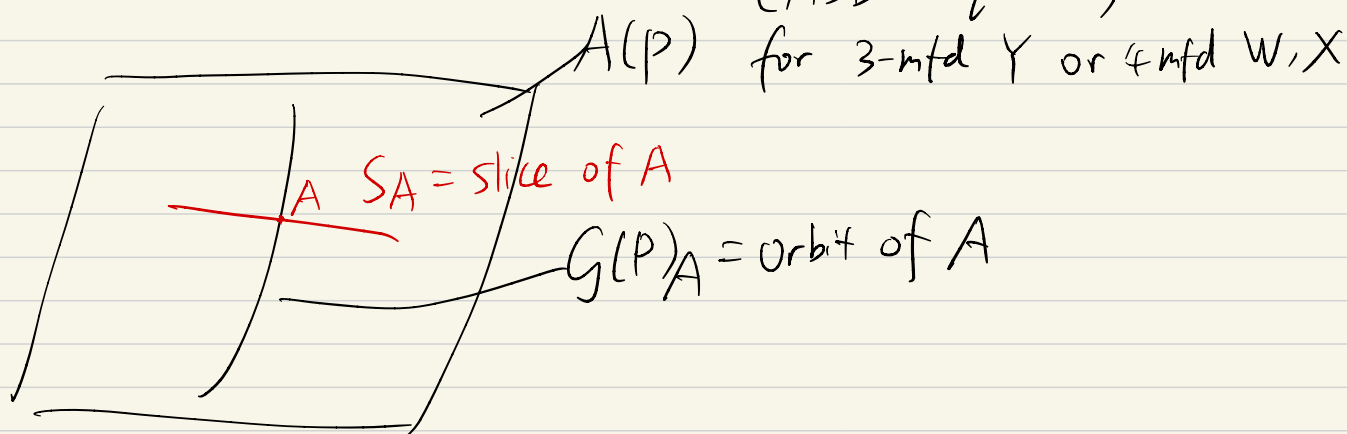
Last time, we introduce Chern-Simons functional for fixed A_0

$$CS(a) = \int_Y \text{tr} \left(a \left(F_{A_0} + \frac{1}{2} d_{A_0} a + \frac{1}{6} [a, a] \right) \right)$$

regarded as $B(Y, P) = A(Y, P) / G(Y, P) \rightarrow \mathbb{R} / 8\pi^2$

$\text{grad}(CS)_A = *F_A$, negative flow line

$$\frac{\partial A(t)}{\partial t} = -*F_A(t) \text{ equivalent to } F_A^+ = 0 \text{ in temporal gauge (ASD-equation)}$$



Action $G(P) \times A(P) \rightarrow A(P)$
 $(u, A) \mapsto A - (d_A u) u^{-1}$
 $G \times A(P) \rightarrow A(P)$

differential $(\zeta, A) \mapsto -d_A \zeta$

$$S_A = \{ a \in \Omega^1(\text{ad } P) \mid \langle d_A \zeta, a \rangle = 0 \quad \forall \zeta \in G \}$$

$$\langle \zeta, d_A^* a \rangle = 0 \quad d_A^* = * d_A *$$

$$= \{ a \in \Omega^1(\text{ad } P), d_A^* a = 0, *a|_{\partial} = 0 \}$$

↑
boundary condition

$$F_{A_0+a} = F_{A_0} + d_{A_0} a + \frac{1}{2} [a, a]$$

$$F_A^\dagger = F_A + * \bar{F}_A \quad F_{A_0+a}^\dagger = F_{A_0}^\dagger + d_{A_0}^\dagger a + \frac{1}{2} [a, a]^\dagger$$

linearization $d_A^\dagger a = 0$

gauge fixing $d_A^* a = 0$

$d_A^\dagger \oplus d_A^*$ elliptic operator while $d_A^\dagger$ isn't.

Consider X closed 4-mfd or especially $X = S^1 \times Y$

$$\mathcal{M}(P) = \{ \text{conn } A \text{ on } P \mid F_A^\dagger = 0 \} / G(P)$$

solutions are called instantons

reducible — stabilizer is nontrivial (S^1 or $SU(2)$)

irreducible — stabilizer is trivial (S^1) — in $SU(2)$ case
 $SO(3) = SU(2)/\{\pm 1\}$

- After Banach completion and small perturbation, we have smooth str near irreducible sol
 stratification near reducible sol

(KM used admissible condition to rule out reducibles)

- dim of moduli space computed by index of Fredholm operator
- Compactness result by Uhlenbeck, in short, some bubbles phenomenon appears over dim 8, compact result for $\dim \leq 7$
- in Floer homology, only use dim 0, 1, 2
- in sutured instanton, use cohomology class $\mu(R)$ $\mu(\text{pt})$
 evaluate on dim 2 3 4 5

↑ ↑
 for independence of choice

Floer's original construction: $Y = \mathbb{Z}H S^3$

all reducibles are abelian, factor through

$$\begin{aligned} \pi_1(Y) &\longrightarrow SU(2) \\ &\nearrow \\ &\supset H_1(Y) = \{0\} \end{aligned}$$

Throw away the unique reducible Θ for construction
(In equivariant instanton theory, needs to use Θ)

For general 3-mfd, Kronheimer-Mrowka introduces
admissible pair (Y, w) , w 1-cycle

where exists surface $\Sigma \subset Y$ s.t. $w \cdot \Sigma$ is odd

(In particular $b_1(Y) > 0$)

Recall that $SO(3)$ bundles P over Y are classified by
 $w_2(P) = PD[w] \in H^2(Y; \mathbb{Z}/2)$

Another way to understand w :

If A is a flat conn on Y , we have $p = p_A$ s.t.

$$\pi_1(Y) \xrightarrow{p} SO(3)$$

Consider the map induced by inclusion $\pi_1(Y \setminus w) \rightarrow \pi_1(Y)$

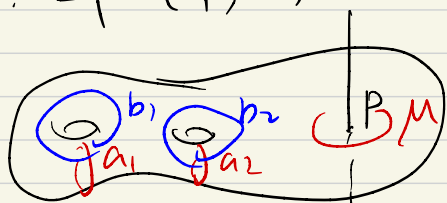
there is no lift, but a lift

$$\begin{array}{ccccc} & & & \tilde{p}' & \longrightarrow SU(2) \\ & \nearrow & & & \downarrow \\ \pi_1(Y) & \xrightarrow{p} & SO(3) & \xleftarrow{\quad} & \pi_1(Y \setminus w) \longrightarrow \pi_1(Y) \longrightarrow SO(3) \end{array}$$

$$w_2(P) = PD[w] \iff \tilde{p}'([\mu_w]) = -1 \text{ for meridians } \mu_w \text{ of } w$$

Lem. If (Y, ω) admissible, then no reducible flat conn

Pf



on Σ ,

$$\mu = [a_1, b_1] \cdot [a_2, b_2] \in \pi_1(\Sigma)$$

commutator

$\rho: \pi_1(Y) \rightarrow SO(3)$ reducible (i.e. ρ not surjective)

$\rho': \pi_1(\Sigma \setminus p) \rightarrow SO(3)$ red

$\tilde{\rho}': \pi_1(\Sigma \setminus p) \rightarrow SU(2)$ red $\Leftrightarrow \text{Im } \tilde{\rho}'$ abelian

$$-1 = \tilde{\rho}'(\mu) = \tilde{\rho}'([a_1, b_1] \cdot [a_2, b_2]) = 1$$

Contradiction! $\square$

$CI^\omega(Y)$: generator critical pts of CS $\Leftrightarrow$ flat conn
(over $\mathbb{Z}$)

differential: for two generators x, y

Define $M(x, y) = \{ A(t) \text{ on } \mathbb{R}_t \times Y \mid \lim_{t \rightarrow -\infty} A(t) = x, \lim_{t \rightarrow +\infty} A(t) = y \} / \text{gauge}$

$\check{M}_1(x, y)$ compactification of $\mathbb{Z}d$ part, has $\mathbb{R}$ translation action

$$dx = \sum_y (\# \check{M}_1(x, y) / \mathbb{R}) y$$

$$\langle d^2 x, z \rangle = 0 \text{ from } \check{M}_2(x, z) / \mathbb{R}$$

which is an oriented 1-mfd with boundary corresponding $\langle d^2 x, z \rangle$

Moreover, there is a relative grading $gr(x, y) \in \mathbb{Z}/8\mathbb{Z}$

$$\text{s.t. } gr(x, y) = \dim M(x, y) \bmod 8$$

Sometimes, consider equivalent setting using vector bundles

- E a $U(2)$ vector bundle over Y
- $\Lambda^2 E \cong$ a fixed line bundle L .
- $C_1(L) = PD[W]$ evaluates odd on some Σ
- Fix A_0 on L

$\mathcal{A}(E) = \{ \text{conn } A \text{ on } E \text{ s.t. } A \text{ induces } A_0 \text{ on } \Lambda^2 E \}$

$\mathcal{G}(E) = \{ \text{gauge transformation fixing } L \}$

Cobordism map for $W: Y_0 \rightarrow Y_1$, $U: W_0 \rightarrow W_1$

again construct $W^* = W \cup (-\infty, 0] \times Y_0 \cup Y_1 \times [0, \infty)$

consider ASD eqn $F_A^+ = 0$ on W^* with limits

$CI^V(W) : CI^W(Y_0) \rightarrow CI^W(Y_1)$ is defined by counting 0-dim moduli space

It is a chain map from counting boundary of 1-dim moduli space

Since the choices of extra data forms

a contractible space, the naturality result of

$CI^W(Y)$ (hence $I^W(Y) = H(CI^W(Y))$ instanton homology)

follows from the cobordism map for $W = Y \times I$ with different data on $Y \times \{\pm 1\}$

Class 14 Sutured instanton homology.

Last time, we construct $I^w(Y)$ for admissible pair (Y, w) (i.e. $\exists \Sigma$ s.t. $w \cdot \Sigma$ odd)

This time, we will construct the μ -action

$$\mu: H_*(Y) \longrightarrow \text{End}(I^w(Y))$$

which is an analogy of cup product for usual homology.

First, consider 4-mfd W (we take $W = \mathbb{R} \times Y$ later) with bundle P . $A^*(P) \subset A(P)$ irreducible conn

$$B^*(P) = A^*(P) / G(P)$$

universal $SO(3)$ bundle $IP = A^* \times P / G$
over $B^* \times W$

we have $p_1(IP) \in H^4(B^* \times W)$ Pontryagin class

$$\tilde{\mu}: H_k(W) \longrightarrow H^{4+k}(B^*) \quad \leftarrow \text{slash product}$$
$$\alpha \longmapsto p_1(IP) / \alpha$$

Prop: Write $M_k^*(P) \in B^*(P)$ for k -dim part of the ASD solutions. For any $\Sigma_1, \dots, \Sigma_n \subset W$, we can find $V_{\Sigma_i} \in B^*(P)$ codim 2 s.t. for $2n \leq k$ the intersection $M_k^* \cap V_{\Sigma_1} \cap \dots \cap V_{\Sigma_n}$ is transverse and dual to

$$\tilde{\mu}(\Sigma_1) \cup \dots \cup \tilde{\mu}(\Sigma_n) \quad \text{on } M_k^* \quad (\text{used in Donaldson polynomial})$$

In $W = \mathbb{R} \times Y$ case, for $\alpha \in H_k(Y) \cong H_k(\mathbb{R} \times Y)$
 find V_α codim $4-k$ whose PD dual represents
 $\tilde{\mu}(\alpha) \in H^{4-k}(B^*(p))$

Define $\mu(\alpha) : CI^w(Y) \rightarrow CI^w(Y)$

$$x \mapsto \sum_Y \#(\underbrace{\check{M}_k(x, y)}_{\text{compactification}} \cap V_\alpha) y$$

Counting $\partial \check{M}_{k+1}(x, y) \cap V_\alpha \leadsto \mu(\alpha)$ is a chain map

Prop $\mu(\alpha)\mu(\beta) = (-1)^{\deg \alpha \deg \beta} \mu(\beta)\mu(\alpha)$ on $I^w(Y)$
 (not hold on $CI^w(Y)$ over $\mathbb{Z}$ similar to cup product) $\square$

Recall in sutured monopole, we find closure (Y, R)
 for balanced (M, γ) and define

$$SHM(M, \gamma) = HM(Y|R) = \bigoplus_{\langle C(\$|R) \rangle = 2g(R)-2} \check{H}M(Y, \$)$$

To define sutured instanton $SHI(M, \gamma)$,
 we need to find analogy of $C(\$)$

KM's idea comes from computation of $Y = S' \times \bar{F}$ w = $S' \times \text{pt}$

Prop (Muñoz) The simultaneously generalized eigenspace
 of $\mu(\bar{F})$ (deg 2) and $\mu(\text{pt})$ (deg 4) (over $\mathbb{C}$) lies in

$$(*) \quad \left\{ (\sqrt{-1})^r \cdot 2k, (-1)^2 \cdot 2 \mid r \in \mathbb{Z}, 0 \leq k \leq g(\bar{F}) - 1 \right\} \text{ when } g(\bar{F}) > 0,$$

Moreover, $(2g-2, 2)$ -eigenspace is 1-dimensional.

(adjunction formula)

Cor: For closed surface $R \subset Y$ with $w \cdot R$ odd, the eigenvalues of $\mu(R)$, $\mu(pt)$ lies in $(*)$ by replacing $g(F)$ by $g(R)$

Pf: Consider a slice $R \times \{t_0\} \subset Y \times I$,

removing its nbhd, we obtain $W: S^1 \times R \sqcup Y \rightarrow Y$

with corresponding $V = w \times I \setminus N(R)$

we have $I^V(W): I^{S^1}(S^1 \times R) \otimes I^w(Y) \rightarrow I^w(Y)$ surj

$$\mu(R) \cdot I^V(W)(a \otimes b) = I^V(W)(\mu(R)a \otimes b)$$

Similar for $\mu(pt)$ and $\mu(R)^2 + \mu(pt)$

Hence eigenvalues are subsets $\square$

Def. $I^w(Y|R) = (2g(R)-2, 2)$ -eigenspace of $(\mu(R), \mu(pt))$ on $I^w(Y)$ over $\mathbb{C}$

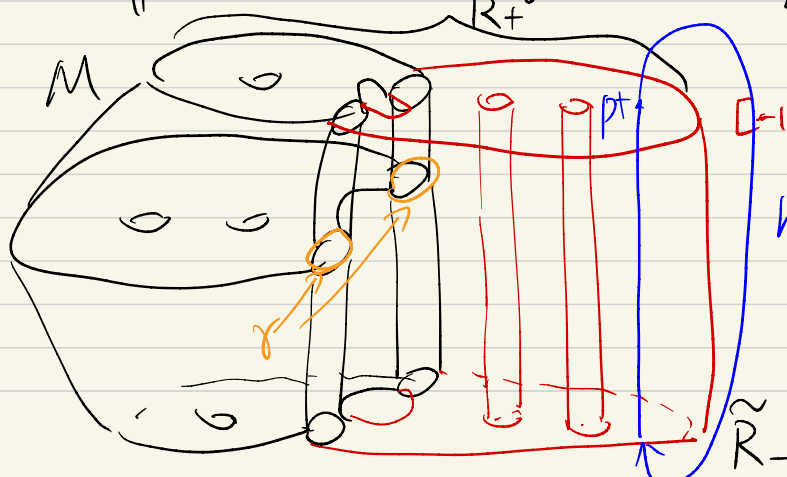
Ex. $I^{S^1}(S^1 \times F|F) \cong \mathbb{C}$ by Muñoz computation

Recall for balanced (M, γ) , we construct preclosure $\tilde{M}$

$$\tilde{M} = M \cup_f [-1, 1] \times T \quad f(\partial T) \xrightarrow{\cong} \gamma$$

$$\partial \tilde{M} = \tilde{R}_+ \cup \tilde{R}_- \quad \tilde{R}_\pm = R_\pm(\gamma) \cup S^\pm 13 \times T$$

A diffeo $h: \tilde{R}_+ \rightarrow \tilde{R}_-$ gives $Y = \tilde{M}/h$, $R = \text{image of } R_\pm$



Now furthermore choose

$pt \in T$, choose h

s.t. $h(S^1 \times pt) = (-1) \times pt$

Let $w = \text{image of } I \times pt$

$$SHI(M, \gamma) = I^w(Y|R)$$

Excision again

- Y closed oriented 3-mfd 1 or 2 components
 - $\Sigma_1, \Sigma_2 \subset Y$ connected surfaces of equal genus
- If Y is connected, $[\Sigma_1], [\Sigma_2]$ independent
 If Y is disconn, Σ_i nonseparating in Y_i
 ($[\Sigma_i] \neq 0$)

Write $\Sigma = \Sigma_1 \cup \Sigma_2$

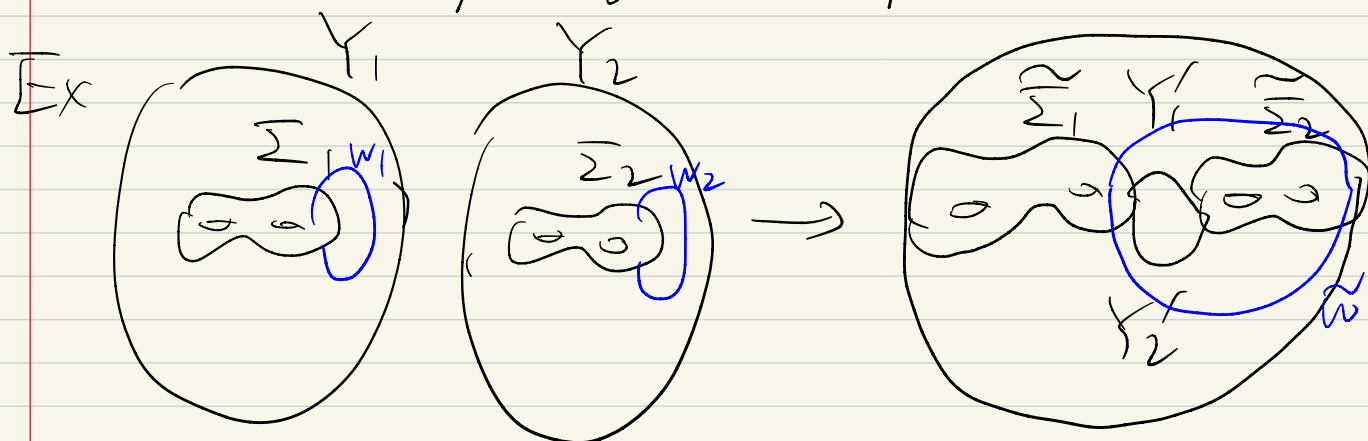
- Fix ori-pre diffeo $h: \Sigma_1 \rightarrow \Sigma_2$

- (new) 1-cycles w_1, w_2 s.t. $w_i \cdot \Sigma_i$ equal and odd

Construct $\tilde{Y}$ by gluing $Y' = Y \setminus N(\Sigma)$

along $\Sigma_1 \xrightarrow{h} \Sigma_2$ $\Sigma_2 \xrightarrow{h^{-1}} \Sigma_1$

obtain $\tilde{\Sigma}$ by image. similarly $\tilde{w}$ from $w = w_1 \cup w_2$



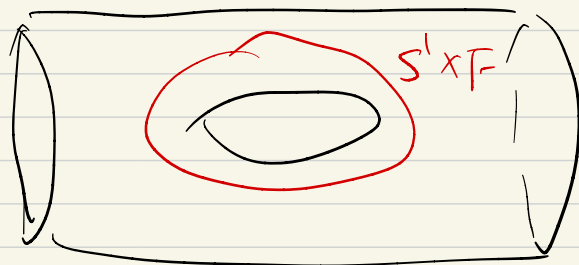
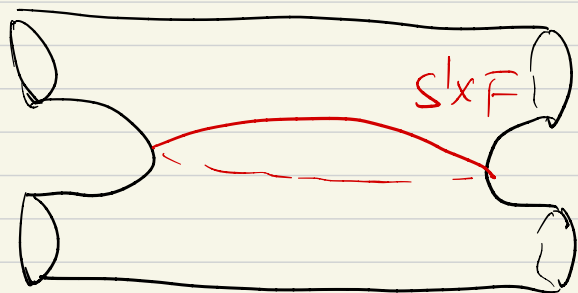
Thm (Floer, Kronheimer-Mrowka)

If $g(\Sigma_i) \geq 1$ ^(≥ 2 in HM), then there exist cobordism maps

$$I^w(Y|\Sigma) \iff I^{\tilde{w}}(\tilde{Y}|\tilde{\Sigma}) \text{ that are iso}$$

The reason now we can assume $g(\Sigma_i) = 1$ is
 because of the computation on $I^{S^1}(\mathbb{T}^3 | \mathbb{T}^2) \cong \mathbb{C}$
 (In HM, need to use local coefficient)

The proof is similar to HM $\square$



Prop $Y = S^1 \times_n \bar{F}$ mapping tori of fixed a pt
 $\bar{F} = pt \times \bar{F}$ $W = [-1, 1] \times pt / h$ $I^W(Y | \bar{F}) \cong \mathbb{C}$

Pf: Excision for $S^1 \times_n \bar{F}$ and $S^1 \times_{h-1} \bar{F}$

Cor. $SHI(M, \mathbb{R})$ indep of h $\square$

Prop $Y = S^1 \times \bar{F}$ α any curve on $\bar{F} = pt \times \bar{F}$
 Then $I^{S^1 \cup \alpha}(Y | \bar{F}) \cong \mathbb{C}$

Pf: By excision along $\bar{F}$, we have

$$\begin{aligned} I^{S^1 \cup \alpha}(Y | \bar{F}) \otimes I^{S^1 \cup \alpha}(Y | \bar{F}) &\cong I^{S^1 \cup 2\alpha}(Y | \bar{F}) \\ &\cong I^{S^1}(Y | \bar{F}) \cong \mathbb{C} \quad \square \end{aligned}$$

Cor For $R \subset Y$, curve $\alpha \subset CR$, we have

$$I^{W \cup \alpha}(Y | R) \cong I^W(Y | R)$$

By excision, used for indep of $g(\mathbb{T})$ next time

Class 15 Properties of SHI

Last time, for balanced (M, γ) , we construct closure (Y, R, w) , and define

$$\text{SHI}(M, \gamma) = I^w(Y|R) = (2g(R)-2, 2)\text{-eigenspace of } \mu(R), \mu(\text{pt}) \text{ on } I^w(Y)$$

We use excision to show its indep of gluing map h in the construction of (Y, R, w) .

We also show that if $\alpha \subset R$ is a curve

$$I^{w \cup \alpha}(Y|R) \cong I^w(Y|R)$$

We use this iso, together with another excision, to show SHI indep of $g(T)$.

Excision: $\Sigma_i \subset Y_i$ surface (gzz in $H(M)$)

$T_i \subset Y_i$ tori $w_i \subset Y_i$ 1-cycles (no w in $H(M)$)

$$T_i \cap \Sigma_i = C_i \quad |T_i \cap w_i| = 1$$

$$h: T_1 \rightarrow T_2 \quad h(C_1) = C_2 \quad h(T_1 \cap w_1) = T_2 \cap w_2$$

T_i cut open Y_i , Σ_i and reglue by h

Thm $I^w(Y|\Sigma) \cong I^w(\tilde{Y}|\tilde{\Sigma})$ by cobordism maps

Pf: similar to the first excision thm, and use

the fact that $I^s(T^2) \cong \mathbb{C} \oplus \mathbb{C}$,

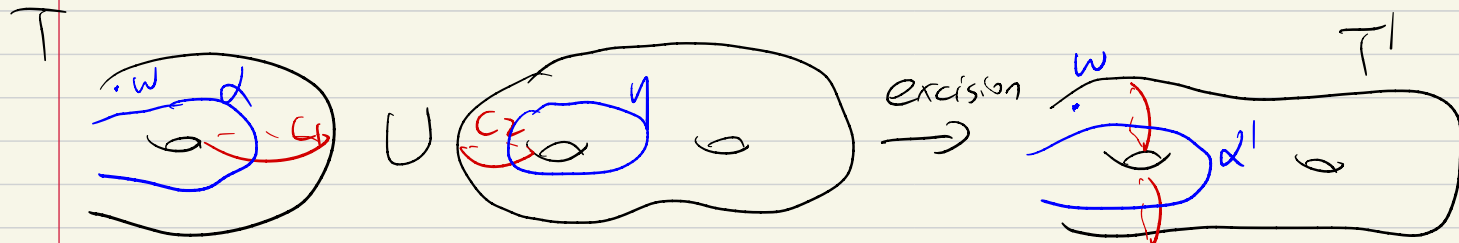
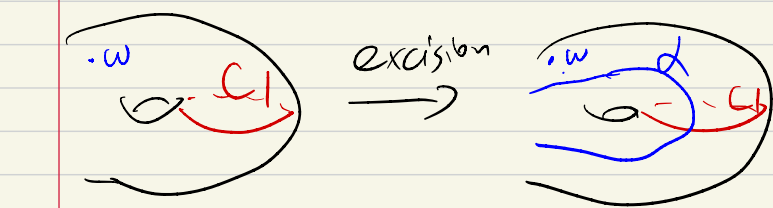
in eigenspace of eigenvalues $(0, +2), (0, -2)$

for $(\mu(T^2), \mu(\text{pt}))$

Pf of $SHI(M, \alpha)$ indep of $g(T)$

iso result by
Kronheimer-Mrowka
(Naturality result by)
Baldwin-Sivek)

$$g(T') = g(T) + 1$$



$$I^{wud}(Y|R) \otimes I^\eta(S' \times \bar{F}_2 | \bar{F}_2) \xrightarrow{\cong} I^{wud}(Y'|R')$$

$$g(R') = g(R) + 1$$

We need to prove $I^\eta(S' \times \bar{F}_2 | \bar{F}_2) \cong \mathbb{C}$ (no w)

Prop: $Y = S' \times \bar{F}$, η a curve on $\bar{F} = pt \times \bar{F}$

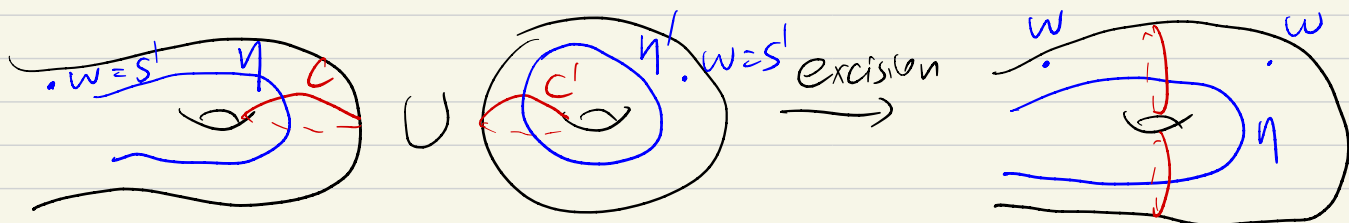
s.t. $\exists$ another curve C , $|\eta \cap C| = 1$ Then $I^\eta(S' \times \bar{F} | \bar{F}) \cong \mathbb{C}$

(I^η is well-defined because $|\eta \cdot S' \times \delta| = 1$)

More generally, $(\lambda, 2)$ -eigen of $I^{S' \cup \eta}(S' \times \bar{F})$ and $I^\eta(S' \times \bar{F})$

are isomorphic

Pf: Excision



$$I^{S' \cup \eta}(S' \times \bar{F} | \bar{F}) \otimes I^{S' \cup \eta'}(S' \times T^2 | T^2) \xrightarrow{\cong} I^{2S' \cup \eta}(S' \times \bar{F} | \bar{F})$$

115

$\mathbb{C}$

115

$\mathbb{C}$

115

$I^\eta(S' \times \bar{F} | \bar{F})$

The general eigen result follows from excision on
+2-eigenspace of $\mu(pt)$

Important properties for SHI (and similar for SHM)

Thm (KM) If M is irreducible, then (M, γ) is taut
iff $\text{SHI}(M, \gamma) \neq 0$.

If M is a homology product, then (M, γ) is product
iff $\text{SHI}(M, \gamma) \cong \mathbb{C}$ (omit pf)

Sketch (not taut $\Rightarrow \text{SHI} = 0$).

If $R_{\pm}(\gamma)$ compressible, then R is compressible

$$\Rightarrow \exists R' \subseteq Y \quad [R'] = [R] \quad 1 \leq g(R') < g(R)$$

$$\Rightarrow \mu(R) = \mu(R') \quad I^w(Y|R) = 0 \text{ because}$$

eigenvalues of $\mu(R')$ on ± 2 eigenspace of $\mu(pt)$
are contained in $\{\pm 2, \dots, \pm 2g(R') - 2\}$

Similar proof for $R_{\pm}(\gamma)$ not norm minimizing

(taut $\Rightarrow \text{SHI} \neq 0$)

taut $\Rightarrow$ hierarchy $(M, \gamma) \xrightarrow{S_1} \dots \xrightarrow{S_n} (M_n, \gamma_n) = \coprod (\mathcal{B}, S')$

$(S' \times \bar{F}, \bar{F}, S')$ is a closure of (M_n, γ_n)

$\text{SHI}(M_n, \gamma_n) \cong \mathbb{C}$ (also used for product $\Rightarrow \text{SHI} \cong \mathbb{C}$)

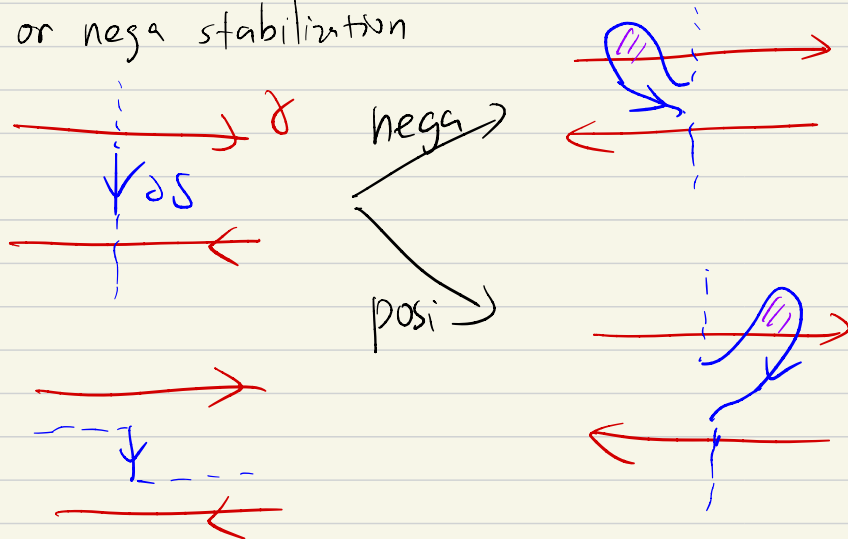
We need to show $\text{SHI}(M_{i+1}, \gamma_i)$ is a direct summand
of $\text{SHI}(M_i, \gamma_i)$ for surface decomposition S_i

Indeed we will define $\mathbb{Z}$ -gr for $S \subset (M, \gamma)$

$$\text{SHI}(M, \gamma) = \bigoplus_{j \in \mathbb{Z} \text{ or } \mathbb{Z} + \frac{1}{2}} \text{SHI}(M, \gamma, S, j)$$

$$\text{SHI}(M_{i+1}, \gamma_{i+1}) \cong \text{SHI}(M_i, \gamma_i, S_i, \frac{1}{2}(\frac{1}{2}|\partial S \cap \gamma| - \chi(S)))$$

We can assume $\frac{1}{2}|\partial S \cap \gamma| - \chi(S)$ is even, if not, we perturb S by posi or nega stabilization



∂S intersects $R_{\pm}(\gamma)$ at circles and arcs. $\partial S_{\pm}$

We find arcs α on T which are linearly indep in homology

Suppose $\widetilde{\partial S}_{\pm} = \partial S_{\pm} \cup \{\pm 1\} \times \alpha$ are circles with the same number

Choose h so that $h(\widetilde{\partial S}_{+}) = \widetilde{\partial S}_{+}$.

Define $\bar{S} = \text{image of } \tilde{S} = S \cup [-1, 1] \times \alpha \text{ in } Y$

$\chi(\bar{S}) = \chi(\tilde{S}) = \chi(S) - \frac{1}{2}|\partial S \cap \gamma|$ even so h exists

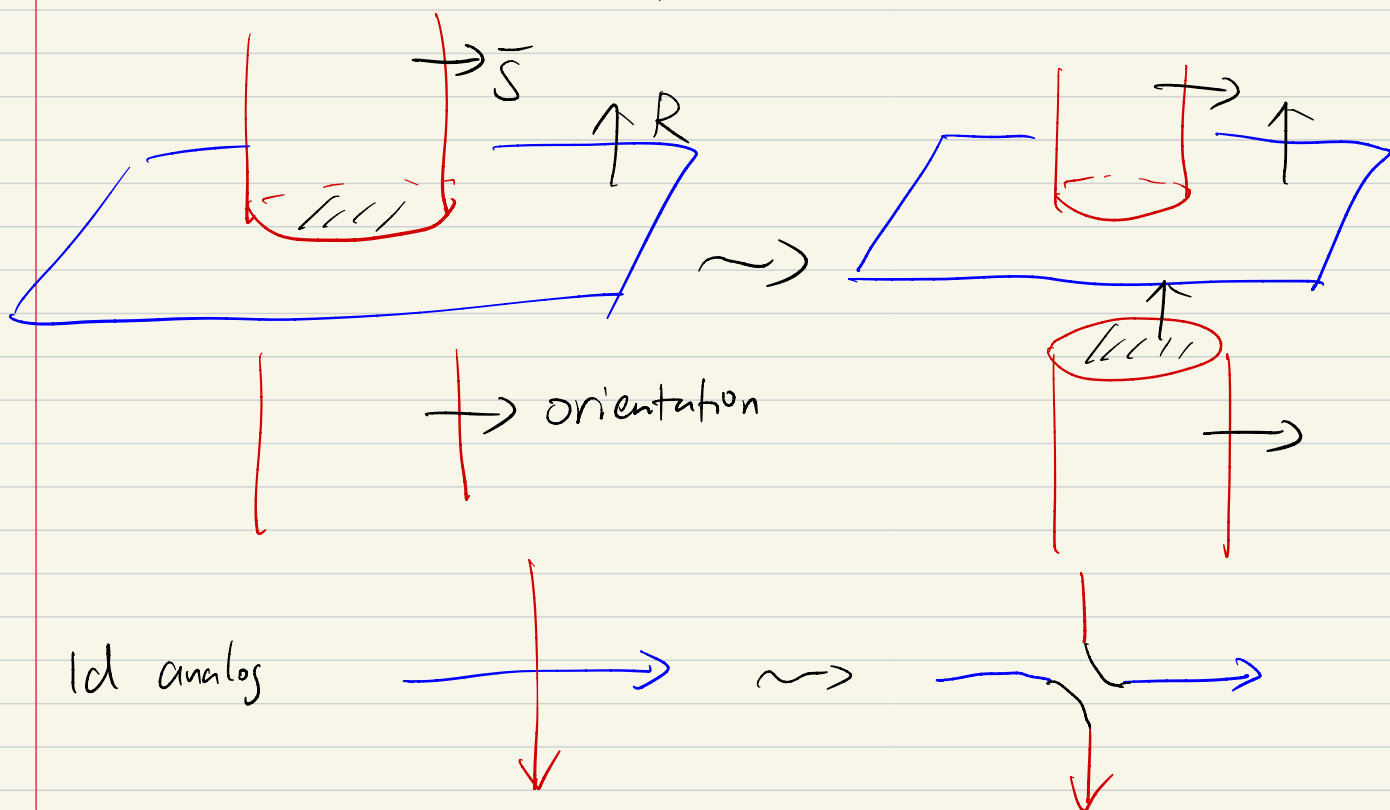
$\text{SHI}(M, \gamma, S, j) = \mathbb{R}^j, 2g(R)-2, 2) - \text{eigenspace of}$

$\mu(\bar{S}), \mu(R), \mu(\text{pt})$ on $I^{\text{wax}}(Y)$

Thm (Zhenkun Li) The grading is independent of all choices (pairing of endpts, arcs, gluing diffeo h)

Rough idea about surface decomposition:

Consider cut and paste operation on $\bar{S}$ and R



We write F for the resulting surface

$$\chi(F) = \chi(\bar{S}) + \chi(R) \quad [F] = [\bar{S}] + [R]$$

(Y, F, ω) is a closure of (M', r') after decomposition

$$\begin{array}{ccc} I^w(Y|F) & = & I^w(Y|\bar{S}) \cap I^w(Y|R) \\ \parallel & & \cap \\ \text{SHI}(M', r') & & \text{SHI}(M, r) \end{array}$$