Class 5 Heegaard Floer homology

General idea: (1) represent 3-mfd Y (closed) or M (w/ ∂)
by data $\mathcal{H}$ called Heegaard diagram
(2) pick up auxiliary data J for $\mathcal{H}$
(3) construct inv for $(\mathcal{H}, J)$
(4) show independence of choices of data.

Def. Let Y be a closed, connected, oriented 3-mfd
A <u>Heegaard diagram</u> $\mathcal{H}$ for Y consists of

- Σ is a closed oriented genus g surface
  (w/ embedding in Y s.t. $Y = H_1 \cup_\Sigma H_2$)
- $\alpha = \{\alpha_1, \ldots, \alpha_g\} \subset \Sigma$ a set of disjoint curves
  linearly indep in $H_1(\Sigma)$ (α_i bounds a disk in H_1)
- $\beta = \{\beta_1, \ldots, \beta_g\} \subset \Sigma$ similar condition as α
  (β_i bounds a disk in H_2)

Later, $\mathcal{H}$ also includes basept z
for pted mfd (Y, z)

- $z \in \Sigma \setminus \alpha \cup \beta$ (w/ embedding in Y)

Such Heegaard diagram comes from a self-indexing
Morse function $f: Y \to \mathbb{R}$ and a metric on Y
(one maximum and minimum)

Morse function: all critical pts ($\nabla f = 0$) nondegenerate

Self-indexing: critical value = index of critical pt

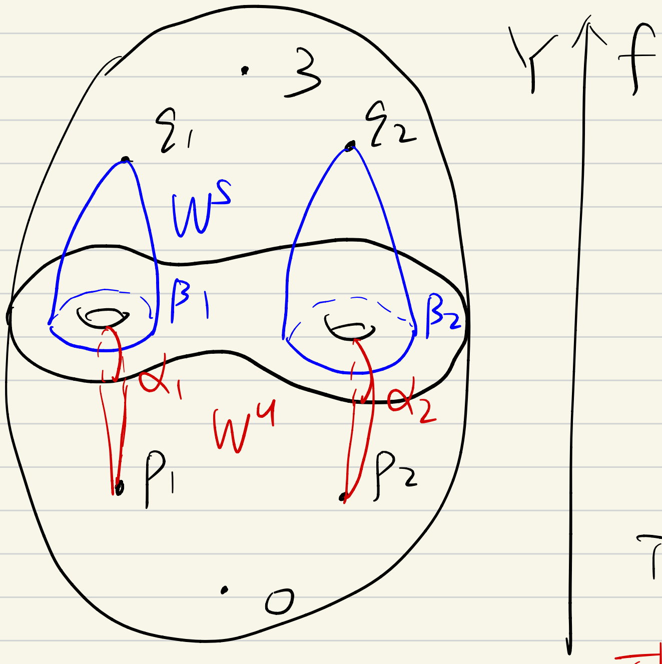
$\Sigma = f^{-1}(\frac{3}{2})$ $H_1 = f^{-1}([0,\frac{3}{2}])$ $H_2 = f^{-1}([\frac{3}{2},3])$

p_i index 1 critical pt

$W^s(p_i)$, $W^u(p_i)$ stable / unstable submfds

$\alpha_i = \Sigma \cap W^u(p_i)$

q_i index 2 critical pt $\beta_i = W^s(q_i) \cap \Sigma$



Rem: Given $\mathcal{H}$,

we can construct a 3-mfd from $\Sigma \times I$ by attaching 2-handles along α on $\Sigma \times \{-1\}$

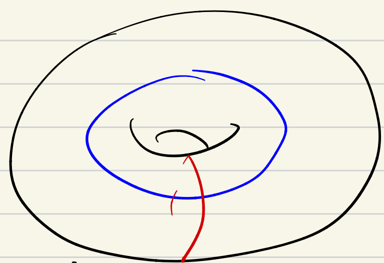
β on $\Sigma \times \{+1\}$

Then fill two spherical ∂ by B^3

The embedding data is for naturality.

Prop $\dfrac{\{\text{closed 3-mfd } Y\}}{\text{diff}} \overset{1-1}{\longleftrightarrow} \dfrac{\{\text{Heegaard diagram } \mathcal{H} \text{ producing } Y\}}{\text{isotopy, handle slide, stabilization}}$

Stabilization : connected sum with

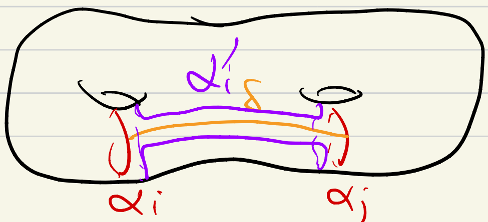


Ex. this is $\mathcal{H}$ for S^3

Handleslide : pick arc δ in $\Sigma \setminus \alpha$ (or $\Sigma \setminus \beta$)

connecting α_i, α_j $\partial N(\alpha_i \cup \delta \cup \alpha_j) = \alpha_i \cup \alpha_i' \cup \alpha_j$

replace α_i by α_i'



To construct $\widehat{HF}(Y)$, we fix a Heegaard diagram $\mathcal{H}$
a pt z on $\Sigma \setminus \alpha \cup \beta$, a cpx str j on Σ
Consider symmetric product

$$\mathrm{Sym}^g \Sigma = \underbrace{\Sigma \times \cdots \times \Sigma}_{g} / S_g \quad S_g \text{ symmetric group}$$

Ex $\mathrm{Sym}^2 \Sigma = \Sigma \times \Sigma / (x,y) \sim (y,x)$

Prop $\mathrm{Sym}^g \Sigma$ is a mfd

Sketch: $\mathrm{Sym}^g \mathbb{C} \cong \mathbb{C}^g$ by fundamental thm of algebra

roots of $z^g + a_1 z^{g-1} + \cdots + a_g \longmapsto (a_1, \ldots, a_g)$ $\square$

Rem smooth str on $\mathrm{Sym}^g \Sigma$ depends on j

But different j gives diffeo strs

j induces a complex str $\mathrm{Sym}^g j$ on $\mathrm{Sym}^g \Sigma$

Define $T_\alpha = \alpha_1 \times \cdots \times \alpha_g / S_g \subset \mathrm{Sym}^g \Sigma$ similar for T_β

Prop (Perutz) There exists a symplectic form ω
on $\mathrm{Sym}^g \Sigma$ s.t. T_α, T_β are Lagrangians
and monotonic with minimal Maslov index 2

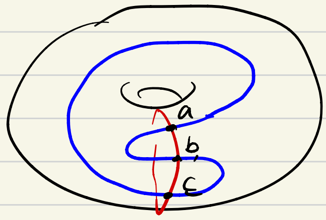
Roughly. HF is the Lagrangian Floer homology of T_α T_β
in $\mathrm{Sym}^g \Sigma$. In original Ozsváth-Szabó construction,
they regard T_α. T_β as totally real submfd.

We define the hat version $\widehat{HF}$ first.

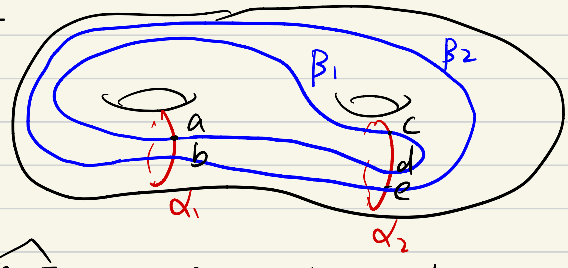
The chain cpx $\widehat{CF}$ is generated by intersections $T_\alpha \cap T_\beta$

explicitly $X = (x_1, \cdots, x_g)$ $x_i \in \alpha_i \cap \beta_{\sigma(i)}$ $\sigma \in S_g$

Ex $g=1$



$g=2$

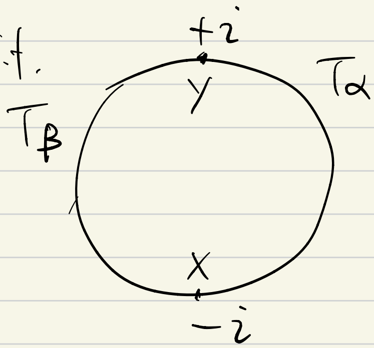


$\widehat{CF} = \mathbb{Z}\langle a, b, c\rangle$ $\widehat{CF} = \mathbb{Z}\langle ae, bc, bd\rangle$

Given $x, y \in \widehat{CF}$, consider $\pi_2(x,y)$ the set of relative homotopy class $u : \mathbb{D} \longrightarrow \mathrm{Sym}^g\Sigma$

$$\begin{array}{c}\mathbb{D}\\ \cap \\ \mathbb{C}\end{array}$$

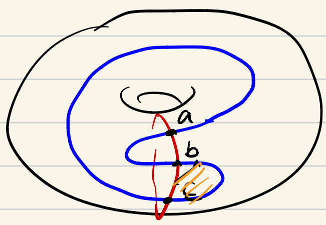
S.t.



$u(-i) = x$ $u(+i) = y$

$u(\mathrm{Re}\, z \geq 0 \cap \partial\mathbb{D}) \subset T_\alpha$

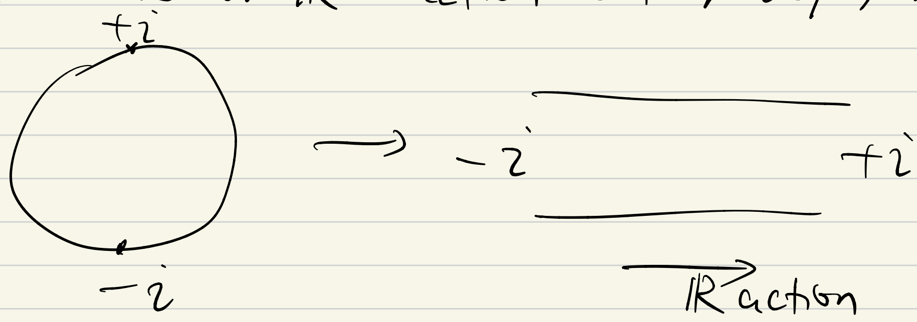
$u(\mathrm{Re}\, z \leq 0 \cap \partial\mathbb{D}) \subset T_\beta$



Given $\phi \in \pi_2(x,y)$, let $\mathcal{M}(\phi)$ be the set of J-holomorphic representatives. i.e. the map $u : \mathbb{D} \longrightarrow \mathrm{Sym}^g\Sigma$ satisfying $J \circ du = du \circ j_{\mathbb{C}}$ for a generic family of almost cpx str $(J_t)_{t\in[0,1]}$ on $\mathrm{Sym}^g\Sigma$

The expected dimension (actual dim when $\mathcal{M}(\phi)$ is smooth mfd) is the Maslov index $\mu(\phi)$ (a formula by Lipshitz)

There is a $\mathbb{R}$-action on $\mathcal{M}(\phi)$ by translation



For the basept z, we have $n_z : \pi_2(x,y) \to \mathbb{Z}$ given by intersection number of $\mathrm{Im}\, u \cap \{z\} \times \mathrm{Sym}^{g-1}\Sigma$

i.e. the signed number of the times that $\mathrm{Im}\, u$ pass z.

Finally, we define

$$\partial x = \sum_{y} \sum_{\substack{\phi \in \pi_2(x,y) \\ \mu(\phi)=1 \\ n_z(\phi)=0}} \left(\# \mathcal{M}(\phi)/\mathbb{R} \right) y$$

where $\#$ is a signed count from the orientation of $\mathcal{M}(\phi)$

(many people works over $\mathbb{Z}/2$ so no sign)

Thm (Ozsváth–Szabó) For $\mathcal{H}$ satisfying some admissible condition, and generic J,

∂ is a finite sum (in particular $\# \mathcal{M}(\phi)/\mathbb{R} < \infty$)

$\partial^2 = 0$ (index 2 bubbles cancel and don't contribute)

$$\widehat{HF}(\mathcal{H}, J) = H(\widehat{CF}, \partial) = \ker \partial / \mathrm{Im}\, \partial$$

Thm (Ozsváth–Szabó) Given two admissible choices $(\mathcal{H}_i, J_i)$ for (Y,p), there exists a quasi-iso

$$\Phi_{12}: \widehat{CF}(\mathcal{H}_1, J_1) \longrightarrow \widehat{CF}(\mathcal{H}_2, J_2) \text{ over } \mathbb{Z}$$

Hence the isomorphism type of $\widehat{HF}$ is an inv of Y

Thm (Juhász–Thurston–Zemke) Over $\boxed{\mathbb{F} = \mathbb{Z}/2}$, for any three choices $(\mathcal{H}_i, J_i)$ of (Y,p) with the embedding data, we have

over $\mathbb{Z}$ hasn't been proved

We write $\mathbb{F} = \mathbb{Z}/2$ Later

$$\Phi_{23} \circ \Phi_{12} \simeq \Phi_{13}$$

In particular, let $3=1$, we obtain $\Phi_{21} \circ \Phi_{12} \simeq Id$ and similarly $\Phi_{21} \circ \Phi_{12} \simeq Id$.

i.e. Φ_{12} is a (canonical) chain homotopy equ.

This is so-called naturality result (rather than iso type)

Def. $$\widehat{CF}(Y,z) = \coprod_{\substack{\text{admissible} \\ (\mathcal{H},J)}} \widehat{CF}(H,J) \Big/ \sim$$

where $x_i \in \widehat{CF}(H_i, J_i)$ are identified if $\Phi_{12}(x_1) = x_2$

$$\widehat{HF}(Y,z) = H(\widehat{CF}(Y,z))$$

Note that the dependence of z is important because a loop in Y based at z can induce nontrivial automorphism

Class 6 More versions of HF

using ϕ with $n_z(\phi) \neq 0$

Let $CF^- = \mathbb{Z}[U]\langle T_\alpha \cap T_\beta \rangle$

$$\partial^- x = \sum_y \sum_{\substack{\phi \in \pi_2(x,y) \\ \mu(\phi)=1}} \left(\# M(\phi)/\mathbb{R}\right) \cdot U^{n_z(\phi)} y$$

$$CF^\infty = CF^- \otimes_{\mathbb{Z}[U]} \mathbb{Z}[U, U^{-1}]$$

$$CF^+ = CF^- \otimes_{\mathbb{Z}[U]} \mathbb{Z}[U, U^{-1}]/\mathbb{Z}[U]$$

(sometimes people use $\mathbb{Z}[[U]]$ and $\mathbb{Z}[[U, U^{-1}]$ for completion)

Similarly, by OS, JTZ results,
we have $(\partial^-)^2 = 0$, isomorphism type over $\mathbb{Z}$
naturality result over $\mathbb{Z}/2$

Prop (OS) long exact sequence

$$HF^- \longrightarrow HF^\infty \longrightarrow HF^+ \longrightarrow HF^-$$

$$HF^- \xrightarrow{U} HF^- \longrightarrow \widehat{HF} \longrightarrow HF^-$$

Prop (OS) Suppose W is a connected oriented 4-mfd with $\partial W = -Y_0 \sqcup Y_1$, then there exists a chain map $\widehat{CF}(W) : \widehat{CF}(Y_0) \longrightarrow \widehat{CF}(Y_1)$ (over $\mathbb{Z}$) obtained by counting holomorphic triangles
similar for other versions $(+, -, \infty)$

Rem. To make $\widehat{CF}(W)$ and $\widehat{HF}(W) = \widehat{CF}(W)_*$ well-defined. $\widehat{HF}(Y_i)$ cannot just be iso type

(e.g. all vector spaces iso to $\mathbb{F}^n$ for some n.

a map $\mathbb{F}^m \to \mathbb{F}^n$ up to iso of $\mathbb{F}^m, \mathbb{F}^n$

doesn't contain many information)

So we need to specify the basept $z_i \in Y_i$

and an arc $\delta \subset W$ connecting z_i

$\widehat{CF}(W,\delta) : \widehat{CF}(Y_0, z_0) \longrightarrow \widehat{CF}(Y_1, z_1)$

well-defined up to homotopy over $\mathbb{F} = \mathbb{Z}/2$ (from naturality)

In general, we consider disconnected 3-mfd

each component has a base pt

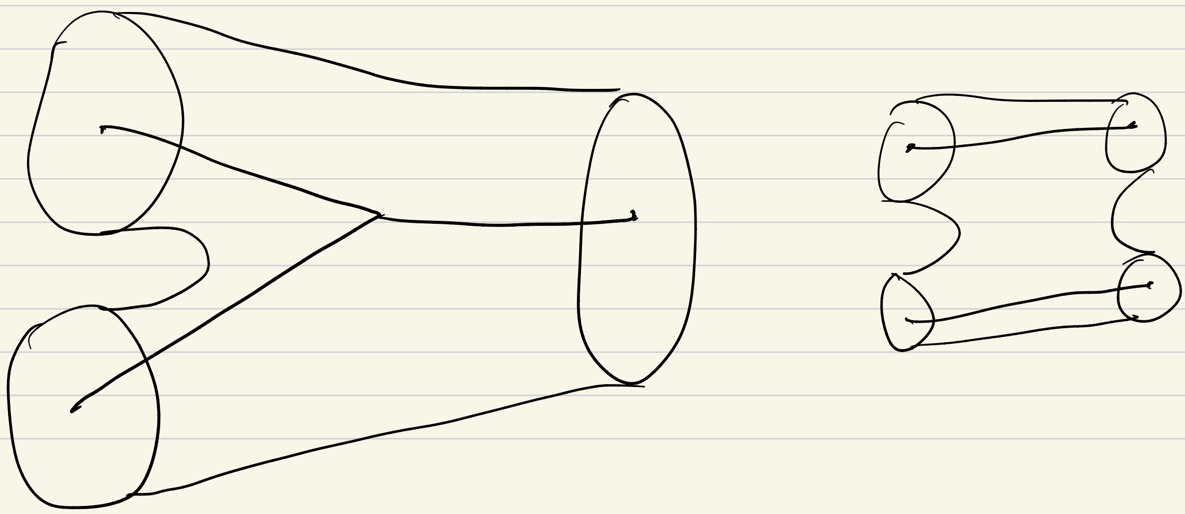
Define $\widehat{CF}(Y_1 \cup Y_2, z_1 \cup z_2) = \widehat{CF}(Y_1, z_1) \otimes_{\mathbb{F}} \widehat{CF}(Y_2, z_2)$

A cobordism (4d mfd W as above) disconnected mfds

together with some embedded graphs gives

well-defined chain maps by work of Zemke

Ex



For multi-based connected 3-mfd $(Y, z_1, \dots, z_n)$ data $\mathcal{H}$ consists of

- Σ connected closed oriented genus g surface
- $\alpha = \{\alpha_1, \dots, \alpha_{g+n-1}\}$ generates g-dim subspace of $H_1(\Sigma)$
- $\beta = \{\beta_1, \dots, \beta_{g+n-1}\}$ 〃 〃 〃
- $z_1, \dots, z_n$ in each component of $\Sigma \setminus \alpha$ and $\Sigma \setminus \beta$

$$\widehat{CF} = \mathbb{F}\langle T_\alpha \cap T_\beta \rangle \quad (\subset \mathrm{Sym}^{g+n-1}\Sigma) \qquad CF^- = \mathbb{F}[U_1, \dots, U_n]\langle T_\alpha \cap T_\beta \rangle$$

$$\widehat{\partial} x = \sum_{y} \sum_{\substack{\phi \in \pi_2(x,y) \\ \mu(\phi)=1 \\ n_{z_i}(\phi)=1 \text{ for all } z_i}} \left(\# \mathcal{M}(\phi)/\mathbb{R} \right) \cdot y$$

$$\partial^- x = \sum_{y} \sum_{\substack{\phi \in \pi_2(x,y) \\ \mu(\phi)=1}} \left(\# \mathcal{M}(\phi)/\mathbb{R} \right) \cdot \left(\prod U_i^{n_{z_i}(\phi)} \right) y$$

Rem over $\mathbb{Z}$, need orientation system for $\mathcal{M}(\phi)$

Canonical orientation by Alishahi–Eftekhary

Prop (OS) $CF^-(Y, z_1, z_2) \simeq$

$$\mathrm{Cone}\left(CF^-(Y, z_1) \otimes_{\mathbb{F}[U_1]} \mathbb{F}[U_1, U_2] \xrightarrow{U_1 - U_2} CF^-(Y, z_2) \otimes_{\mathbb{F}[U_2]} \mathbb{F}[U_1, U_2] \right)$$

Cor $\widehat{HF}(Y, z_1, \dots, z_n) \simeq \widehat{HF}(Y, z_1) \otimes \mathbb{F}^{2^{n-1}}$

$HF^-(Y, z_1, \dots, z_n) \simeq HF^-(Y, z_1)$ all U_i act as U_1

Prop (Zemke) There exists graph TQFT for multibased disconnected 3-mfd and cob with graphs

Rough idea of TQFT (proposed by Atiyah)

Cob_3 category with

obj closed oriented 3-mfd Y (possibly disconnected)

$Y_1 \otimes Y_2 = Y_1 \sqcup Y_2$ unit = empty mfd

mor cobordism W w/ $\partial W = -Y_0 \amalg Y_1$

$Vect_{\mathbb{F}}$ or $Mod_{\mathbb{Z}}$

Cat of $\mathbb{F}$-vector spaces or $\mathbb{Z}$-modules

A TQFT is a symmetric monoidal functor

$F : Cob_3 \longrightarrow Vect_{\mathbb{F}}$ or $Mod_{\mathbb{Z}}$

i.e. $F(Y)$ a vector space

$F(Y_1 \cup Y_2) = F(Y_1) \otimes_{\mathbb{F}} F(Y_2)$ $F(\phi) = \mathbb{F}$

$F(W) : F(Y_0) \longrightarrow F(Y_1)$ a linear map

$F(W_1 \cup_{Y_1} W_2) = F(W_2) \circ F(W_1)$

In HF theory, we need to add extra data (basepts, graph)

to obj and mor of Cob_3 to get Cob_3^{Γ}

$\widehat{HF} : Cob_3^{\Gamma} \longrightarrow Vect_{\mathbb{F}}$

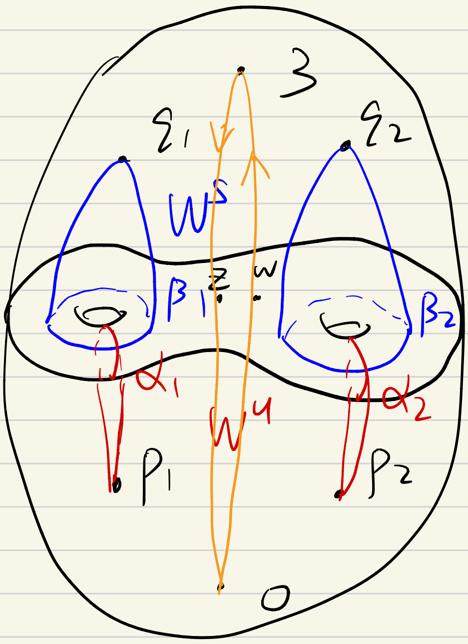
$HF^- : Cob_3^{\Gamma} \longrightarrow Mod_{\mathbb{F}[U]}$ when set all U_i be the same on CF^-

In this case $HF^-(Y, z_1, z_2) = HF^-(Y, z_1) \otimes \mathbb{F}[U]^2$

Class 7 Knot Floer homology

Generalization to knot $K \subset Y$ (OS, Rasmussen)

$$\underbrace{\{(Y,K)\}}_{\text{diff}} \xrightarrow{1-1} \underbrace{\{\text{doubly-pted Heegaard diagram producing } (Y,K)\}}_{\text{isotopy, handleslide, stabilization avoid basepts}}$$



$Y \uparrow f$ K = union of flow lines through z and w

orientation by $0 - w - 3 - z - 0$

equivalently, find an arc δ_1 in $\Sigma \setminus \alpha$ connecting z to w, push it into H_1

also arc δ_2 in $\Sigma \setminus \beta$, push it into H_2

$K = \delta_1 \cup (-\delta_2)$

Prop (OS) When K is null-homologous $([K]=0 \in H_1(Y;\mathbb{Z})$

or more generally rationally null-homologous (in $H_1(Y;\mathbb{Q})$),

the function $n_w : \pi_2(x,y) \longrightarrow \mathbb{Z}$

$$\phi \longmapsto \phi \cap \{w\} \times \mathrm{Sym}^{g-1}\Sigma$$

induces a filtration on $\widehat{CF}(Y)$ and $CF^-(Y)$

Let $\widehat{CFK}(Y,K)$ be the E_0 page for $\widehat{CF}(Y)$

for graded $gCFK^-(Y,K)$ be the E_0 page for $CF^-(Y)$

More explicitly. $$\partial_K x = \sum_y \sum_{\substack{\mu(\phi)=1 \\ n_z=n_w=0}} \#(\mathcal{M}(\phi)/\mathbb{R})\, y$$

$$\partial_K^- x = \sum_y \sum_{\substack{\mu(\phi)=1 \\ n_w=0}} \#(\mathcal{M}(\phi)/\mathbb{R})\, U^{n_z(\phi)} y$$

Let the homology (E_1 page) be denoted by

$\widehat{HFK}(Y,K)$ $HFK^-(Y,K)$

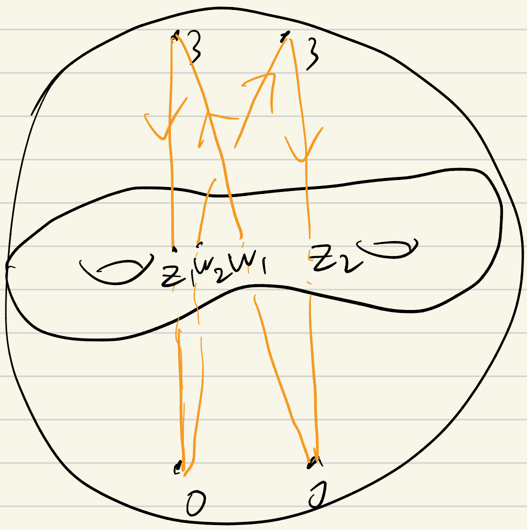
Sometimes we also consider the filtered cpx

$CFK^-(Y,K)$ which contains most information

We have similar isomorphism type result (OS) over $\mathbb{Z}$

and naturality result (JTZ) over $\mathbb{F}$ (specify z, w)

We can also consider multibased pt case



$Z = \{z_1, \cdots, z_n\}$ $W = \{w_1, \cdots, w_n\}$

$\alpha = \{\alpha_1, \cdots, \alpha_{g+n}\}$ $\beta = \{\beta_1, \cdots, \beta_{g+n}\}$

or link case (basepts on each component)

l - component has

$|Z| = |W| = l + n$

$|\alpha| = |\beta| = g + l - 1 + n$

originally defined over $\mathbb{F}$ by OS

upgrade to $\mathbb{Z}$ by Alishahi-Eftekhary

To achieve TQFT, Zemke introduce a (master) cpx

CFL^- or CFK^- over $\mathbb{F}[U_i, V_j]$

$$\partial_{UV} x = \sum_y \sum_{\mu(\phi)=1} \#(M(\phi)/\mathbb{R})\, U^{n_z(\phi)} V^{n_w(\phi)} y$$

However $(\partial_{\bar{UV}})^2$ is not always zero

For knot K with $z_1, w_1, z_2, w_2, \cdots$ in cyclic order

$$(\partial_{\bar{UV}})^2 = U_1 V_1 - V_1 U_2 + \cdots + U_n V_n - V_n U_1$$

Sign comes from AE's orientation

For link, similar term for each component

We may set $V_i = V$ (people usually do)

or $V_i = \dfrac{U_i^n - U_{i+1}^n}{U_i - U_{i+1}}$ (by Dowlin for sl_n-like HF)

to make $(\partial_{\bar{UV}})^2 = 0$

$V_i = 0$ recover OS's case

Zemke also constructed TQFT for (Y, K, z, w)

and graph cobordism (W, Σ, Γ) (over $\mathbb{F}$

Σ embedded oriented surface in W with $\partial\Sigma = -K_0 \sqcup K_1$

Now we move to sutured mfd (M, γ)

recall M is a compact oriented 3-mfd with boundary. $\gamma = A(\gamma) \cup T(\gamma) \subset \partial M$

a set of annuli and tori.

$s(\gamma)$ core of $A(\gamma)$, oriented, called <u>suture</u>

$R(\gamma) = \partial M \setminus \mathrm{int}\,\gamma = R_+(\gamma) \cup R_-(\gamma)$. determined by ori of $s(\gamma)$, ∂M.

Def. A sutured mfd (M, γ) is <u>balanced</u> if

- M has no closed component
- $T(\gamma) = \emptyset$, and all components of ∂M have sutures
  write γ for $s(\gamma)$
- (balanced condition) on each component.

$$\chi(R_+(\gamma)) = \chi(R_-(\gamma))$$

The reason to introduce such condition is because

$$\frac{\{\text{balanced } (M, \gamma)\}}{\text{diff}} \xleftrightarrow{1-1} \frac{\left\{\begin{matrix}\text{Sutured diagram } \mathcal{H} \\ \text{producing } (M,\gamma)\end{matrix}\right\}}{\text{iso, slide, stab}}$$

where a sutured diagram $\mathcal{H}$ consists of

Σ compact oriented surface with ∂ genus g
no closed component

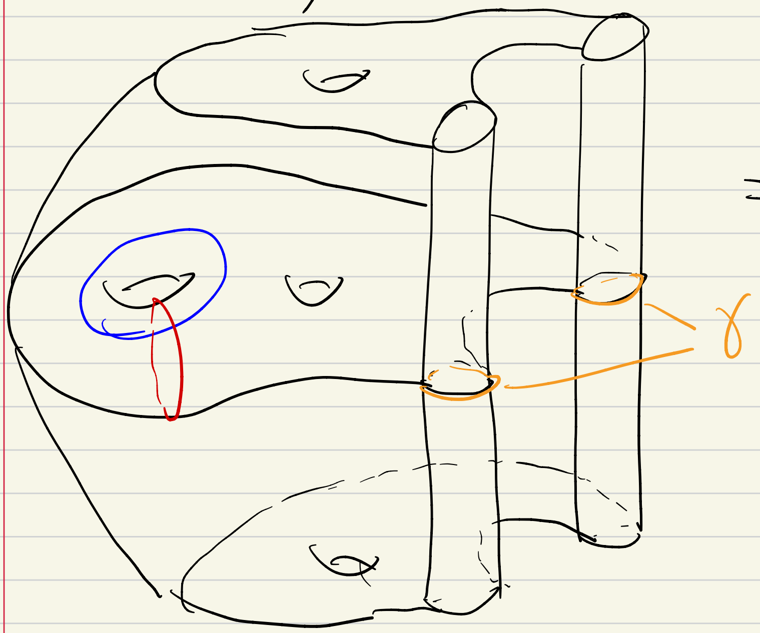
$|\alpha| = |\beta| = n$ (can different from $g(\Sigma)$ if Σ connected)

α_i linear indep in $H_1(\Sigma)$ similar for β_i

$M = \Sigma \times I \cup$ 2handles along α on $\Sigma \times \{-1\}$

$\cup$ 2 handles along β on $\Sigma \times \{+1\}$

γ (actually $s(\gamma)$) $= \partial \Sigma \times \{0\}$



$SFC(\mathcal{H}, J)$ $\quad \curvearrowright \mathrm{Sym}^n \Sigma$

$= \mathbb{Z}\langle T_\alpha \cap T_\beta \rangle$

with diff ∂ counting holo disk away from boundary.

Def (Juhász) $SFH(M, \gamma) = H(SFC(\mathcal{H}, J))$

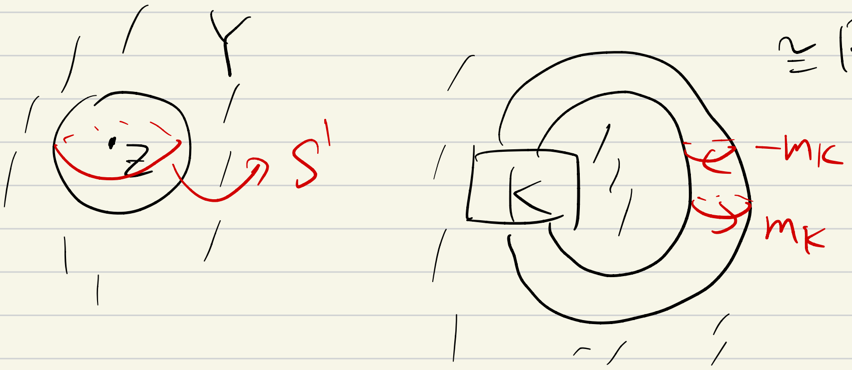
similar iso type and natural results apply.

Check the definition. we have

$SFH(Y \setminus \mathrm{int}\, B^3, S^1) \cong \widehat{HF}(Y, z)$

$SFH(Y \setminus \mathrm{int}\, N(K), \gamma_K = m_K \cup -m_K)$

$\cong \widehat{HFK}(Y, K, z, w)$



m_K meridian in positions of z, w.

Rem. It's possible to recover $HFK^-(Y, K)$ and even $CFK^-_{UV=0}(Y, K)$ by SFH, but currently unknown for CFK^-_{UV}

Class 8 Sutured Floer homology

Previously, we construct

- $HF^{\wedge,+,-,\infty}$ for (Y,z) and cobordism (W,Γ)
- $HFK^{\wedge,-}$, CFK^-, CFK_{UV} for (Y,K,z,w)
- SFH for balanced (M,γ)

Today, we first mention important properties of SFH, then state applications and generalization of HF theory

Though the construction is based on Heegaard diagram, many properties are shared by other Floer theories.

First, we recall a surface decomposition

$$(M,\gamma) \overset{S}{\rightsquigarrow} (M',\gamma') \text{ for some embedded } S$$

gives another sutured mfd.

Def A decomposition surface is balanced-admissible if

- S has no closed component
- For every component R_0 of R_+ or R_- the set of closed components in $S \cap R_0$ consists of parallel curves $\Big\{$ either representing nontrivial class in $H_1(R(\gamma))$ or as oriented boundary of a compact subsurface $R_1 \subset R(\gamma)$ with canonical ori

called boundary-coherent

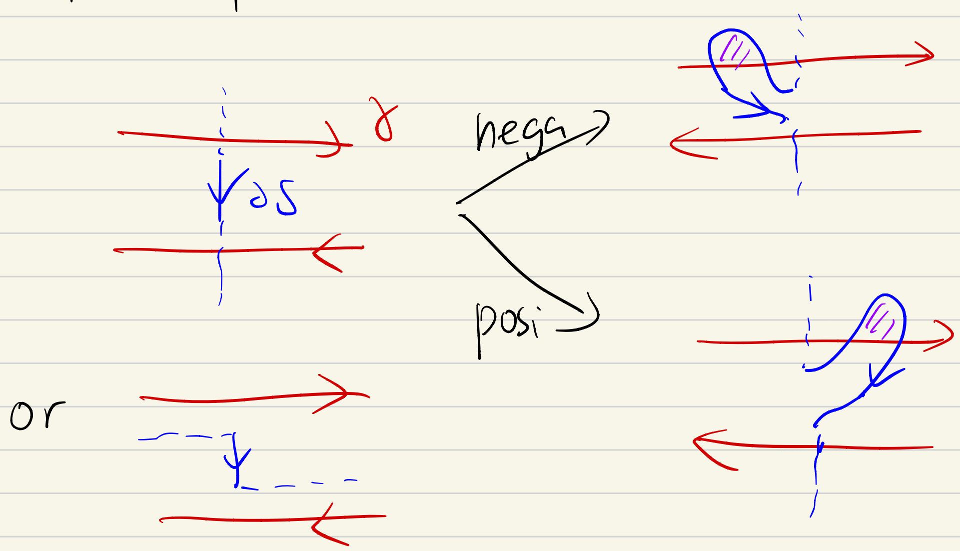
Ex. product disk $S \cong D^2$, $|\partial S \cap \gamma| = 2$

· product annulus $S \cong S^1 \times I$ $\partial S = C_+ \cup C_-$

$C_\pm \subset R_\pm(\gamma)$

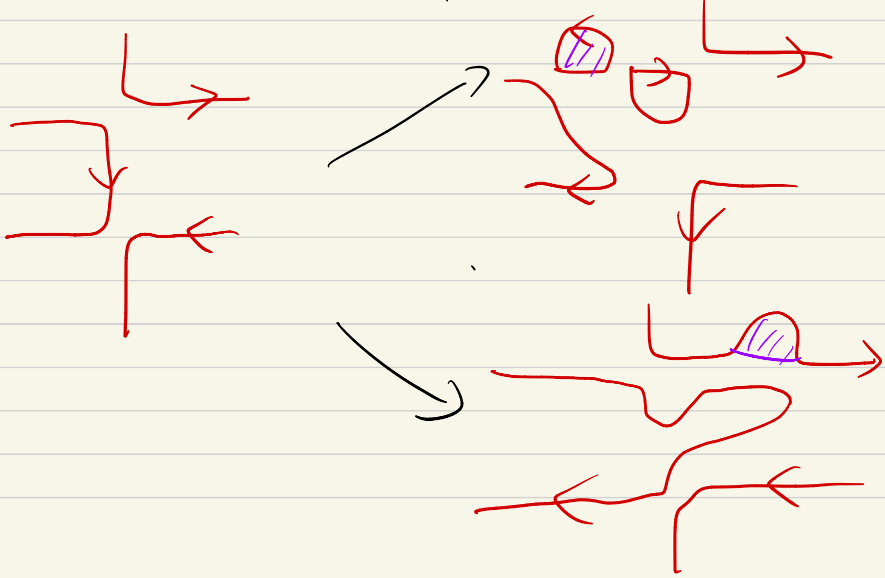
Lem If S is balanced-admissible in a balanced (M, γ), then (M', γ') after decomposition is also balanced. (omit pf)

Def. a positive/negative stabilization of S is



Lem Decomposing along S and S^+ produce diffeo sutured mfd. Decomposing along S^- produces nontaut sutured mfd since both $R_\pm$ compressible

Pf



We can always pertub S so that $\partial S \cap R_0$ are just arcs and hence boundary-coherent

(Juhász called S <u>good</u> if every component of ∂S intersects both $R_\pm$, which means $\partial S \cap R_0$ are arcs)

Thm (Juhász) If S is a balanced admissible surface in a balanced (M,γ) and we have $(M,\gamma) \overset{S}{\rightsquigarrow} (M',\gamma')$ then $SFH(M',\gamma')$ is a direct summand of $SFH(M,\gamma)$

If (M,γ) is taut (M has no closed component), there is a sutured hierarchy

$(M,\gamma) = (M_0,\gamma_0) \overset{S_1}{\rightsquigarrow} (M_1,\gamma_1) \overset{S_2}{\rightsquigarrow} \cdots \overset{S_n}{\rightsquigarrow} (M_n,\gamma_n)$

s.t. each S_i is connected and $S_i \cap$ a component of $R(\gamma_{i-1})$ consists of parallel oriented <u>nonseparating</u> curves or arcs

↳ imply nonzero in H_1 because $\exists$ curve δ intersects it once.

$(M_n,\gamma_n) = \bigsqcup (B^3, S^1)$

Cor. If (M,γ) is balanced and taut, then $SFH(M,\gamma) \neq 0$

Pf: $SFH(B^3,S^1) = \mathbb{Z}$ by direct computation

$SFH(M_n,\gamma_n) = \mathbb{Z}$ by tensor product

Juhász' thm implies $SFH(M_n,\gamma_n) \subset SFH(M,\gamma)$ $\square$

Rem. if (M,γ) is not taut but irreducible, we can construct an admissible Heegaard diagram s.t. $T_\alpha \cap T_\beta = \phi$ so $SFH(M,\gamma) = 0$ (due to Yi Ni in Juhász paper)

Thm If M irr, $SFH(M,\gamma) \neq 0$ iff (M,γ) is taut. $\square$

The second important thm is about product sutured mfd

$$(M,\gamma) = (R\times I, \partial R\times\{0\})$$

By direct computation $SFH(M,\gamma) \cong \mathbb{Z}$.

Thm (Yi Ni, Juhász) Suppose (M,γ) is balanced and a homology product ($i_*: H_*(R_\pm;\mathbb{Z}) \to H_*(M;\mathbb{Z})$ iso).

Then $SFH(M,\gamma) \cong \mathbb{Z}$ iff (M,γ) is product sutured mfd

These two thms provide alternative proofs of important properties of $\widehat{HFK}(K)$ for $K\subset S^3$ (or general Y)

Thm (OS) There exists a $\mathbb{Z}\oplus\mathbb{Z}$-grading on $\widehat{HFK}(K)$

One is Maslov grading from homological grading

The other is called Alexander grading

$$\sum \chi(\widehat{HFK}(K,j))\, t^j = \Delta_K(t) \quad \text{Alexander polynomial}$$

Moreover, it detects the genus $g(K)$

$$g(K) = \max\{\text{Alexander gr } j \mid \widehat{HFK}(K,j) \neq 0\}$$

Thm (Yi Ni) $\widehat{HFK}(K, g(K)) \cong \mathbb{Z}$ iff K is fibered

i.e. $S^3\setminus K$ fibered over S^1.

or equivalent $S^3\setminus N(S)$ for minimal genus Seifert surface is a product $S\times I$.

To see the relation more explicitly, we introduce spin^c-decomposition on SFH (and also HF, HFK)

We consider unit norm vector fields v on (M,γ)

s.t. • on R_+ v point out of M

• on R_- v point into M

• along $A(\gamma)$, v tangent to ∂M

Two such vector fields are <u>homologous</u> if $v_1|_{M \setminus B^3}$ is isotopic to $v_2|_{M \setminus B^3}$

Let $\mathrm{Spin}^c(M,\gamma)$ be the set of homology class of such vector fields.

We define $\mathrm{Spin}^c(Y) = \mathrm{Spin}^c(Y \setminus \mathrm{int} B^3, S^1)$

$\mathrm{Spin}^c(Y,K) = \mathrm{Spin}^c(Y \setminus \mathrm{int} N(K), \gamma_K)$

Lem. $\mathrm{Spin}^c(M,\gamma)$ is an affine space over

$$H^2(M,\partial M;\mathbb{Z}) \cong H_1(M;\mathbb{Z})$$

i.e. fix $\mathfrak{s}_0 \in \mathrm{Spin}^c(M,\gamma)$, we obtain a 1-1 correspondence

$$\mathrm{Spin}^c(M,\gamma) \longrightarrow H^2(M,\partial M)$$
$$\mathfrak{s} \longmapsto \mathfrak{s} - \mathfrak{s}_0$$

(a unit norm vector field determines a map $Y \to S^2$, consider the pull back of generator of $H^2(S^2)$)

Lem (OS, Juhász) There exists a map
$\mathfrak{s} : \mathbb{T}_\alpha \cap \mathbb{T}_\beta \longrightarrow \mathrm{Spin}^c(M,\gamma)$ s.t.

$\langle \partial x, y \rangle \neq 0$ only if $\mathfrak{s}(x) = \mathfrak{s}(y)$

Cor $SFH(M,\gamma) = \bigoplus_{\mathfrak{s} \in Spin^c(M,\gamma)} SFH(M,\gamma,\mathfrak{s})$

In the case for knot $H_1(Y \setminus N(K)) \cong \mathbb{Z}$

the spin^c decomposition is Alexander grading

(fixed by symmetry. i.e. $\widehat{HFK}(K,j) \cong \widehat{HFK}(K,-j)$)

In general, there is only $\mathbb{Z}/2d$ Maslov grading for a fixed $SFH(M,\gamma,\mathfrak{s})$.

and relative $\mathbb{Z}/2$ grading for $SFH(M,\gamma)$

enough for Euler characteristic χ

Thm (Juhász-Friedl-Rasmussen) Fix $\mathfrak{s}_0 \in Spin^c(M,\gamma)$

$$\sum_{\mathfrak{s} \in Spin^c(M,\gamma)} \chi(SFH(M,\gamma,\mathfrak{s}))(\mathfrak{s}-\mathfrak{s}_0) = \tau(M,\gamma)$$

$$\in \mathbb{Z}[H_1(M)] / \pm H_1(M)$$

$\tau(M,\gamma)$ some Turaev-type torsion

computed by fundamental group

This generalizes $\chi(\widehat{HFK}(K)) = \Delta_K(t)$

For $\mathfrak{s} \in Spin^c(M,\gamma)$ and some trivialization t of $v^\perp|\partial M$,

we define $c_1(\mathfrak{s},t) = c_1(v^\perp, t) \in H^2(M,\partial M)$

be the relative Euler class (or first chern class)

A surface S induces a trivialization t_S

Define $SFH(M,\gamma,S,i) = \bigoplus_{\langle c_1(\mathfrak{s},t_S),[S]\rangle = -2i} SFH(M,\gamma,\mathfrak{s})$

sign only for convention

Then Juhász's thm actually says for $(M,\gamma) \overset{S}{\rightsquigarrow} (M',\gamma')$

$$SFH(M',\gamma') \cong SFH(M,\gamma,S,\tfrac{1}{2}(\underbrace{\tfrac{1}{2}|\partial S\cap\gamma|}_{2}-\underbrace{\chi(S)}_{1-2g}))$$

Thm (Juhász) $SFH(M,\gamma,S,j)=0$

when $j > \frac{1}{2}(\frac{1}{2}|\partial S\cap\gamma|-\chi(S))$

This is so-called adjunction inequality

(use stabilization S^- nontaut)

In particular, when S is Seifert surface of K

with $g(S)=g(K)$

$$SFH(M',\gamma') \cong SFH(M,\gamma,S,g(S))$$
$$\cong \widehat{HFK}(K,g(K))$$

(M',γ') is taut. adjunction + taut $\neq 0$ implies

$\widehat{HFK}$ detects the genus

On the other hand, Yi Ni shows $\Delta_K(t)$ is monic implies

(M',γ') is homology product,

then $SFH(M',\gamma') \cong \mathbb{Z} \Leftrightarrow (M',\gamma')$ product

$\Leftrightarrow K$ fibered

More topics in HF theory:

- bordered Floer for M with parametrization of ∂M and gluing result for $\widehat{HF}(Y=M_1\cup_{\Sigma} M_2)$
- involutive HF for spin^c $\mathfrak{s}$ s.t. $\bar{\mathfrak{s}}=\mathfrak{s}$ (i.e. the vector field $-v$ homologous to v)