Class 9 HM theory

In these two weeks, we introduce another Floer homology for closed 3-mfd from gauge theory for Seiberg–Witten equ and generalize it to balanced sutured mfd.

Indeed, we will only focus on properties similar to HF theory, so the concrete construction is not so important.

In HF theory, for closed 3-mfd Y,

we have $HF^{\wedge,+,-,\infty}$ as iso type over $\mathbb{Z}$ or $\mathbb{Z}[U]$

actual space over $\mathbb{F}$ or $\mathbb{F}[U]$ when specify basept z.

In HM (monople) theory, Kronheimer–Mrowka constructed

- $\widehat{HM}$ (from) analog to HF^-
- $\check{HM}$ (to) analog to HF^+
- $\overline{HM}$ (bar) analog to HF^∞

all as actual space over $\mathbb{Z}$ without basept

but $\mathbb{Z}[U]$ structure depends on the basept z

Bloom constructed $\widetilde{HM}$ tilde as $\mathrm{Cone}(\hat{HM} \xrightarrow{U} \hat{HM})$

which depends on z.

Also for cobordism W (can have disconnected boundaries)

There are $HM(W)$ as maps for $\wedge \vee - \sim$

Hence it also satisfies some TQFT property

(with decoration about basepts)

Thm (Kutluhan–Lee–Taubes, or Taubes + Colin–Ghiggini–Honda)

$$\begin{tikzcd} \check{HM} \arrow[rr, "\cong KLT"] \arrow[dr, dash, "T \cong"'] & & HF^+ \\ & ECH \arrow[ur, dash, "\cong CGH"'] & \end{tikzcd}$$

for any closed 3-mfd Y (with a spin^c str $\mathfrak{s}$)

but not known for W

ECH: embedded contact homology by Hutchings

(There is also a sutured ECH theory but similar properties like SFH haven't been proved)

To define the Seiberg–Witten equ for Y with metric we need to specify a spin^c str $\mathfrak{s}$

Other than homology class of unit norm vector fields, we use another description of $\mathfrak{s}$

Def A spin^c str $\mathfrak{s}$ on a Riemannian closed 3-mfd Y is an Hermitian rank 2 bundle $S \to Y$ together with a Clifford multiplication

$\rho : TY \to Hom(S, S)$, which means

a bundle map satisfying $\rho(v)^2 = -|v|^2 Id_S$ for each $v \in TY$

use metric

More concretely, this means given any oriented frame e_1, e_2, e_3 at T_yY, we can find a basis of S_y so that $\rho(e_i)$ is the Pauli matrices

$$\sigma_1 = \begin{pmatrix} i & \\ & -i \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} & i \\ i & \end{pmatrix}$$

which forms standard basis of $SU(S_y)$

Rem. ρ is a bundle isometry if $SU(2)$ with metric $\frac{1}{2}tr(ab^*)$

S is called <u>spinor bundle</u> and its sections are called spinors

Rem

- Spin^c str in this def always exists since

  $TY \cong Y \times \mathbb{R}^3$ for any closed 3-mfd Y

  we can pick a global trivialization e_1, e_2, e_3

  and let them act on $Y \times \mathbb{C}^2$ by Pauli matrices

- we can construct a unit norm vector from such def

  of $\mathfrak{s}$: Fix a unit norm spinor $\Psi : Y \to S$

  there exists a unique vector field $X(\Psi) : Y \to TY$

  s.t. at each pt $\mathbb{C}\Psi$ and its orthogonal complement $\mathbb{C}\Psi^\perp$

  are $\pm i$ eigenspaces of $\rho(X(\Psi))$

Lem. $\mathrm{Spin}^c(Y)$ is an affine space over $H^2(Y;\mathbb{Z})$

Idea: $\{\text{complex line bundle}\} \xrightarrow{1-1} H^2(Y;\mathbb{Z})$

$L \longleftrightarrow c_1(L)$

Given $\mathfrak{s}_0 = (S_0, \rho_0)$ $\mathfrak{s} = (S_0 \otimes L, \rho_0 \otimes \mathrm{Id}_L)$ is a new spin^c

representation of $SU(2)$ on S is irreducible

the space of bundle maps of (S, ρ) and (S', ρ')

intertwines the Clifford operation is a complex line bundle (Schur Lem)

Def a connection B on spin^c bundle S is called a <u>spin^c conn</u>

if $\nabla_B(\rho(X)\Psi) = \rho(\nabla_{LC} X)\Psi + \rho(X)\nabla_B \Psi$ X v.f. Ψ spinor

$\uparrow$

Levi-Civita

The space of spin^c conn is an affine space over $\Omega^1(Y; i\mathbb{R})$

by irreducibility of ρ. Consider induced B^t on $\det(S)$

$\tilde{B} - B = b \otimes 1_S$ then $\tilde{B}^t - B = 2b$ imaginary 1-form

Def The configuration space $C(Y,\mathfrak{s})$ of $(Y,\mathfrak{s})$ consists of the pair (B,Ψ) B spin^c conn Ψ spinor

We extend Clifford mul ρ to 1-form by metric and then k-forms by

$$\rho(\alpha\wedge\beta)=\frac{1}{2}\left(\rho(\alpha)\rho(\beta)+(-1)^{\deg\alpha\deg\beta}\rho(\beta)\rho(\alpha)\right)$$

Note that for the Hodge star $*$ induced by metric we have $\rho(*\alpha)=-\rho(\alpha)$

Define <u>Dirac operator</u> D_B by composition

$$\Psi\in\Gamma(S)\xrightarrow{\nabla_B}\Gamma(T^*X\otimes S)\xrightarrow{\rho}\Gamma(S)$$

which is a self-adjoint elliptic operator

gauge group $\mathcal{G}(Y,\mathfrak{s})=\{u:Y\to S^1\}$

Consists of automorphism of spin^c str

u called <u>gauge transformation</u>

$$u\cdot(B,\Psi)=(\underline{B-u^{-1}du},u\cdot\Psi)$$

recall u as unitary bundle auto intertwining ρ

formula from pull-back on S

If $\Psi\neq 0$, then stabilizer is trivial, call (B,Ψ) <u>irreducible</u>

If $\Psi=0$, then stabilizer is S^1, call (B,Ψ) reducible

We fix a basept y_0 in Y, consider $\mathcal{G}_0(Y,\mathfrak{s})\subset\mathcal{G}(Y,\mathfrak{s})$ with value 1 at y_0. It acts freely on $C(Y,\mathfrak{s})$

Def $\mathcal{B}(Y,\mathfrak{s})=C(Y,\mathfrak{s})/\mathcal{G}(Y,\mathfrak{s})$ moduli configuration space

$\mathcal{B}_0(Y,\mathfrak{s})=C(Y,\mathfrak{s})/\mathcal{G}_0(Y,\mathfrak{s})$ based moduli conf space

Rem The cohomology class of u is $\frac{1}{2\pi i} u^{-1} du$

when $\frac{1}{2\pi i} u^{-1} du = 0$ $u = e^{\xi}$ for $\xi \in \Gamma(Y, i\mathbb{R})$

Now we introduce Chern-Simons-Dirac functional

Fix a base ($spin^c$) conn B_0.

$$\mathcal{L}(B, \Psi) = -\frac{1}{8}\int_Y (B^t - B_0^t) \wedge (F_{B^t} + F_{B_0^t})$$

$$+ \frac{1}{2}\int_Y \langle D_B \Psi, \Psi \rangle dvol \quad \in \mathbb{R}$$

F_{B^t} imaginary valued 2-form D_B self-adj $\langle D_B\Psi, \Psi\rangle \in \mathbb{R}$

$C(Y, \mathfrak{s})$ affine space, so its tangent space at each pt

is identified w/ $\Omega^1(Y; i\mathbb{R}) \times \Gamma(S)$

There is a L^2 inner product

by adding metric on Y, Hermitian metric on S

We compute formal gradient about this L^2 inner product

Let $(b \otimes Id_S, \psi)$ be small change in (B, Ψ)

B^t change by $2b$.

$$\lim_{t \to 0} \frac{\mathcal{L}(B + tb\otimes Id, \Psi + t\psi) - \mathcal{L}(B, \Psi)}{t}$$

$$-\frac{1}{8}\int_Y (2b \wedge (F_{B^t} + F_{B_0^t}) + (B^t - B_0^t \wedge 2db))$$

$$+\frac{1}{2}\int_Y \langle \rho(b)\Psi, \psi \rangle dvol + \int_Y Re\langle \psi, D_B\Psi\rangle dvol$$

Stokes $\quad$ imaginary-valued 1-form

$$= \underbrace{-\frac{1}{2}\int_Y b \wedge F_{B^t}}_{\parallel} + \int_Y \langle b, \rho^{-1}(\Psi\Psi^*)_0 \rangle dvol$$

$$\frac{1}{2}\int_Y \langle b, * F_{B^t} \rangle dvol + \int_Y Re\langle \Psi, D_B \Psi \rangle \, dvol$$

$(\Psi\Psi^*)_0$ is traceless part of $\Psi\Psi^*$ because of def of ρ^{-1}

So $\operatorname{grad} \mathcal{L} = \left(\left(\frac{1}{2} * F_{B^t} + \rho^{-1}(\Psi\Psi^*)_0 \otimes Id_S , D_B \Psi \right) \right)$

as a section $C(Y, S) \longrightarrow TC(Y, S)$

(change B_0 only change $\mathcal{L}$ by constant)

By $\rho(*\alpha) = -\rho(\alpha)$, we obtain Seiberg–Witten equs

$$\begin{cases} \frac{1}{2}\rho(F_{B^t}) - (\Psi\Psi^*)_0 = 0 \\ D_B \Psi = 0 \end{cases}$$

Class 10 Monopole Floer homology

Last time, we introduce Chern-Simons-Dirac functional

$$\mathcal{L}(B,\Psi) = -\frac{1}{8}\int_Y (B^t - B_0^t)\wedge(F_{B^t} + F_{B_0^t})$$

$$+\frac{1}{2}\int_Y \langle D_B\Psi, \Psi\rangle \, d\mathrm{vol} \quad \in \mathbb{R}$$

for a base (spin^c) conn B_0.

whose formal gradient gives Seiberg-Witten equs

$$\begin{cases} \frac{1}{2}\rho(F_{B^t}) - (\Psi\Psi^*)_0 = 0 \\ D_B\Psi = 0 \end{cases}$$

$(\Psi\Psi^*)_0$ is traceless part of $\Psi\Psi^*$ because of def of ρ^{-1}

If $\Psi = (\alpha,\beta)$, then $(\Psi\Psi^*)_0 = \begin{pmatrix} \frac{1}{2}(|\alpha|^2 - |\beta|^2) & \alpha\bar{\beta} \\ \bar{\alpha}\beta & \frac{1}{2}(|\beta|^2 - |\alpha|^2) \end{pmatrix}$

Note that $c_1(S) = [-\frac{1}{2\pi i}F_{B_0^t}] = [-\frac{1}{2\pi i}F_{B^t}]$

For $u \in \mathcal{G}(Y,\mathfrak{s}) = \{u: Y \to S^1\}$ gauge group

$u\cdot(B,\Psi) = (B - u^{-1}du, u\cdot\Psi)$

Since $S^1 = K(\mathbb{Z},1)$, $\pi_0(\mathcal{G}(Y,\mathfrak{s})) = H^1(Y,\mathbb{Z})$

$$= [u] \longmapsto \frac{1}{2\pi i}u^{-1}du$$

$\mathcal{L}(u\cdot(B,\Psi)) - \mathcal{L}(B,\Psi)$

$$-\frac{1}{8}\int_Y (-2u^{-1}du\wedge(F_{B^t} + F_{B_0^t})) = \frac{1}{2}\int_Y u^{-1}du\wedge F_{B_0^t}$$

$$= \frac{1}{2}((2\pi i[u]\cup(-2\pi i c_1(S)))[Y]$$

$$= 2\pi^2([u]\cup c_1(S))[Y]$$

Hence $\mathcal{L}$ is only invariant under identity component of $\mathcal{G}(Y,\mathfrak{s})$ if $c_1(S) \neq 0$ (such $\mathfrak{s}$ called <u>nontorsion</u>)

It induces $\bar{\mathcal{L}} : \mathcal{C}(Y,\mathfrak{s}) \longrightarrow \mathbb{R}/2\pi^2\mathbb{Z}$

which is inv under full $\mathcal{G}(Y,\mathfrak{s})$

solutions of SW equs are called <u>monopole</u>

Prop Reducible monopole ($\Phi = 0$) exists only when $c_1(S)$ is torsion. In such case, the space of reducible monopoles are identified with

$$H^1(Y;i\mathbb{R})/2\pi i H^1(Y;\mathbb{Z})$$

(flat conn up to gauge)

Pf. in SW equ., if $\Phi = 0$, then $F_{B^t} = 0$

Note $c_1(S) = \left[-\frac{1}{2\pi i} F_{B^t}\right] = 0$

Fix base conn B_0. $F_{B_0^t + 2b} = F_{B_0^t} + 2db$

(recall $b \wedge b$ vanish since it is 1-form rather than matrix)

Since $F_{B_0^t}$ is exact by $c_1(S) = 0$, we can find b

s.t. $F_{B_0^t + 2b} = 0$.

other flat conn obtained by adding closed form.

The quotient comes from $[u] = \frac{1}{2\pi i} u^{-1} du$

Prop Suppose scalar curv of (Y, g) is nonnegative at each pt. Then there is no irreducible monopoles.

Pf. We use Weitzenböck formula

$$D_B^2 \Psi - \nabla_B^* \nabla_B \Psi = \frac{1}{2} \rho(F_{B^t}) \Psi + \frac{1}{4} s \Psi$$

∇_B^* formal adjoint of ∇_B s scalar curvature

If (B, Ψ) satisfies SW equs

we have $\nabla_B^* \nabla_B \Psi + (\Psi \Psi^*)_0 \Psi + \frac{1}{4} s \Psi = 0$

take L^2 product with Ψ, we get

$$\|\nabla_B \Psi\|_{L^2}^2 + \langle (\Psi \Psi^*)_0 \Psi, \Psi \rangle + \frac{1}{4} s \|\Psi\|_{L^2}^2 = 0$$

$$\begin{pmatrix} \frac{1}{2}(|\alpha|^2 - |\beta|^2) & \alpha \bar{\beta} \\ \bar{\alpha} \beta & \frac{1}{2}(|\beta|^2 - |\alpha|^2) \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(|\alpha|^2 - |\beta|)\alpha + \alpha |\beta|^2 \\ |\alpha|^2 \beta + \frac{1}{2}(|\beta|^2 - |\alpha|^2)\beta \end{pmatrix}$$

$$\frac{1}{2}(|\alpha|^2 - |\beta|)|\alpha|^2 + |\alpha|^2 |\beta|^2 + |\alpha|^2 |\beta|^2 + \frac{1}{2}(|\beta|^2 - |\alpha|^2)|\beta|^2$$

$$= \frac{1}{2}(|\alpha|^4 + |\beta|^4 + 2|\alpha|^2|\beta|^2) = \frac{1}{2}(|\alpha|^2 + |\beta|^2)^2 = \frac{1}{2} \|\Psi\|_{L^2}^4$$

$s \geq 0$ implies $\|\Psi\|_{L^2}^4 = 0$. $\Psi = 0$ $\square$

Similar idea can be applied to prove

Prop The space of solutions is compact

Rem The compact result is the main advantage of monopole theory comparing to instanton theory

monopole has S^1 gauge, instanton has $SO(3)$, $SU(2)$ gauge

The monopole Floer homology is a ∞-dim analog of Morse homology for mfd with boundary.

We consider the blow-up configuration space

$$C^{\sigma}(Y,\mathfrak{s}) = \Omega^1(Y, i\mathbb{R}) \times \mathbb{R}_{\geq 0} \times \mathbb{S}(\Gamma(\psi))$$

$$(B \ , \ r \ , \ \psi \ )$$

$$\|\psi\| = 1 \quad \Psi = r\psi$$

This is ∞-dim "mfd" with boundary consists of $(B, 0, \phi)$ which projects to reducible sol

$\mathcal{G}(Y,\mathfrak{s})$ acts freely on $C^{\sigma}(Y,\mathfrak{s})$

(We need to take completion of C^{σ} and $\mathcal{G}$ by Sobolev norms. to apply Sard-Smale thm for Banach mfds)

we consider gradient flow equ on $Y \times I$ variable τ on I

because B^t is used

$$\begin{cases} \frac{1}{2}\frac{d}{d\tau}B^t = -\frac{1}{2} * F_{B^t} - r^2\rho^{-1}(\psi\psi^*)_0 \\ \frac{d}{d\tau} r = -\langle \psi, D_B\psi\rangle_{L^2} r \\ \frac{d}{d\tau}\psi = -D_B\psi + \langle \psi, D_B\psi\rangle_{L^2}\psi \end{cases}$$

The RHS is $(\operatorname{grad}\mathcal{L})^{\sigma}$

which is the same as $\operatorname{grad}\mathcal{L}$ for $r \neq 0$. tangent to locus $r=0$.

$(\operatorname{grad}\mathcal{L})^{\sigma}$ desends to

$$\mathcal{B}^{\sigma}(Y,\mathfrak{s}) = C^{\sigma}(Y,\mathfrak{s}) / \mathcal{G}(Y,\mathfrak{s})$$

Class 11 Sutured monopole homology

In the finite dim Morse homology for mfd B with boundary.

we can construct complexes $\check{C}, \hat{C}, \bar{C}$ s.t.

$H_*(\check{C}) = H_*(B)$, $H_*(\hat{C}) = H_*(B, \partial B)$, $H_*(\bar{C}) = H_*(\partial B)$

There is also a long exact seq

$$H_*(\bar{C}) \longrightarrow H_*(\check{C}) \longrightarrow H_*(\hat{C}) \longrightarrow H_*(\bar{C})$$

We omit actual construction of monopole Floer homology

generators are from critical pts of $(\text{grad } \mathcal{L})^6$ + perturbation

differential are from 1d moduli space of flow lines on $Y \times I$

with perturbation, quotient by translation $\mathbb{R}$

The compactness is used to show the sum in differential is finite

$\partial^2 = 0$ comes from 2d moduli space $/ \mathbb{R}$

For a cobordism $W: Y_0 \to Y_1$,

we consider $W^* = W \cup (-\infty, 0] \times Y_0 \cup [0, \infty) \times Y_1$

with cylindrical metric $dr^2 + g_i$

The cobordism map is from 4d SW equ moduli space on W^+

$\langle CM(W)(x), y \rangle$ counting signed number

in 0-dim moduli space with boundary conditions from

critical pts x, y (also need compactness)

Counting 1-dim moduli space $\Rightarrow$ chain map.

4d SW: $\begin{cases} \frac{1}{2}\rho(F_{A^t}^+) = (\Phi \Phi^*)_0 \\ D_A^+ \Phi = 0 \end{cases}$

F^+ comes from +1 eigenspace of Hodge star $*$ on 2-form

D^+ comes from spinor bundle S^+ S^-

To define monopole homology for balanced sutured mfd (M,γ)

we need to use $\check{HM}(Y) = \bigoplus_{\mathfrak{s}\in Spin^c(Y)} \check{HM}(Y,\mathfrak{s})$

for closed 3-mfd Y. (iso to HF^+)

For an embedded oriented surface $R \subset Y$

Define $HM(Y|R) = \bigoplus_{\langle c_1(\mathfrak{s}), R\rangle = 2g(R)-2} \check{HM}(Y,\mathfrak{s})$

where $c_1(\mathfrak{s}) = c_1(S)$ for spinor bundle S

If $g(R) > 1$, $\langle c_1(\mathfrak{s}), R\rangle = 2g(R)-2 > 0$ $\mathfrak{s}$ nontorsion.

There is no reducible solutions, hence $\overline{HM} = 0$

$\overline{HM} \to \check{HM} \to \hat{HM} \to \overline{HM}$ implies $\check{HM} \cong \hat{HM}$

Note that by adjunction inequality, if $\langle c_1(\mathfrak{s}), R\rangle > 2g(R)-2$,

then $\check{HM}(Y,\mathfrak{s}) = 0$. so it is the top grading for R.

For balanced (M,γ), we are going to construct (Y,R) called closure

and define sutured monopole homology as

$SHM(M,\gamma) = HM(Y|R)$

Then we show the indep of choices of (Y,R)

(iso type by Kronheimer-Mrowka,
naturality up to scalar multiplication by Baldwin-Sivek)

First we construct preclosure $\widetilde{M}$ by picking

- an auxiliary surface T compact connected $|\partial T| = |\gamma|$
- $f: \partial T \to \gamma$ orientation reversing diffeo

$\widetilde{M} = M \cup_f [-1,1] \times T \quad \partial \widetilde{M} = \widetilde{R}_+ \cup \widetilde{R}_-$

$\widetilde{R}_\pm = R_\pm(\gamma) \cup \{\pm 1\} \times T$ we need $g(\widetilde{R}_\pm) \geq 2$

Furthermore, pick

- orientation preserving diff $h : \widetilde{R}_+ \longrightarrow \widetilde{R}_-$

Let $Y = \widetilde{M}/h \quad R = $ image of $\widetilde{R}_\pm$

We need to show indep of $g(T)$, h.

Class 12 Floer's excision theorem.

$\widehat{M} = M \cup_f [-1,1] \times T \quad \partial \widetilde{M} = \widetilde{R}_+ \cup \widetilde{R}_-$

$\widetilde{R}_\pm = R_\pm(\gamma) \cup \{\pm 1\} \times T$ we need $g(\widetilde{R}_\pm) \geq 2$

Furthermore, pick

- orientation preserving diff $h : \widetilde{R}_+ \to \widetilde{R}_-$

Let $Y = \widetilde{M}/h \quad R = \text{image of } \widetilde{R}_\pm$

We need to show indep of $g(T)$, h.

The proof is based on Floer's excision thm.

(generalized by KM to any genus)

We start with the following lem.

Lem Let F be closed connected oriented surface with $g(F) \geq 2$. Let $Y = F \times S^1$ $F = F \times pt$

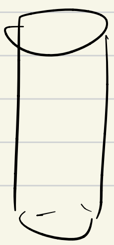
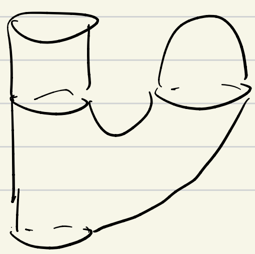
Then $HM(Y|F) \cong \mathbb{Z}$.

Indeed, $CM(Y|F) \cong \mathbb{Z}$ for constant negative metric on F and product metric on Y.

Lem Let $W = F \times D^2 : \phi \to Y$ for Y, F as above. Then the cobordism map is ± 1

$HM(W) : HM(\phi) \cong \mathbb{Z} \longrightarrow HM(Y|F) \cong \mathbb{Z}$

Pf. This follows from TQFT property.

Consider $F \times$  $\simeq F \times$  left Id right nonzero

Setting of Floer's excision thm.

- Y closed oriented 3-mfd 1 or 2 components
- $\Sigma_1, \Sigma_2 \subset Y$ connected surfaces of equal genus

If Y is connected, $[\Sigma_1], [\Sigma_2]$ independent

If Y is disconn, Σ_i nonseparating in Y_i

$([\Sigma_i] \neq 0)$

Write $\Sigma = \Sigma_1 \cup \Sigma_2$

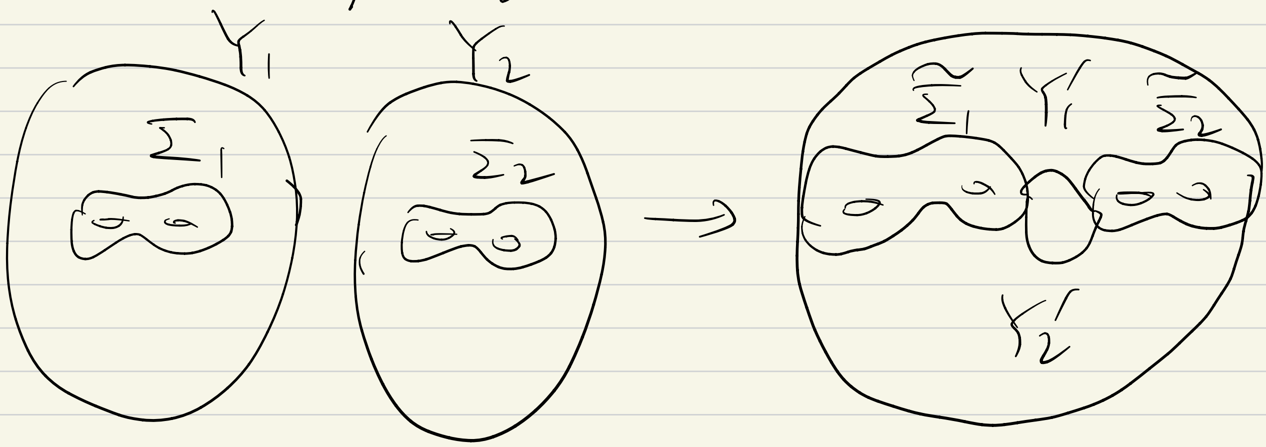
- Fix ori-rever diffeo $h: \Sigma_1 \to \Sigma_2$

Construct $\tilde{Y}$ by gluing $Y' = Y \setminus N(\Sigma)$

along $\Sigma_1 \xrightarrow{h} -\Sigma_2 \quad \Sigma_2 \xrightarrow{h^{-1}} -\Sigma_1$

obtain $\tilde{\Sigma}$ by image

Ex

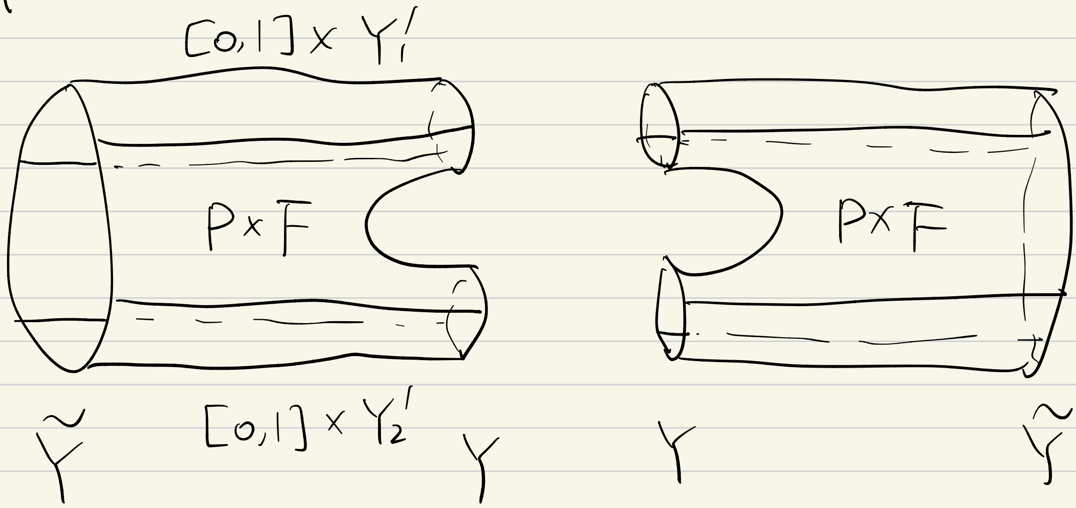


Thm (Floer, Kronheimer–Mrowka)

If $g(\Sigma_i) \geq 2$, then there exist cobordism maps

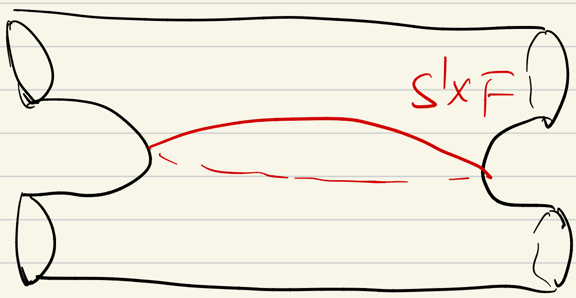
$HM(Y|\Sigma) \rightleftarrows HM(\tilde{Y}|\tilde{\Sigma})$ that are iso

Pf. construct cobordism $W: \tilde{Y} \to Y$ $\quad \overline{W}: Y \to \tilde{Y}$



$g(F) = g(\Sigma_i)$

$W \circ \overline{W}$:

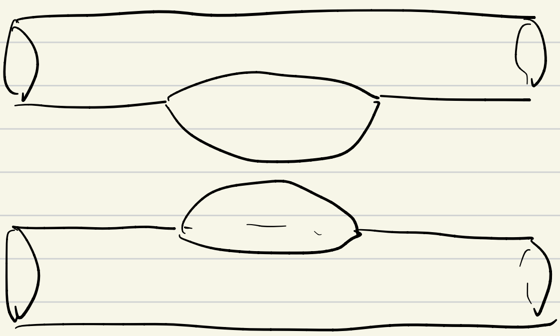


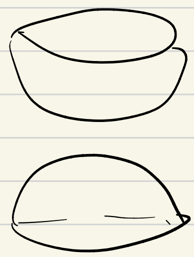
$HM(F \times S^1 | F) \cong \mathbb{Z}$

Id

$\|$ $\qquad \|$

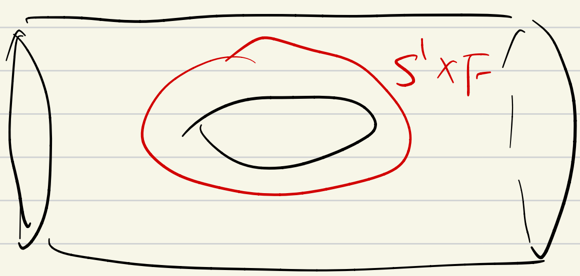
$\pm \mathrm{Id}_Y$





$\pm$Id

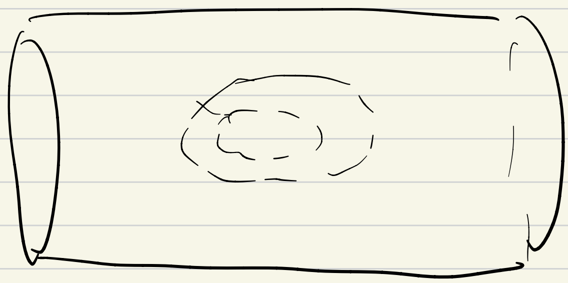
$\overline{W} \circ W$



$\|$

similar

$\pm$Id



Prop. Let $Y = F \times_h S^1 = F \times I / F \times \{-1\} \sim h(F) \times \{1\}$ be the mapping torus and $F = F \times pt$.

Then $HM(Y|F) \cong \mathbb{Z}$

pf: By excision on $F \times_h S^1$ and $F \times_{h^{-1}} S^1$ along F.

$$HM(F \times_h S^1 \cup F \times_{h^{-1}} S^1 | F \cup F) \cong HM(F \times S^1 | F) \cong \mathbb{Z}$$

$$\|$$

$$H_*(CM(F \times_h S^1 | F) \otimes_{\mathbb{Z}} CM(F \times_{h^{-1}} S^1 | F))$$

no Tor term in Kunneth formula. so both iso $\mathbb{Z}$ □

Prop $SHM(M,\gamma)$ indep of diffeo h.

Pf: By excision on (Y, R) and $(F \times_{h_1 \circ h_2^{-1}} S^1, F)$ with $g(R) = g(F)$

To prove $SHM(M,\gamma)$ ind of $g(T)$, we need HM with local coefficient to prove excision in genus 1.

We can also fix a large genus g and define $SHM^g(M,\gamma)$ for $g = g(T)$

(Baldwin-Sivek prove naturality of SHM^g over $\mathbb{Z}/\pm 1$ and SHM without g over local coeff)

Similar to SFH, we have surface decomp thm,

irr ⇒ taut iff nonzero, homology product ⇒ product iff $\mathbb{Z}$

(we will define grading for $(S, \partial S) \subset (M, \partial M)$ after introducing sutured instanton homology)