Class 18 surgery exact triangle

Recall Dehn surgery on knot:

$\partial(S^3 \setminus \mathrm{int}\, N(K)) \cong T^2$ has canonical basis

$m = \partial(pt \times D^2) \subset S^1 \times D^2 \cong N(K)$ called <u>meridian</u>

$l = \partial S$ for any Seifert surface called <u>longitude</u>

(It is also in the kernel of $H_1(\partial M) \hookrightarrow H_1(M)$)

Orient m, l s.t. $m \cdot l = -1$ (point into M)

Then we have

$$\mathbb{Q} \cup \{\tfrac{1}{0} = \infty\} \xleftrightarrow{1-1} \{\text{simple closed curves on } \partial(S^3 \setminus \mathrm{int}\, N(K))\}$$

$$p/q \qquad\qquad\qquad pm + ql$$

In particular $0 = \frac{0}{1} \longmapsto l = \partial S$

$$S^3_{p/q}(K) = (S^3 \setminus \mathrm{int}\, N(K)) \cup_\phi S^1 \times D^2$$

$$\phi(pt \times \partial D^2) = pm + ql$$

(doesn't depend on $\phi(S^1 \times pt)$)

$$H_1(S^3_{p/q}(K)) = \begin{cases} \mathbb{Z}/p & p \neq 0 \\ \mathbb{Z} & p = 0 \end{cases}$$

Thm (Property R conj. Gabai)

If $K \subset S^3$ is not the unknot $\bigcirc$.

then $S^3_0(K) \not\cong S^1 \times S^2$

Use taut $\Rightarrow$ $SHI \neq 0$ and excision, K/M proved

Thm (K/M) Y irreducible, $w \cdot R$ odd

Then $I^w(Y|R) \neq 0$

Cor $Y = S^3_0(K)$ R = Seifert surface $S \cup$ disk in $S^1 \times D^2$ along ∂S

w = meridian of $K \subset S^3_0(K)$

If $K \neq O$, then $I^w(Y|R) \neq 0$. In particular $I^w(Y) \neq 0$

Thm (Property P conj) If $K \neq O$, we have $S^3_{p/q}(K) \not\cong S^3$.

Sketch: $p/q \neq \pm 1$ by homology. $p/q = -1$ by mirror

$p/q = +1$, we use surgery exact triangle (introduce later)

$$\begin{tikzcd}
I^w(S^3_0(K)) \arrow[rr] & & I(S^3_1(K)) \arrow[dl] \\
& I(S^3) \arrow[ul] &
\end{tikzcd}$$

Floer homology using $SU(2)$ irreducible solutions

$I(S^3) = 0 \Rightarrow I(S^3_1(K)) \neq 0$

$\Rightarrow$ there must be an irreducible homomorphism (Im not abelian)

$\rho : \pi_1(S^3_1(K)) \to SU(2)$ (as generator of $I(S^3_1(K))$)

$\Rightarrow \pi_1(S^3_1(K)) \neq \{e\} \Rightarrow S^3_1(K) \not\cong S^3$. $\square$

We can generalize Dehn surgery to knot K in general 3-mfd Y. There are still two canonical curves on $\partial(Y \setminus N(K))$: write $M = Y \setminus N(K)$

homological longitude l : generator of $\ker(H_1(\partial M; \mathbb{Q}) \to H_1(M; \mathbb{Q}))$ (use half lives half dies)

meridian μ : $pt \times \partial D^2 \subset S^1 \times D^2 \cong N(K)$

But $\mu \cap l$ can have more than one pts.

We can either choose

- homological meridian m: a curve with $m \cdot l = -1$

or

- longitude λ: a curve with $\mu \cdot \lambda = -1$

Both are not canonical because $(m+l)\cdot l = -1$ $\mu\cdot(\mu+\lambda) = -1$

We will consider (μ, λ) as basis of ∂M.

λ is called a <u>framing</u> of K

Dehn surgery on a framed knot $K \subset Y$ is

$$Y_{p/q}(K) = Y \setminus N(K) \cup_\phi S^1 \times D^2$$

$$\phi(pt \times \partial D^2) = p\mu + q\lambda$$

- In the case K is null-homologous $([K] = 0 \in H_1(Y; \mathbb{Z}))$, there is a canonical choice of λ which is just l

- If $q = \pm 1$, it is called integral surgery. Let $n = p/q$

There is a standard surgery cobordism $W_n : Y \to Y_n(K)$ which is obtained from $Y \times I$ by attaching n-framed 4d 2-handle along $K \times \{1\}$.

Def a <u>surgery triad</u> is a triple of slopes

$(r_1, r_2, r_3) = (p_1/q_1, p_2/q_2, p_3/q_3) \in (\mathbb{Q} \cup \{\infty\})^3$

s.t. it is possible to orient the curves so that

$r_1 \cdot r_2 = r_2 \cdot r_3 = r_3 \cdot r_1 = -1$ $\quad r_i \cdot r_{i+1} = -\det\begin{pmatrix} p_1 & p_2 \\ q_1 & q_2 \end{pmatrix}$ because $\mu\cdot\lambda = -1$

Ex. $\left(\frac{1}{0}, \frac{n}{1}, \frac{-(n+1)}{-1}\right)$, $\left(\frac{0}{-1}, \frac{1}{n}, \frac{1}{n+1}\right)$

Thm (surgery exact triangle, Floer, OS, KM, $\cdots$)

Suppose K is a framed knot in Y.

Let (r_1, r_2, r_3) be a surgery triad.

Then there is a surgery exact triangle

in Floer homology over $\mathbb{F} = \mathbb{Z}/2$

$$\begin{tikzcd}
\widehat{HF}(Y_{r_1}(K)) \arrow[rr, "f_1"] & & \widehat{HF}(Y_{r_2}(K)) \arrow[dl, "f_2"] \\
& \widehat{HF}(Y_{r_3}(K)) \arrow[ul, "f_3"] &
\end{tikzcd}$$

where f_i are cobordism maps $\widehat{HF}(W_i)$

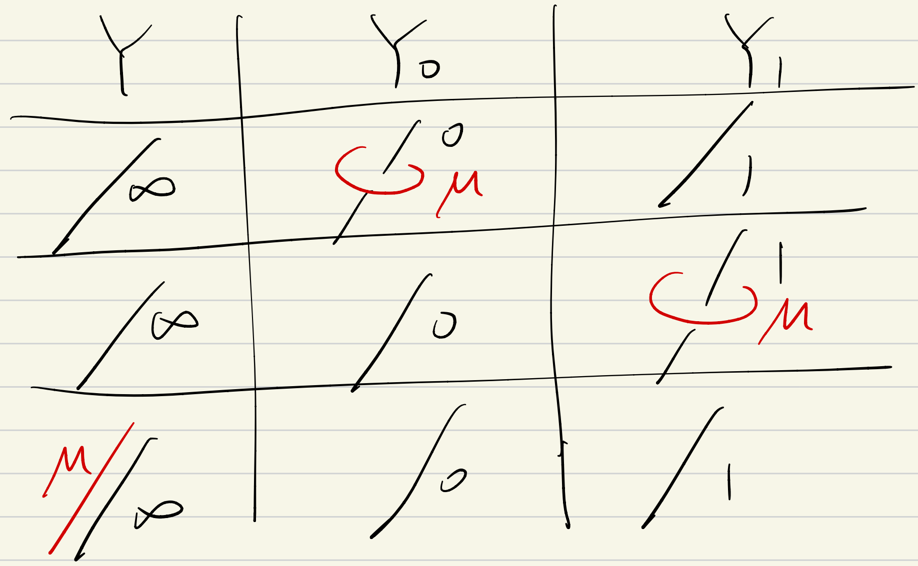
for surgery cobordisms.

$\widehat{HF}$ can be replaced by $\overset{\mathbb{F}}{HF^+}$, $\underbrace{\widetilde{HM}, \widehat{HM}}_{\mathbb{F} \text{ or } \mathbb{Q}}$

when consider instanton (over $\mathbb{Z}$), we need to add extra

1-cycle to one of the Floer homology, and also

extra 2-cycle to cobordism maps. (3 choices)



μ extra 1-cycle

Idea of pf: use the following triangle detection lemma

Lem (OS) Suppose we have chain maps (over $\mathbb{F}, \mathbb{Z}, \mathbb{Q}$)

$$\begin{tikzcd} C_1 \arrow[rr, "f_1"] & & C_2 \arrow[dl, "f_2"] \arrow[ll, bend left, "h_1"] \\ & C_3 \arrow[ul, "f_3"] \arrow[ur, bend right, "h_2"'] & \end{tikzcd}$$

$C_1 \xrightarrow{h_3} C_3$

and homotopies h_i and maps k_i

s.t. $f_{i+1} \circ f_i = \partial_{i+2} \circ h_{i+2} + h_{i+2} \circ \partial_i$

$f_i \circ h_i + h_{i+1} \circ f_{i+1} = \mathrm{Id} + \partial_{i+1} \circ k_{i+1} + k_{i+1} \circ \partial_{i+1}$

for all $i \in \mathbb{Z}/3$

Then there exists a canonical quasi-isomorphism from $\mathrm{Cone}(C_i \xrightarrow{f_i} C_{i+1})$ to C_{i+2} for $i \in \mathbb{Z}/3$

pf: We prove for $i=1$. We construct

$$F = (h_3, f_2) . \quad \mathrm{Cone}(C_1 \xrightarrow{f_1} C_2) \longrightarrow C_3$$

$$G = \begin{pmatrix} f_3 \\ h_2 \end{pmatrix} \quad C_3 \longrightarrow \mathrm{Cone}(C_1 \xrightarrow{f_1} C_2)$$

$F \circ G \simeq \mathrm{Id}$ by k_3

$$G \circ F = \begin{pmatrix} f_3 h_3 & f_3 f_2 \\ h_2 h_3 & h_2 f_2 \end{pmatrix} =$$

$$\begin{pmatrix} \mathrm{Id} & \\ * & \mathrm{Id} \end{pmatrix} + \begin{pmatrix} \partial & \\ -f_1 & -\partial \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} + \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \partial & \\ -f_1 & -\partial \end{pmatrix}$$

$f_3 f_2 = 0 + \partial B + B \partial \Rightarrow B = h_1$

$f_3 h_3 = \mathrm{Id} + \partial A + A \partial - B f_1 \Rightarrow A = k_1$

$h_2 f_2 = \mathrm{Id} - f_1 B - \partial D - D \partial \Rightarrow D = -k_2$

Class 19 Surgery triangle

Last time, we introduce surgery exact triangle for a triad

E.g.
$$\begin{tikzcd} \widehat{HF}(Y_{r_1}) \arrow[rr] & & \widehat{HF}(Y_{r_2}) \arrow[dl] \\ & \widehat{HF}(Y_{r_3}) \arrow[ul] & \end{tikzcd}$$

$Y_{r_i} = Y_{r_i}(K)$ Dehn surgery for framed knot K

Similar for HF^+, $\widetilde{HM}$, $\check{HM}$, I^ω (with extra μ)

There are multiple proofs in HF, I

one by energy filtration (omit), the other is by triangle detection lemma

Lem (OS) Suppose we have chain maps (over $\mathbb{F}, \mathbb{Z}, \mathbb{Q}$)

$$\begin{tikzcd} C_1 \arrow[rr, shift left, "f_1"] \arrow[dr, shift left, "h_3"] & & C_2 \arrow[ll, shift left, "h_1"] \arrow[dl, shift left, "f_2"] \\ & C_3 \arrow[ul, shift left, "f_3"] \arrow[ur, shift left, "h_2"] & \end{tikzcd}$$

and homotopies h_i and maps k_i

s.t. $f_{i+1}\circ f_i = \partial_{i+2}\circ h_{i+2} + h_{i+2}\circ \partial_i$

$f_i\circ h_i + h_{i+1}\circ f_{i+1} = \mathrm{Id} + \partial_{i+1}\circ k_{i+1} + k_{i+1}\circ \partial_{i+1}$

for all $i \in \mathbb{Z}/3$

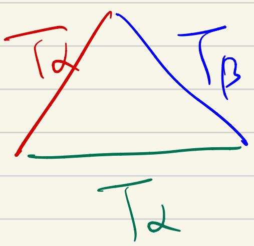
Then there exists a canonical quasi-isomorphism from $\mathrm{Cone}(C_i \xrightarrow{f_i} C_{i+1})$ to C_{i+2} for $i \in \mathbb{Z}/3$

$C_i = \widehat{CF}(Y_{r_i})$ $\quad f_i = \widehat{CF}(W_i)$ cobordism map

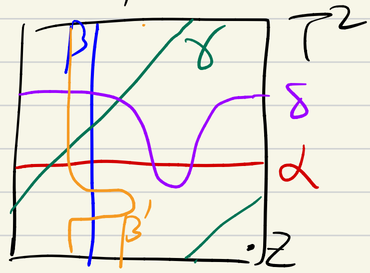
Concretely, we can construct doubly-pted Heegaard diagram $\mathcal{H}$ for (Y_{r_1}, K_{r_1}), where $K_{r_1} \subset S^1\times D^2$ is the core of the gluing solid torus. (core knot)

$\mathcal{H} = (\Sigma, \alpha, \beta, z, w)$ $\quad \beta = \{\beta_1, \cdots, \beta_g\}$ $\quad \beta_g$ meridian of K_{r_1}

Attaching 2-handles on $\Sigma \times I$ along α and $\beta \setminus \beta_g$,
we obtained 3-mfd with two boundaries,
one is a sphere (can be capped by B^3)
the other is a torus ($\partial N(K_{r_1})$)
We can find γ_g and δ_g on $\partial N(K_{r_1})$
as meridians of K_{r_2}, K_{r_3}
and isotopies of β_i for $i<g$ as γ_i, δ_i
so that $(\Sigma, \alpha, \gamma, z, w)$, $(\Sigma, \alpha, \delta, z, w)$
are diagrams for (Y_{r_2}, K_{r_2}), (Y_{r_3}, K_{r_3})
The cobordism map $\widehat{CF}(W_1)$ is obtained by
counting holomorphic triangle for $(\Sigma, \alpha, \beta, \gamma, z)$



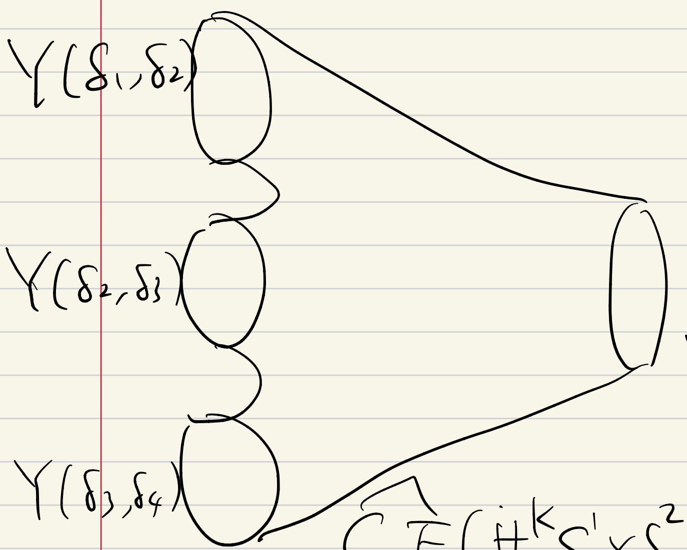
$\longrightarrow \mathrm{Sym}^g \Sigma$



The map h_3 is from 0d moduli space of
holomorphic 4-gons for $(\Sigma, \alpha, \beta, \gamma, \delta, z)$
The map k_1 is from 0d moduli space of
holomorphic 5-gons for $(\Sigma, \alpha, \beta, \gamma, \delta, \beta', z)$
β' isotopy of β.
boundary of 1-d moduli space
+ model cal for $(\Sigma, \beta, \gamma, \delta, \beta')$ $\Rightarrow$ equations in Lemma

In general, given $(\Sigma, \delta_1, \cdots, \delta_n, z)$,
δ_i are linear indep curves $|\delta_i| = g(\Sigma)$
We can consider moduli space of holomorphic n-gons to obtain a map

$$F: \widehat{CF}(\delta_1,\delta_2)\otimes\cdots\otimes\widehat{CF}(\delta_{n-1},\delta_n)\longrightarrow\widehat{CF}(\delta_1,\delta_n)$$



$Y(\delta_i,\delta_j)$ are mfds from $(\Sigma,\delta_i,\delta_j)$

usually
$Y(\delta_i,\delta_{i+1}) \cong \#^k S^1\times S^2$ $i>1$

$$\widehat{CF}(\#^k S^1\times S^2) = \widehat{HF}(\#^k S^1\times S^2) = \bigotimes_k H_*(S^1)$$

by direct computation. If we fix the top grading generator Θ_k of $\widehat{CF}(\#^k S^1\times S^2)$, then we define

$$f(x) = F(x\otimes\Theta_{k_1}\otimes\Theta_{k_2}\otimes\cdots)$$

Such map F satisfies A_∞ relation by boundary of 1d moduli space (associativity law with higher terms

e.g. $F_{\alpha\gamma\delta}(F_{\alpha\beta\gamma}(-\otimes-)\otimes-) - F_{\alpha\beta\delta}(-\otimes F_{\beta\gamma\delta}(-\otimes-))$
$= \partial F_{\alpha\beta\gamma\delta}(-\otimes-\otimes-) + F_{\alpha\beta\gamma\delta}\,\partial(-\otimes-\otimes-)$

model calculation means computing $F_{\beta\gamma\delta}(\Theta\otimes\Theta) = 0$ (mod 2)
$h = F_{\alpha\beta\gamma\delta}(-\otimes\Theta\otimes\Theta)$

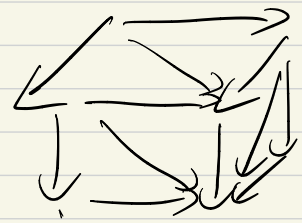
Class 20 Iterated mapping cone and spectral sequence

Give Y with a link $L = \coprod K_i \hookrightarrow Y$,

we can construct quasi-isomorphisms (not chain homotopy equ)

$$\widehat{CF}(Y) \overset{q.i.}{\simeq} \mathrm{Cone}\big(\widehat{CF}(Y_0(K_1)) \to \widehat{CF}(Y_1(K_1))\big)$$

$$\overset{q.i.}{\simeq} \mathrm{Cone}\left(\begin{tikzcd} \widehat{CF}(Y_{00}(K_1,K_2)) \arrow[r] \arrow[d] \arrow[dr] & \widehat{CF}(Y_{10}(K_1,K_2)) \arrow[d] \\ \widehat{CF}(Y_{01}(K_1,K_2)) \arrow[r] & \widehat{CF}(Y_{11}(K_1,K_2)) \end{tikzcd}\right)$$

$\overset{q.i.}{\simeq}$ Cone () $\overset{q.i.}{\simeq} \cdots$

$Y_{ij}(K_1,K_2)$ are i surgery on K_1, j surgery on K_2.

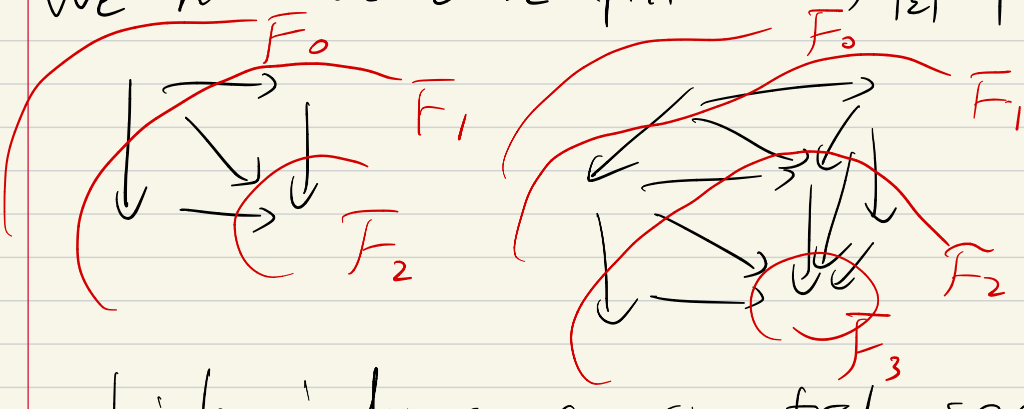
edge map comes from cobordism. (holo 3-gon)

face map comes from homotopy (holo 4-gon)

and so on.

For $\varepsilon \in [0,1]^n$, define $|\varepsilon|$ = number of 1.

we have a cube filtration $F_{|\varepsilon|}$ from $|\varepsilon|$



which induces a spectral sequence

with $E_0 = \bigoplus_{\varepsilon} (\widehat{CF}(Y_\varepsilon), d)$ sometimes computable.

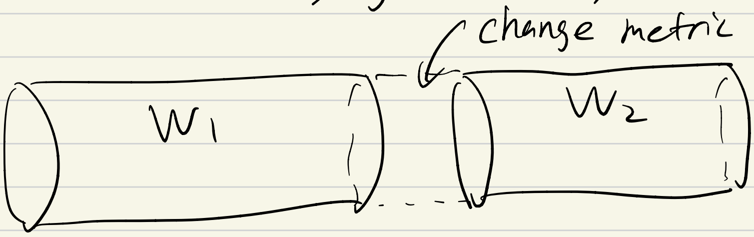
$E_1 = \left(\bigoplus_{\varepsilon} \widehat{HF}(Y_\varepsilon), \text{cobordism maps}\right)$ $\dim E_2 \geq \dim E_\infty = \widehat{HF}(Y)$

The analogs of holo n-gon maps in gauge theory (HM, I) are cobordism with families of metric.

e.g. for two cobordisms $W_1 : Y_0 \to Y_1$ $W_2 : Y_1 \to Y_2$

we construct a family of metric $g_t|_{t\in[0,\infty)}$ on $W = W_1 \cup_{Y_1} W_2$, which are identical away from $N(Y_1)$

when $t \mapsto \infty$, geometrically we obtain broken cobordisms.



We consider 0d moduli space of tuple (g_t, monopole/instanton), which also induces a map $h : C(Y_0) \to C(Y_2)$

Consider 1d moduli space $\Rightarrow$ h is a homotopy btw

$C(W)$ and $C(W_2) \circ C(W_1)$

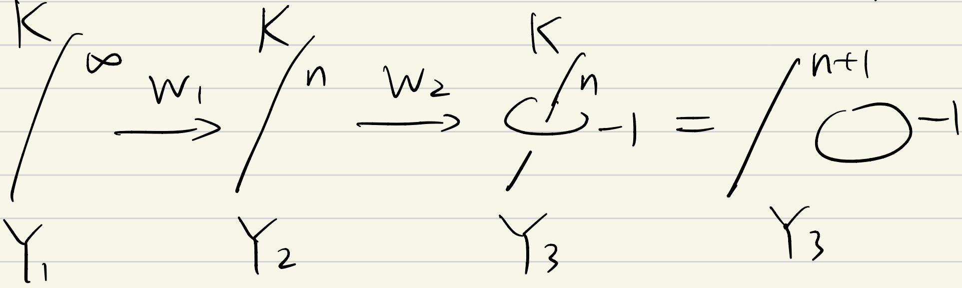
In the proof of surgery triangle, we notice

$$\begin{tikzcd} Y_1 \arrow[r, "W_1"] & Y_2 \arrow[dl, "W_2"] \\ Y_3 \arrow[u, "W_3"] & \end{tikzcd}$$

$W_1 \cup W_2 \cong (-W_3) \# \overline{\mathbb{CP}}^2$

$-W_3 : Y_1 \to Y_3$. opposite orientation.

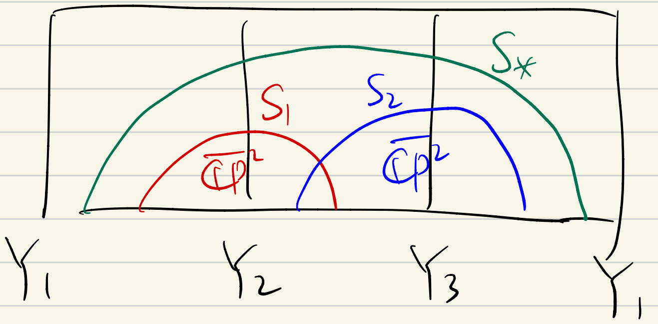
This follows from Kirby calculus



We needs to use the fact that there is no solution on $\overline{\mathbb{CP}}^2$ and hence $(-W_3) \# \overline{\mathbb{CP}}^2$. i.e. $C(-W_3 \# \overline{\mathbb{CP}}^2) = 0$

Then the family of metric along Y_1 gives the homotopy btw $f_2 \circ f_1$ and 0.

For K, we consider a family of metric on $W_1 \cup W_2 \cup W_3$

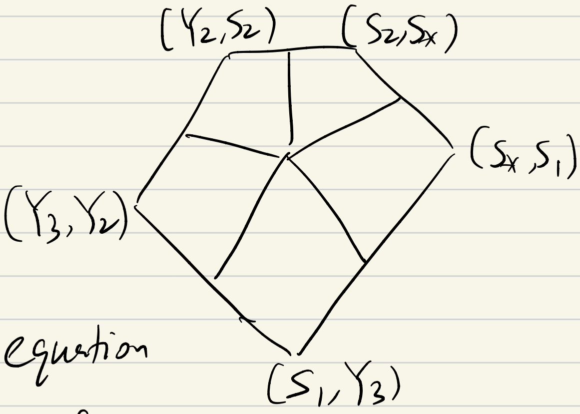


$S_1, S_2 \cong S^3 \quad S_* \cong S^1 \times S^2$

there are five pairs

(Y_2, S_2), (S_2, S_*), (S_*, S_1), (S_1, Y_3), (Y_3, Y_2)

We construct g_t^i for $i = 1, \dots, 5$, $t \in [0,\infty)^2$ and glue them together



We use 0d moduli space to construct the map k in detection lemma.

1d moduli space to show the equation

Here we need model calculation for

$$S_* \cong \partial N(\ {}_{-1}\ \ {}_{-1}) \cong \partial N(\ {}^{-1}\ \ {}^{0}) \cong S^1 \times S^2$$

This term contributes to Id

Similarly, we also have iterated mapping cone for L using family of metric with higher dimension.

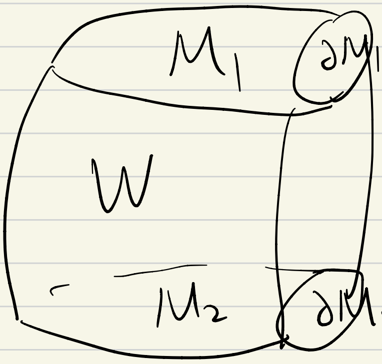
Note that the proofs only use the results near $N(L)$

So we can generalize the triangle to sutured homology if K or L is inside the 3-mfd M.

Class 22 Contact structure

Floer theory for closed 3-mfds and cobordisms satisfies some TQFT property. For sutured mfd (M,γ), there are two kinds of cobordism maps:

① 4d cobordism happens inside M, i.e. $\exists$ 4-mfd W with $\partial W = -M_1 \cup \partial M_2 \times I \cup M_2$ where $\partial M_1 \cong \partial M_2$.



In this case, we can construct the cobordism map similarly in the closed case i.e. in SFH, counting holo triangle in SHM, SHI, using cobordisms btw closures

② 3d cobordism happens near ∂M.

i.e. a 3-mfd N with $\partial N = -\partial M_1 \cup \partial M_2$

s.t. $M_2 = M_1 \cup_{\partial M_1} N$, together with sutures $\gamma_i \subset \partial M_i$

In particular. $N = \partial M \times I$, but $\gamma_1 \neq \gamma_2$

In the case, we need extra data on N to define cobordism map $SFH(M_1,\gamma_1) \to SFH(M_2,\gamma_2)$ which is a contact structure ξ with dividing set γ_i

For simplicity, we won't mention details in contact geometry and the construction but focus on properties shared by all three sutured homologies

Def. A 3d <u>contact mfd</u> (N, ξ) consists of

- N 3d mfd (possibly with boundary)
- $\xi \subset TY$ an (oriented) plane field bundle (rk 2 subbuddle) that can be written as $\ker\alpha$ for a 1-form α satisfying $\alpha\wedge d\alpha \neq 0$ such α is called <u>contact form</u>

Note that for and $\lambda : N \to \mathbb{R}_{>0}$

$$\lambda\alpha\wedge d(\lambda\alpha) = \overbrace{\lambda\alpha\wedge d\lambda\wedge\alpha}^{0} + \lambda^2\alpha\wedge d\alpha \neq 0$$

We consider ξ rather than α.

When Y is oriented, we usually need $\alpha\wedge d\alpha > 0$ (called <u>positive</u>)

Ex. $(\mathbb{R}^3, \xi_{std} = \ker(dz - y\,dx))$ standard contact str.

Rem. Similar to symplectic geometry, there is Darboux's thm in contact geometry roughly saying nbhd of a pt in a contact mfd is contactomorphic to the standard one in $\mathbb{R}^3$

Def A <u>contact vector field</u> v is a vector field whose local flow preserves ξ. An embedded surface $\Sigma \subset N$ is called <u>convex</u> if there exists a nbhd $N(\Sigma)$ in which a contact vector field exists and transverse to Σ.

Given v, Σ, let $\Gamma_\Sigma = \{x \in \Sigma \mid v(x) \in \xi_x\}$ be the <u>dividing set</u>, which are a set of closed curves

Rem. Any Σ is isotopic to a convex surface.

In particular, when $\partial N \neq \phi$, we can make ∂N convex

The condition of convex surface implies Contact structure in

$N(\Sigma) \cong \Sigma \times \mathbb{R}$ is invariant in $\mathbb{R}$ direction

Γ_Σ is roughly the set of pts where ξ perpendicular to Σ

Given Γ_Σ, we can construct $\mathbb{R}$-inv contact strcuture

on $\Sigma \times \mathbb{R}$ with dividing set Γ_Σ.

Def. Given a sutured mfd (M, γ) (or $(N, \gamma_1 \cup \gamma_2)$),

a contact str ξ is called compatible if $\Gamma_{\partial M} = \gamma$.

Def. A knot $K \subset (N, \xi)$ is called Legendrian if

it is tangent to ξ. It is called transverse if

it is transeverse to ξ.

Contact structure induces a framing on knot,

by Eliashberg, (-1)-surgery on a Legendrian knot

produces another contact mfd.

Def A contact str ξ on a closed Y is called Stein fillable

if there is a Stein domain (holomorphic subset of $\mathbb{C}^2$)

with boundary Y, and the induced contact str ξ.

There are also other fillable conditions (weak/strong symp fillable)

Rem A stein fillable (Y, ξ) is the result of contact (-1)-surgery

on some Legendrian link in $S^1 \times S^2$ with a standard ξ.

There are definitions for ξ called tight and overtwisted.
We omit def and only metion that
- ξ is either tight or overtwisted (OT)
- In each homotopy class of plane field, there exists a unique OT ξ. (classification by Eliashberg)
- Classification of tight contact str are hard in general, but done on $S^3, S^1\times S^2, L(p,q), T^3$
$S^1\times D^2, T^2\times I, \Sigma\times I$ (all with some restrictions on $\Gamma_{\partial M}$)

Results about Floer homology (by OS, KM, HKM, BS, etc)
Given (N,ξ), there is an element
$c(N,\xi)\in SFH(-N,-\Gamma_{\partial N})$ called contact elements.
opposite orientation

Similar for $HF^+(-Y)$ (can have twisted coefficients)
$\widecheck{HM}(-Y)$ $SHM(-N,-\Gamma_{\partial N})$, $SHI(-N,-\Gamma_{\partial N})$

Note that we need naturality result to specify the element, or we can only say $c=0$ / $c\neq 0$ in isomorphism class

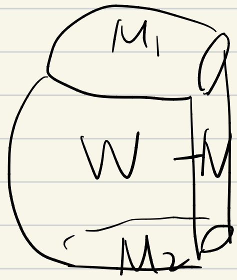
It satisfies the following properties.
- $c=0$ for OT ξ (more vanishing result for Giroux torsion)
- $c\neq 0$ for Stein-fillable ξ (more nonvanishing results for different versions)
- (N',ξ') is the result of $(+1)$-surgery on K from (N,ξ)
the cobordism map $F: SFH(-N,-\Gamma)\to SFH(-N',-\Gamma')$ sends $c(N,\xi)$ to $c(N',\xi')$

Class 23 contact gluing map

Def. A sutured cobordism from (M_1,γ_1) to (M_2,γ_2) consists of $\mathcal{W}=(W,N,\xi)$

- W cpt 4-mfd $\partial W \setminus \text{int} N = -M_1 \cup M_2$
- N cpt 3-mfd $\partial N = \partial M_1 \cup (-\partial M_2)$
- ξ (oriented) contact str on N.

  dividing set $-\gamma_1 \cup \gamma_2$ on ∂N



A cobordism map

$$SFH(\mathcal{W}): SFH(M_1,\gamma_1) \longrightarrow SFH(M_2,\gamma_2)$$

is the composition $F_W \circ \Phi_{-\xi}$

$$\Phi_{-\xi}: SFH(M_1,\gamma_1) \longrightarrow SFH(M_1 \cup (-N), \gamma_2)$$

$$F_W: SFH(M_1 \cup (-N), \gamma_2) \longrightarrow SFH(M_2,\gamma_2)$$

We can define F_W similar to cobordism map for closed 3-mfd. The map $\Phi_{-\xi}$ is called a contact gluing map, with multiple equivalent definitions

Last time, we mention contact element $c(N,\xi) \in SFH(-N, -\Gamma_{\partial N})$ which is preserved by (+1)-contact surgery on Legendrian knot, vanishes for OT, is nonvanishing for Stein fillable ξ

In contact geometry, it is used to distinguish contact strs.

There are also multiple ways to define

- in HF by Oszváth-Szabó using fibered knot K in open book decomposition compatible with ξ

$$\left(\begin{array}{l} \widehat{HFK}(-Y,K,-g(K))\cong\mathbb{Z}\longrightarrow\widehat{HF}(-Y) \\ \text{or } HF^+(-Y_0(K),g-1)\cong\mathbb{Z}\longrightarrow HF^+(-Y) \end{array}\right)$$

- in SFH by Honda-Kazez-Matić using Heegaard diagram compatible with (partial) open book decomposition of ξ

 (pick up specific generator)

- in HM by Kronheimer-Mrowka using monopole solutions on a 4-mfd W with $\partial W=Y$, constructed by ξ.

- in SHM by Baldwin-Sivek using closure (Y,R) of $(N,\Gamma_{\partial N})$ with contact structure ξ_Y extends ξ and apply KM's construction

- in SHI by BS using 3d contact handle attaching maps sending generator of $SHI(\text{product})\cong\mathbb{C}$ to $SHI(-N,-\xi)$

We focus on the last way (which can also be applied to SFH, SHM) because it is also used for the contact gluing map Φ_ξ

More explicitly, given (N,ξ), we can decompose it into contact 0,1,2,3-handles $C_1,\cdots,C_n$ in the order

We construct map Φ_{C_i} for each handle attachments.
and then $\Phi_\xi = \Phi_{C_n} \circ \Phi_{C_{n-1}} \circ \cdots \circ \Phi_{C_1}$

Prop (Zhenkun Li). The above construction is independent of the choice of the decomposition.

Concrete construction. Let (M_i, γ_i) be the resulting for i-handle from (M,γ), $H_i : SHI(-M,-\gamma) \to SHI(-M_i, \gamma_i)$

$\boxed{\text{0-handle}}$ $(M,\gamma) \longrightarrow (M_1,\gamma_1) = (M,\gamma) \sqcup (B^3, S^1)$

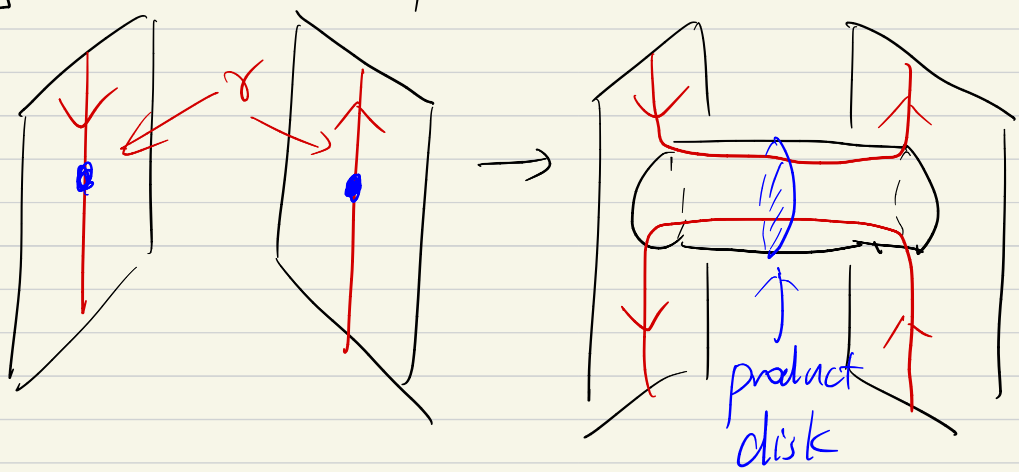
$H_0 = id : SHI(-M,-\gamma) \longrightarrow SHI(-M,-\gamma) \otimes \underbrace{SFH(-B^3, S^1)}_{\mathbb{C}}$

$\boxed{\text{3-handle}}$ $(M,\gamma) \xrightarrow{\text{diffeo}} (M',\gamma') = (M_3,\gamma_3) \# (B^3, S^1)$

$H_3 =$ composition $\xrightarrow{\text{4d 1-handle}} (M_3,\gamma_3) \sqcup (B^3, S^1) \xrightarrow{H_0^{-1}} (M_3, \gamma_3)$

The bottom two are more important

$\boxed{\text{1-handle}}$ Choose two pts on γ

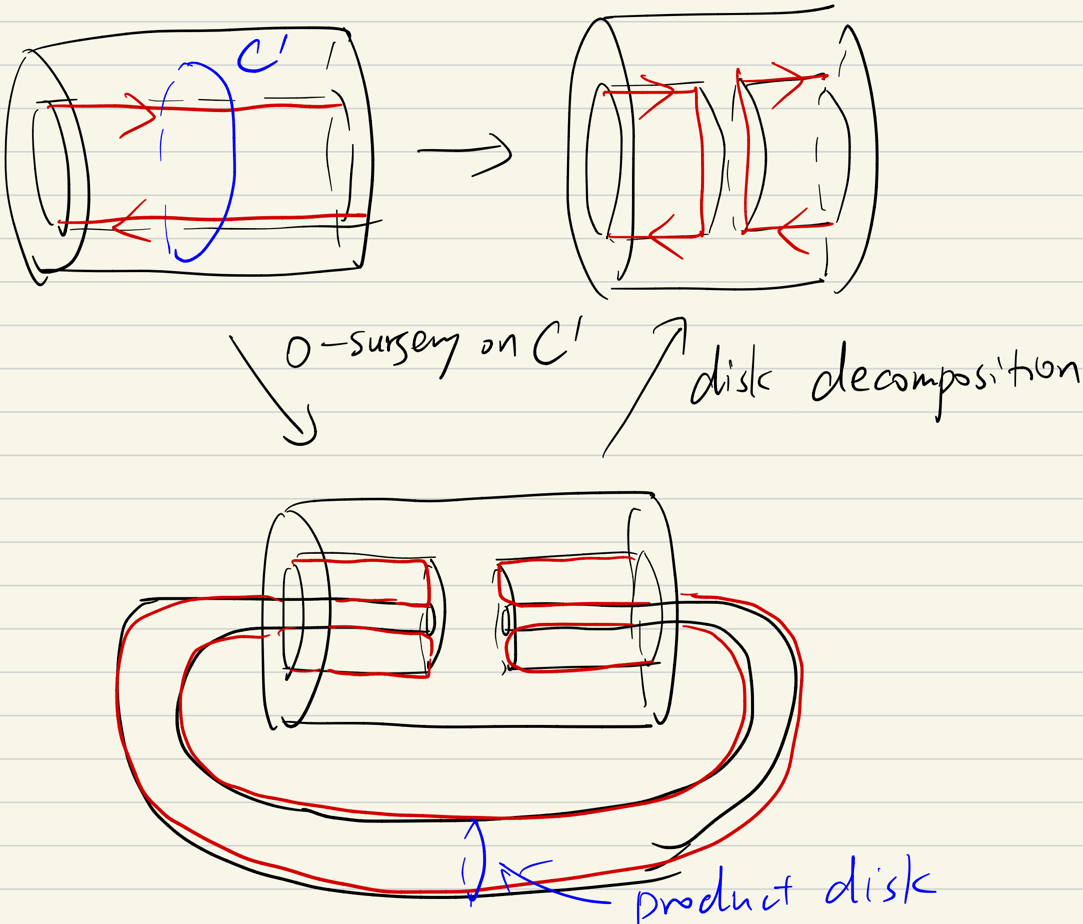


inverse of product disk decomposition

We can find a closure of both (M,γ) and (M_1,γ_1)

so H_1 is the canonical map btw the closures

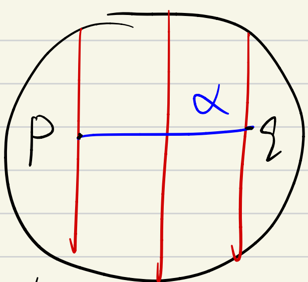
2-handle $|C \cap \gamma| = 2$ push C into M, get C'



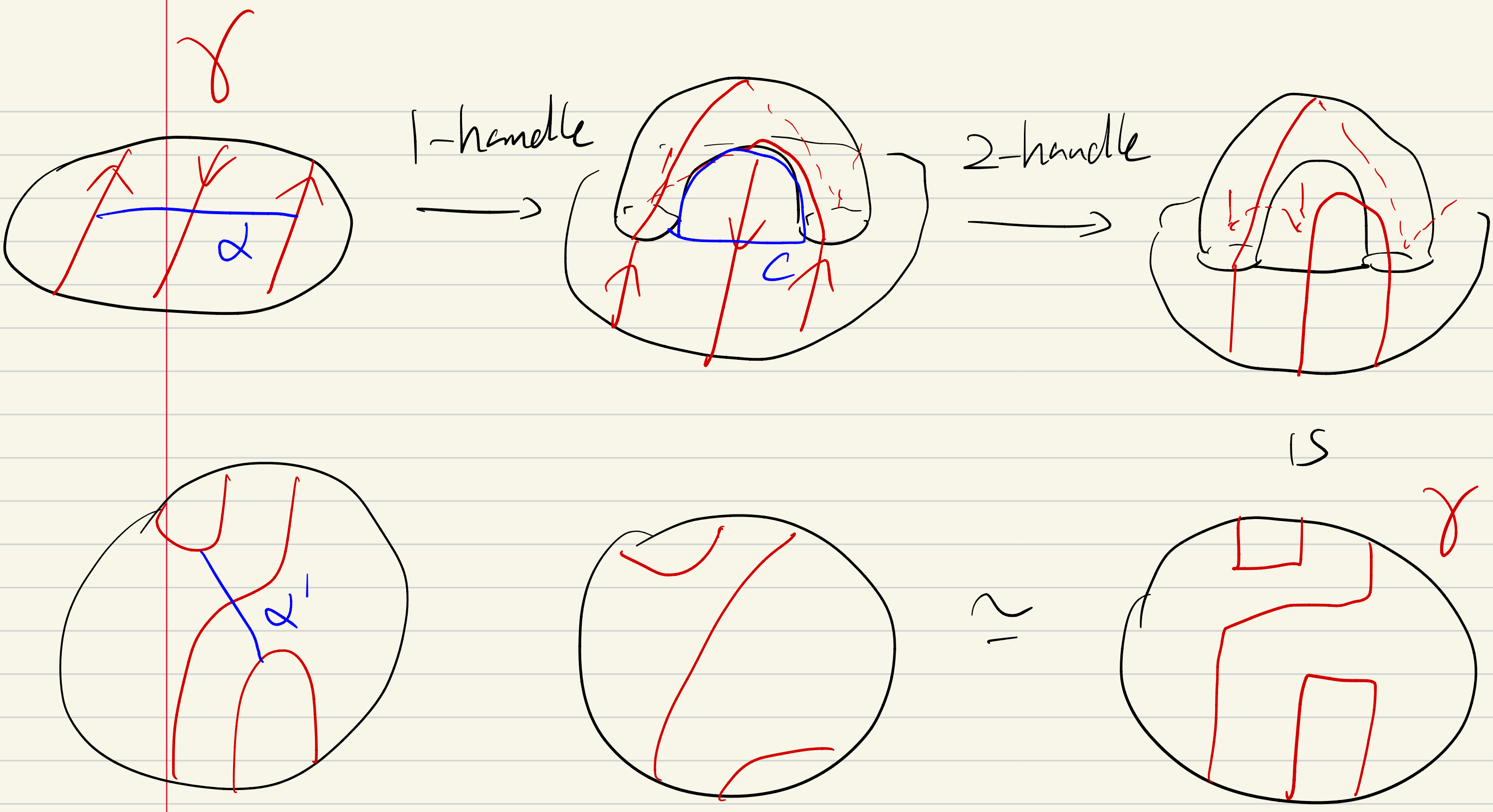
H_2 is the composition of 4d 0-surgery cobordism and the disk decomposition.

Bypass attachment: Suppose $\alpha \subset \partial M$ is an arc with endpts p, q on γ, $|\alpha \cap \gamma| = 3$

locally, we have



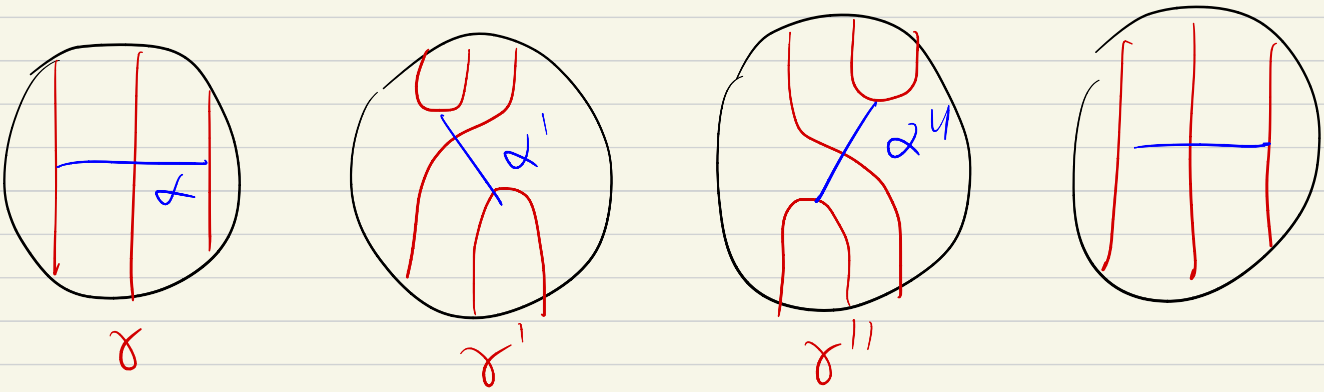
A <u>bypass attachment</u> along α is the composition of contact 1-handle along p, q, and contact 2-handle along $\alpha \cap$ core of 1 handle.



Note that the 3-mfd M is the same but γ are different.

Apply the bypass attachment again to the new arc α', we obtain the third suture γ'', and arc α''.

Apply the bypass attachment to γ'', α'', we obtain the original suture γ



Thm (Baldwin-Sivek) There exists an exact triangle

$$\begin{tikzcd} SHI(-M,-\gamma) \arrow[rr] & & SHI(-M',-\gamma') \arrow[dl] \\ & SHI(-M'',-\gamma'') \arrow[ul] & \end{tikzcd}$$

Similiar for SHM

Idea of pf: in suitable closure, a bypass attaching map is just a surgery cobordism map.

We can find good closure s.t. the above triangle is just the surgery exact triangle.

For instanton, we need to use some trick to remove the extra bundle data μ

Rem Similar result holds for SFH, but since it is not defined by closure, the proof is different. Indeed, it uses bordered sutured Floer homology and local computation by Etnyre-Vela-Vick-Zarev.