

Class 18 Surgery exact triangle

Recall Dehn surgery on knot:

$\partial(S^3 \setminus \text{int } N(K)) \cong T^2$ has canonical basis

$m = \partial(pt \times D^2) \subset S^1 \times D^2 \cong N(K)$ called meridian

$L = \partial S$ for any Seifert surface called longitude

(It is also in the kernel of $H_1(\partial M) \hookrightarrow H_1(M)$)

Orient m, L s.t. $m \cdot L = -1$ (point into M)

Then we have

$$\mathbb{Q} \cup \{\frac{1}{0} = \infty\} \xleftrightarrow{1-1} \left\{ \begin{array}{c} \text{simple closed curves on } \partial(S^3 \setminus \text{int } N(K)) \\ p m + q L \end{array} \right.$$

In particular $0 = \frac{0}{1} \mapsto L = \partial S$

$$S^3_{p/q}(K) = (S^3 \setminus \text{int } N(K)) \cup_{\phi} S^1 \times D^2$$

$$\phi(pt \times \partial D^2) = p m + q L$$

(doesn't depend on $\phi(S^1 \times pt)$)

$$H_1(S^3_{p/q}(K)) = \begin{cases} \mathbb{Z}/p & p \neq 0 \\ \mathbb{Z} & p = 0 \end{cases}$$

Thm (Property R conj. Gabai)

If $K \subset S^3$ is not the unknot $\bigcirc$.

then $S^3_0(K) \not\cong S^1 \times S^2$

Use taut $\Rightarrow$ $\text{SHI} \neq 0$ and excision, K/U proved
 Thm (K/U) Y irreducible, $w \cdot R$ odd

Then $I^w(Y/R) \neq 0$

Cor $Y = S^3_0(K)$ $R =$ Seifert surface $S \cup \text{disk in } S^1 \times D^2$
 along ∂S
 $w =$ meridian of $K \subset S^3_0(K)$

If $K \neq \emptyset$, then $I^w(Y/R) \neq 0$. In particular $I^w(Y) \neq 0$

Thm (Property P conj) If $K \neq \emptyset$, we have $S^3_{p/q}(K) \neq S^3$.

Sketch: $p/q \neq \pm 1$ by homology. $p/q = -1$ by mirror

$p/q = +1$, we use surgery exact triangle (introduce later)

$$\begin{array}{ccc} I^w(S^3_0(K)) & \rightarrow & I(S^3_1(K)) \\ \uparrow & & \downarrow \\ & I(S^3) & \end{array} \quad \begin{array}{l} \leftarrow \text{Floor homology using } SU(2) \\ \text{irreducible solutions} \end{array}$$

$$I(S^3) = 0 \Rightarrow I(S^3_1(K)) \neq 0$$

$\Rightarrow$ there must be an irreducible homomorphism (I is not abelian)

$$\rho = \pi_1(S^3_1(K)) \rightarrow SU(2) \quad (\text{as generator of } I(S^3_1(K)))$$

$$\Rightarrow \pi_1(S^3_1(K)) \neq \{e\} \Rightarrow S^3_1(K) \neq S^3. \quad \square$$

We can generalize Dehn surgery to knot K in general 3-mfd Y . There are still two canonical curves on $\partial(Y \setminus N(K))$: write $M = Y \setminus N(K)$

homological longitude L : generator of $\ker(H_1(\partial M; \mathbb{Q}) \rightarrow H_1(M; \mathbb{Q}))$

meridian μ : $\text{pt} \times \partial D^2 \subset S^1 \times D^2 \cong N(K)$ (use half lives half dies)

But $\mu \cap L$ can have more than one pts.

We can either choose

- homological meridian m : a curve with $m \cdot l = -1$
- or • longitude λ : a curve with $\mu \cdot \lambda = -1$

Both are not canonical because $(m+l) \cdot l = -1$ $\mu \cdot (\mu + \lambda) = -1$

We will consider (μ, λ) as basis of ∂M .

λ is called a framing of K

Dehn surgery on a framed knot $K \subset Y$ is

$$Y_{p/q}(K) = Y \setminus N(K) \cup_{\phi} S^1 \times D^2$$

$$\phi(pt \times \partial D^2) = p\mu + q\lambda$$

- In the case K is null-homologous ($[K] = 0 \in H_1(Y; \mathbb{Z})$), there is a canonical choice of λ which is just l
- If $q = \pm 1$, it is called integral surgery. Let $n = p/q$. There is a standard surgery cobordism $W_n: Y \rightarrow Y_n(K)$ which is obtained from $Y \times I$ by attaching n -framed 4d 2-handle along $K \times \{1\}$.

Def a surgery triad is a triple of slopes

$$(r_1, r_2, r_3) = (p_1/q_1, p_2/q_2, p_3/q_3) \in (\mathbb{Q} \cup \{\infty\})^3$$

s.t. it is possible to orient the curves so that

$$r_1 \cdot r_2 = r_2 \cdot r_3 = r_3 \cdot r_1 = -1 \quad r_i \cdot r_{i+1} = -\det \begin{pmatrix} p_1 & p_2 \\ q_1 & q_2 \end{pmatrix}$$

because $\mu \cdot \lambda = -1$

$$\text{Ex. } \left(\frac{1}{0}, \frac{n}{1}, \frac{-(n+1)}{-1} \right), \left(\frac{0}{-1}, \frac{1}{n}, \frac{1}{n+1} \right)$$

Thm (surgery exact triangle, Floer, OS, KM, ...)

Suppose K is a framed knot in Y .

Let (r_1, r_2, r_3) be a surgery triad.

Then there is a surgery exact triangle in Floer homology over $\mathbb{F} = \mathbb{Z}/2$

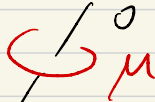
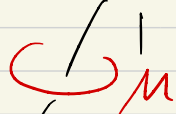
$$\begin{array}{ccc} \widehat{HF}(Y_{r_1}(K)) & \xrightarrow{f_1} & \widehat{HF}(Y_{r_2}(K)) \\ & \nwarrow f_3 \quad \searrow f_2 & \\ & \widehat{HF}(Y_{r_3}(K)) & \end{array}$$

where f_i are cobordism maps $\widehat{HF}(W_i)$

for surgery cobordisms.

$\widehat{HF}$ can be replaced by $\widehat{HF}^+$, $\widehat{HM}$, $\widehat{HM}$ For $\mathbb{Q}$

When consider instanton (over $\mathbb{Z}$), we need to add extra 1-cycle to one of the Floer homology, and also extra 2-cycle to cobordism maps. (3 choices)

Y	Y_0	Y_1
∞		1
∞	0	
μ/∞	0	1

μ extra
1-cycle

Idea of pf: use the following triangle detection lemma

Lem (OS) Suppose we have chain maps (over $\mathbb{F}, \mathbb{Z}, \mathbb{Q}$)

$$\begin{array}{ccc}
 C_1 & \xrightarrow{f_1} & C_2 \\
 \nwarrow f_3 & \searrow f_2 & \\
 & C_3 &
 \end{array}
 \quad
 \begin{array}{c}
 \text{and homotopies } h_i \text{ and maps } k_i \\
 \text{sit. } f_{i+1} \circ f_i = \partial_{i+2} \circ h_{i+2} + h_{i+2} \circ \partial_i \\
 f_i \circ h_i + h_{i+1} \circ f_{i+1} = \text{Id} + \partial_{i+1} \circ k_{i+1} + k_{i+1} \circ \partial_{i+1} \\
 \text{for all } i \in \mathbb{Z}/3
 \end{array}$$

Then there exists a canonical quasi-isomorphism from $\text{Core}(C_i \xrightarrow{f_i} C_{i+1})$ to C_{i+2} for $i \in \mathbb{Z}/3$

pf. We prove for $i=1$. We construct

$$F = (h_3, f_2): \text{Core}(C_1 \xrightarrow{f_1} C_2) \rightarrow C_3$$

$$G = \begin{pmatrix} f_3 \\ h_2 \end{pmatrix}: C_3 \rightarrow \text{Core}(C_1 \xrightarrow{f_1} C_2)$$

$$F \circ G \simeq \text{Id} \text{ by } k_3$$

$$G \circ F = \begin{pmatrix} f_3 h_3 & f_3 f_2 \\ h_2 h_3 & h_2 f_2 \end{pmatrix} =$$

$$\begin{pmatrix} \text{Id} & \\ * & \text{Id} \end{pmatrix} + \begin{pmatrix} \partial & \\ -f_1 & -\partial \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} + \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \partial & \\ -f_1 & -\partial \end{pmatrix}$$

$$f_3 f_2 = 0 + \partial B + B \partial \Rightarrow B = h_1$$

$$f_3 h_3 = \text{Id} + \partial A + A \partial - B f_1 \Rightarrow A = k_1$$

$$h_2 f_2 = \text{Id} - f_1 B - \partial D - D \partial \Rightarrow D = -k_2$$

Class 19 Surgery triangle

Last time, we introduce surgery exact triangle for a triad

$$\text{Eg. } \widehat{HF}(Y_1) \rightarrow \widehat{HF}(Y_2) \quad Y_i = Y_i(K) \text{ Dehn surgery for framed knot } K$$

$$\quad \quad \quad \nwarrow \quad \quad \quad \searrow$$

$$\quad \quad \quad \widehat{HF}(Y_3)$$

Similar for HF^+ , $\widetilde{HM}$, $\widetilde{HM}$, I^w (with extra μ)

There are multiple proofs in HF , I
one by energy filtration (omit), the other is by
triangle detection lemma

Lem (OS) Suppose we have chain maps (over $\mathbb{F}, \mathbb{Z}, \mathbb{Q}$)

$$C_1 \xrightarrow{f_1} C_2 \quad \text{and homotopies } h_i \text{ and maps } k_i$$

$$\quad \quad \quad \nwarrow \quad \quad \quad \searrow$$

$$f_3 \quad C_3 \quad f_2$$

s.t. $f_{i+1} \circ f_i = \partial_{i+2} \circ h_i + h_{i+2} \circ \partial_i$

$$f_i \circ h_i + h_{i+1} \circ f_{i+1} = Id + \partial_{i+1} \circ k_i + k_{i+1} \circ \partial_{i+1}$$

for all $i \in \mathbb{Z}/3$

Then there exists a canonical quasi-isomorphism
from $\text{Cone}(C_i \xrightarrow{f_i} C_{i+1})$ to C_{i+2} for $i \in \mathbb{Z}/3$

$$C_i = \widehat{CF}(Y_{r_i}) \quad f_i = \widehat{CF}(W_i) \text{ cobordism map}$$

Concretely, we can construct doubly-pted Heegaard
diagram $\mathcal{H}$ for (Y_{r_i}, K_{r_i}) , where $K_{r_i} \subset S^1 \times D^2$
is the core of the gluing solid torus. (core knot)

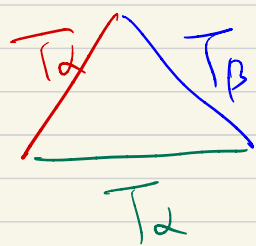
$$\mathcal{H} = (\Sigma, \alpha, \beta, z, w) \quad \beta = [\beta_1, \dots, \beta_g] \quad \beta_g \text{ meridian of } K_{r_i}$$

Attaching 2-handles on $\Sigma \times I$ along α and $\beta \setminus \beta_g$,
 we obtained 3-mfld with two boundaries,
 one is a sphere (can be capped by B^3)
 the other is a torus ($\partial N(K_{r_1})$)

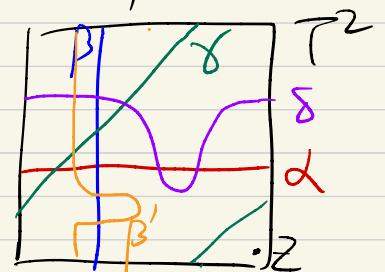
We can find γ_g and δ_g on $\partial N(K_{r_1})$
 as meridians of K_{r_2}, K_{r_3}
 and isotopies of β_i for $i < g$ as γ_i, δ_i
 so that $(\Sigma, \alpha, \gamma, z, w), (\Sigma, \alpha, \delta, z, w)$

are diagrams for $(Y_{r_2}, K_{r_2}), (Y_{r_3}, K_{r_3})$

The cobordism map $\widehat{CF}(W_1)$ is obtained by
 counting holomorphic triangle for $(\Sigma, \alpha, \beta, \gamma, z)$



$\longrightarrow \text{Sym}^S \Sigma$



The map h_3 is from 0-d moduli space of
 holomorphic 4-gons for $(\Sigma, \alpha, \beta, \gamma, \delta, z)$

The map k_1 is from 0-d moduli space of
 holomorphic 5-gons for $(\Sigma, \alpha, \beta, \gamma, \delta, \beta', z)$
 β' isotopy of β .

boundary of 1-d moduli space

+ model cal for $(\Sigma, \beta, \gamma, \delta, \beta')$ $\Rightarrow$ equations in Lemma

In general, given $(\Sigma, \delta_1, \dots, \delta_n, z)$,

δ_i are linear indep curves $|\delta_i| = g(\Sigma)$

We can consider moduli space of holomorphic n -gons to obtain a map

$$F: \widehat{CF}(\delta_1, \delta_2) \otimes \dots \otimes \widehat{CF}(\delta_{n-1}, \delta_n) \rightarrow \widehat{CF}(\delta_1, \delta_n)$$

$Y(\delta_i, \delta_j)$ are mfd's from $(\Sigma, \delta_i, \delta_j)$

$Y(\delta_i, \delta_{i+1}) \cong \#^k S^1 \times S^2 \quad i \geq 1$

$$\widehat{CF}(\#^k S^1 \times S^2) = \widehat{HF}(\#^k S^1 \times S^2) = \bigotimes_k H_*(S^1)$$

by direct computation. If we fix the top grading generator Θ_k of $\widehat{CF}(\#^k S^1 \times S^2)$, then we define

$$f(x) = F(x \otimes \Theta_{k_1} \otimes \Theta_{k_2} \otimes \dots)$$

Such map F satisfies A_∞ relation

by boundary of 1d moduli space

(associativity law with higher terms

$$\begin{aligned} \text{e.s. } F_{\alpha\beta\gamma\delta} (F_{\alpha\beta\gamma}(- \otimes -) \otimes -) - F_{\alpha\beta\gamma\delta} (- \otimes F_{\beta\gamma\delta}(- \otimes -)) \\ = \partial F_{\alpha\beta\gamma\delta} (- \otimes - \otimes -) + F_{\alpha\beta\gamma\delta} \partial (- \otimes - \otimes -) \end{aligned}$$

$$\begin{aligned} \text{model calculation means computing } F_{\beta\gamma\delta}(\Theta \otimes \Theta) &= 0 \\ h = F_{\alpha\beta\gamma\delta}(- \otimes \Theta \otimes \Theta) &\quad (\text{mod } 2) \end{aligned}$$

Iterated mapping cone and spectral sequence

Give Y with a link $L = \coprod K_i \hookrightarrow Y$,

We can construct quasi-isomorphisms (not chain homotopy eqn)

$$\begin{aligned} \widehat{CF}(Y) &\stackrel{q.i.}{\simeq} \text{Cone}(\widehat{CF}(Y_0(K_1)) \rightarrow \widehat{CF}(Y_1(K_1))) \\ &\stackrel{q.i.}{\simeq} \text{Cone} \left(\begin{array}{ccc} \widehat{CF}(Y_{00}(K_1, K_2)) & \rightarrow & \widehat{CF}(Y_{10}(K_1, K_2)) \\ \downarrow & \searrow & \downarrow \\ \widehat{CF}(Y_{01}(K_1, K_2)) & \rightarrow & \widehat{CF}(Y_{11}(K_1, K_2)) \end{array} \right) \\ &\stackrel{q.i.}{\simeq} \text{Cone} \left(\begin{array}{ccc} & \rightarrow & \\ \swarrow & & \searrow \\ \downarrow & \rightarrow & \downarrow \\ & \rightarrow & \end{array} \right) \stackrel{q.i.}{\simeq} \dots \end{aligned}$$

$Y_{ij}(K_1, K_2)$ are i surgery on K_1 , j surgery on K_2 .

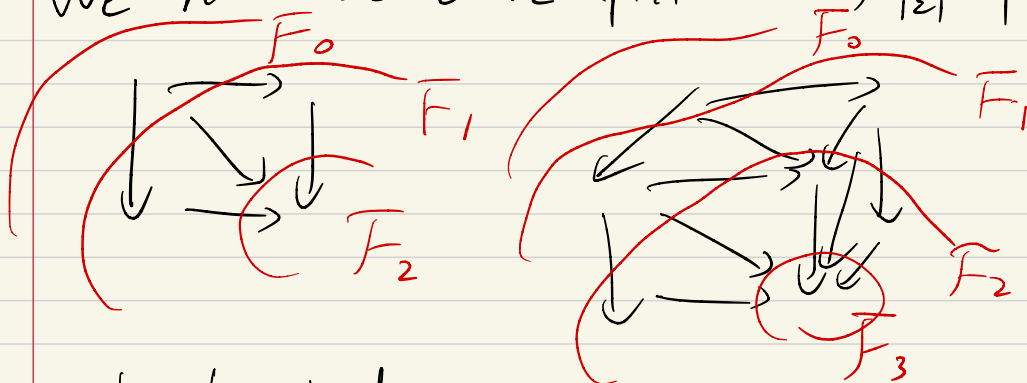
edge map comes from cobordism. (holo 3-gon)

face map comes from homotopy (holo 4-gon)

and so on.

For $\Sigma \in [0, 1]^n$, define $|\Sigma|$ = number of 1.

We have a cube filtration $F_{|\Sigma|}$ from $|\Sigma|$



which induces a spectral sequence

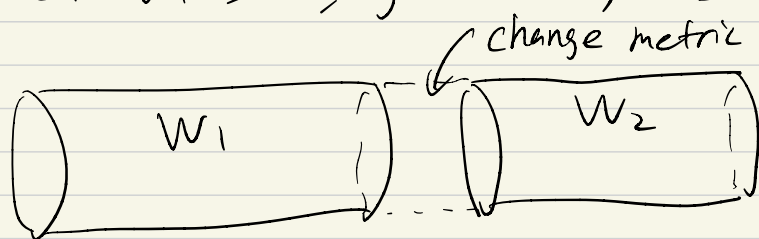
with $E_0 = \bigoplus_{\Sigma} (\widehat{CF}(Y_{\Sigma}), d)$ sometimes computable.

$E_1 = (\bigoplus_{\Sigma} \widehat{HF}(Y_{\Sigma}), \text{cobordism meps})$ $\dim E_2 \geq \dim E_{\infty} = \widehat{HF}(Y)$

The analogs of holo n -gon maps in gauge theory (HM, I) are cobordism with families of metric.

e.g. for two cobordisms $W_1: Y_0 \rightarrow Y_1$ $W_2: Y_1 \rightarrow Y_2$ we construct a family of metric $g_t|_{t \in [0, \infty)}$ on $W = W_1 \cup_{Y_1} W_2$, which are identical away from $N(Y_1)$

when $t \rightarrow \infty$, geometrically we obtain broken cobordisms.



We consider $0d$ moduli space of tuple $(g_t, \text{monopole/instanton})$, which also induces a map $h: C(Y_0) \rightarrow C(Y_2)$

consider $1d$ moduli space $\Rightarrow h$ is a homotopy btw

$$C(W) \text{ and } C(W_2) \circ C(W_1)$$

In the proof of surgery triangle, we notice

$$Y_1 \xrightarrow{W_1} Y_2 \quad W_1 \cup W_2 \cong (-W_3) \# \overline{\mathbb{CP}}^2$$

$$\begin{matrix} W_3^K \\ \swarrow \\ Y_3 \end{matrix} \quad \searrow W_2 \quad -W_3: Y_1 \rightarrow Y_3 \quad \text{opposite orientation.}$$

This follows from Kirby calculus

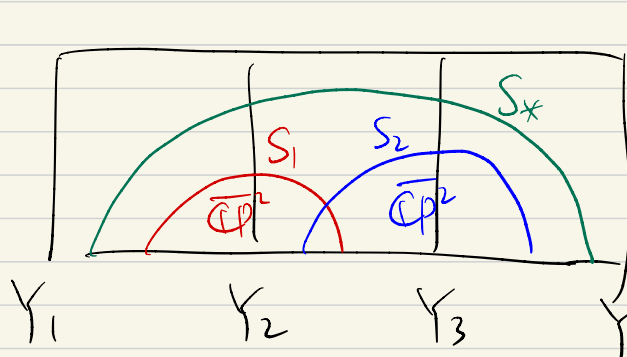
$$\begin{matrix} K \\ \swarrow \\ Y_1 \end{matrix} \xrightarrow{\infty, W_1} \begin{matrix} K \\ \swarrow \\ Y_2 \end{matrix} \xrightarrow{n, W_2} \begin{matrix} K \\ \swarrow \\ Y_3 \end{matrix} \xrightarrow{-1} \begin{matrix} K \\ \swarrow \\ Y_3 \end{matrix} = \begin{matrix} n+1 \\ \swarrow \\ Y_3 \end{matrix}$$

We need to use the fact that there is no solution on $\overline{\mathbb{CP}}^2$ and hence

$$(-W_3) \# \overline{\mathbb{CP}}^2, \text{ i.e. } C(-W_3 \# \overline{\mathbb{CP}}^2) = 0$$

Then the family of metric along Y_1 gives the homotopy btw $f_2 \circ f_1$ and 0.

For K , we consider a family of metric on $W_1 \cup W_2 \cup W_3$



$$S_1, S_2 \cong S^3 \quad S_* \cong S^1 \times S^2$$

there are five pairs

$$(Y_2, S_2), (S_2, S_*), (S_*, S_0)$$

$$(S_0, Y_3), (Y_3, S_1)$$

We construct g_t^i for $i=1, \dots, 5$, $t \in [0, \infty)^2$

and glue them together

We use Id moduli space to construct the map k in detection lemma.

Id moduli space to show the equation

Here we need modal calculation for

$$S_* \cong \partial N(\bigcirc_{-1}) \cong \partial N(\bigcirc^{-1} \bigcirc^0) \cong S^1 \times S^2$$

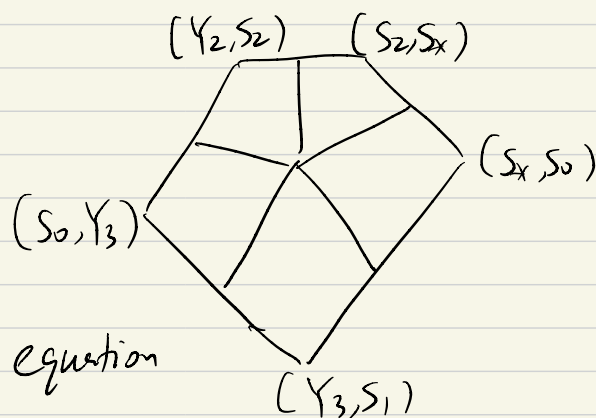
This term contributes to Id

Similarly, we also have iterated mapping cone for L using family of metric with higher dimension.

Note that the proofs only use the results near $N(L)$

So we can generalize the triangle to Sutured homology

if K or L is inside the 3-mfd M .



Class 18 contact structure

Floer theory for closed 3-mfd and cobordisms

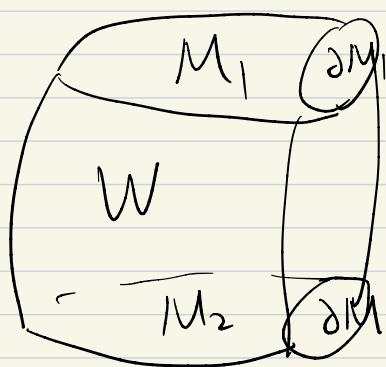
satisfies some TQFT property. For sutured mfd (M, γ) ,

there are two kinds of cobordism maps:

① 4d cobordism happens inside M ,

i.e. $\exists$ 4-mfd W with $\partial W = -M_1 \cup \partial M_2 \times I \cup M_2$

where $\partial M_1 \cong \partial M_2$.



In this case, we can construct the cobordism map similarly in the closed case i.e. in SFH, counting holomorphic triangle in SHM, SHI, using cobordisms btw closures

② 3d cobordism happens near ∂M .

i.e. a 3-mfd N with $\partial N = -\partial M_1 \cup \partial M_2$

s.t. $M_2 = M_1 \cup_{\partial M_1} N$, together with sutures $\gamma_i \subset \partial M_i$

In particular. $N = \partial M \times I$, but $\gamma_1 \neq \gamma_2$

In the case, we need extra data on N to

define cobordism map $SFH(M_1, \gamma_1) \rightarrow SFH(M_2, \gamma_2)$

which is a contact structure γ with dividing set γ_i

For simplicity, we won't mention details in contact

geometry and the construction but focus on

properties shared by all three sutured homologies

Def. A 3d contact mfd $(N, \{ \})$ consists of

- N 3d mfd (possibly with boundary)
- $\{ \subset TY$ an (oriented) plane field bundle (rk 2 subbundle) that can be written as $\ker \alpha$ for a 1-form α satisfying $\alpha \wedge d\alpha \neq 0$ such α is called contact form

Note that for any $\lambda: N \rightarrow \mathbb{R}$

$$\lambda \alpha \wedge d(\lambda \alpha) = \lambda \alpha \wedge d\lambda \wedge \alpha + \lambda^2 \alpha \wedge d\alpha \neq 0$$

We consider $\{ \}$ rather than α .

When Y is oriented, we usually need $\alpha \wedge d\alpha > 0$
(called positive)

Ex. $(\mathbb{R}^3, \{ \}_{\text{std}} = \ker(dz - ydx))$ standard contact str.

Rem. Similar to symplectic geometry, there is Darboux's thm in contact geometry roughly saying nbhd of a pt in a contact mfd is contactomorphic to the standard one in $\mathbb{R}^3$

Def A contact vector field V is a vector field whose local flow preserves $\{ \}$. An embedded surface $\Sigma \subset N$ is called convex if there exists a nbhd $N(\Sigma)$ in which a contact vector field exists and transverse to Σ . Given V, Σ , let $\Gamma_\Sigma = \{x \in \Sigma \mid V(x) \in \{x\}\}$ be the dividing set, which are a set of closed curves

Rem. Any Σ is isotopic to a convex surface:

In particular, when $\partial N \neq \emptyset$, we can make ∂N convex

The condition of convex surface implies contact structure in

$N(\Sigma) \cong \Sigma \times \mathbb{R}$ is invariant in $\mathbb{R}$ direction

Γ_Σ is roughly the set of pts where ζ perpendicular to Σ

Given Γ_Σ , we can construct $\mathbb{R}$ -inv contact structure

on $\Sigma \times \mathbb{R}$ with dividing set Γ_Σ .

Def. Given a sutured mfd (M, γ) (or $(N, \gamma_1 \cup \gamma_2)$), a contact str ζ is called compatible if $\Gamma_{\partial M} = \gamma$.

Def. A knot $K \subset (N, \zeta)$ is called Legendrian if it is tangent to ζ . It is called transverse if it is transverse to ζ .

Contact structure induces a framing on knot,

by Eliashberg, (-1) -surgery on a Legendrian knot produces another contact mfd.

Def A contact str ζ on a closed Y is called Stein fillable if there is a Stein domain (holomorphic subset of $\mathbb{C}^2$) with boundary Y , and the induced contact str ζ .

There are also other fillable conditions (weak/strong symplectic fillable)

Rem A Stein fillable (Y, ζ) is the result of contact (-1) -surgery on some Legendrian link in $S^1 \times S^2$ with a standard ζ .

There are definitions for $\{$ called tight and overtwisted.
We omit def and only mention that

- $\{$ is either tight or overtwisted (OT)
- In each homotopy class of plane field, there exists a unique OT $\{$. (classification by Eliashberg)
- Classification of tight contact str are hard in general, but done on $S^3, S^1 \times S^2, L(p, q), T^3$
 $S^1 \times D^2, T^2 \times I, \Sigma \times I$ (all with some restrictions on $\Gamma_{\partial M}$)

Results about Floer homology (by OS, KM, HKM, BS, etc)
Given $(N, \{)$, there is an element

$c(N, \{) \in SFH(-N, -\Gamma_{\partial N})$ called contact elements.
↑ opposite orientation

Similar for $HF^+(-Y)$ (can have twisted coefficients)

$\widetilde{HM}(-Y), SHM(-N, -\Gamma_{\partial N}), SHI(-N, -\Gamma_{\partial N})$

Note that we need naturality result to specify the element,
or we can only say $c=0 / c \neq 0$ in isomorphism class

It satisfies the following properties.

- $c=0$ for OT $\{$ (more vanishing result for Giroux torsion)
 - $c \neq 0$ for Stein-fillable $\{$ (more nonvanishing results for different versions)
 - $(N', \{')$ is the result of $(+1)$ -surgery on K from $(N, \{)$
- the cobordism map $F: SFH(-N, \Gamma) \rightarrow SFH(-N', \Gamma')$
sends $c(N, \{)$ to $c(N', \{')$

Class 19 contact gluing map

Def. A saturated cobordism from (M_1, γ_1) to (M_2, γ_2)

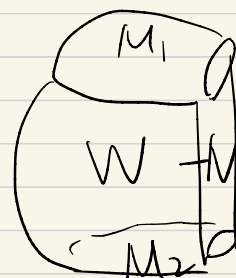
consists of $\mathcal{W} = (W, N, \zeta)$

- W opt 4-mfd $\partial W \setminus \text{int } N = -M_1 \cup M_2$

- N opt 3-mfd $\partial N = \partial M_1 \cup (-\partial M_2)$

- ζ (oriented) contact str on N .

dividing set $-\gamma_1 \cup \gamma_2$ on ∂N



A cobordism map

$$\text{SFH}(W) : \text{SFH}(M_1, \gamma_1) \rightarrow \text{SFH}(M_2, \gamma_2)$$

is the composition $F_W \circ \Phi_{-\zeta}$

$$\Phi_{-\zeta} : \text{SFH}(M_1, \gamma_1) \rightarrow \text{SFH}(M_1 \cup (-N), \gamma_2)$$

$$F_W : \text{SFH}(M_1 \cup (-N), \gamma_2) \rightarrow \text{SFH}(M_2, \gamma_2)$$

We can define F_W similar to cobordism map for closed 3-mfd. The map $\Phi_{-\zeta}$ is called a contact gluing map, with multiple equivalent definitions

Last time, we mention contact element

$c(N, \zeta) \in \text{SFH}(-N, -\Gamma_{\partial N})$ which is preserved by (+)-contact surgery on Legendrian knot, vanishes for OT, is nonvanishing for Stein fillable ζ .

In contact geometry, it is used to distinguish contact str.

There are also multiple ways to define

- in HF by Ozsváth-Szabó using fibered knot K
in open book decomposition compatible with ξ
$$\left(\widehat{HF}(K, -Y, -g(K)) \cong \mathbb{Z} \rightarrow \widehat{HF}(-Y) \right)$$

(or $HF^+(-Y_0(K), g-1) \cong \mathbb{Z} \rightarrow HF^+(-Y)$)
- in SFH by Honda-Kazez-Matić using Heegaard diagram
compatible with (partial) open book decomposition of ξ
(pick up specific generator)
- in HM by Kronheimer-Mrowka using monopole solutions
on a 4-manifold W with $\partial W = Y$, constructed by ξ .
- in SHM by Baldwin-Sivek using closure (Y, R)
of (N, ξ_0) with contact structure ξ extends ξ_0
and apply KM's construction
- in SHI by BS using 3d contact handle attaching maps
sending generator of $SHI(\text{product}) \cong \mathbb{C}$ to $SHI(-N, -\xi)$

We focus on the last way (which can also be applied
to SFH, SHM) because it is also used for the
contact gluing map Φ_ξ

More explicitly, given (N, ξ) , we can decompose it
into contact 0, 1, 2, 3-handles $C_1, \dots, C_n$ in the order

We construct map Φ_{C_i} for each handle attachments.
and then $\Phi_{\{ \} } = \Phi_{C_n} \circ \Phi_{C_{n-1}} \circ \dots \circ \Phi_{C_1}$

Prop (Zhenkun Li). The above construction is independent of the choice of the decomposition.

Concrete construction. Let (M_i, γ_i) be the resulting for i -handle from (M, γ) , $H_i: SHZ(-M, -\gamma) \rightarrow SHZ(-M_i, -\gamma_i)$

0-handle $(M, \gamma) \rightarrow (M_1, \gamma_1) = (M, \gamma) \sqcup (B^3, S')$

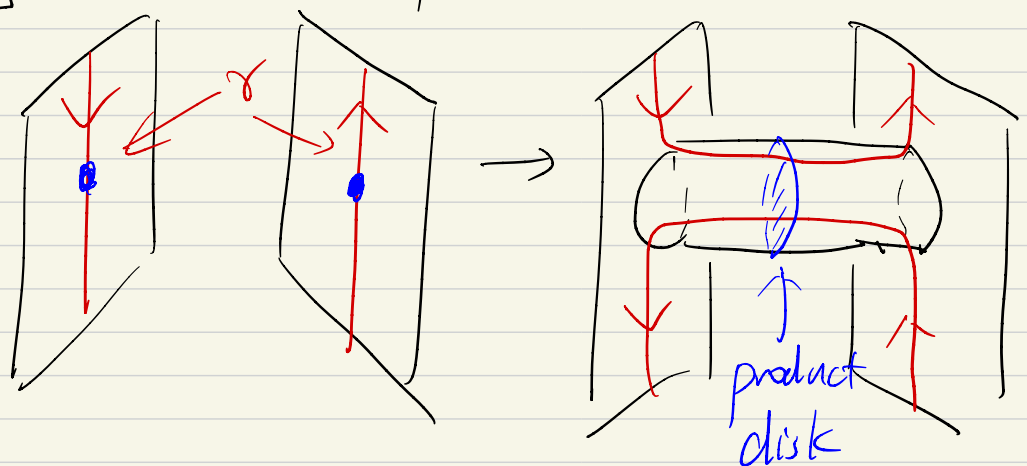
$$H_0 = \text{id}: SHZ(-M, -\gamma) \rightarrow SHZ(-M, -\gamma) \otimes \underline{SFH(-B^3, S')}$$

3-handle $(M, \gamma) \xrightarrow{\text{diffes}} (M', \gamma') = (M_3, \gamma_3) \# (B^3, S')$

$$H_3 = \text{composition} \xrightarrow{\text{4d 1-handle}} (M_3, \gamma_3) \sqcup (B^3, S') \xrightarrow{H_0^{-1}} (M_3, \gamma_3)$$

The bottom two are more important

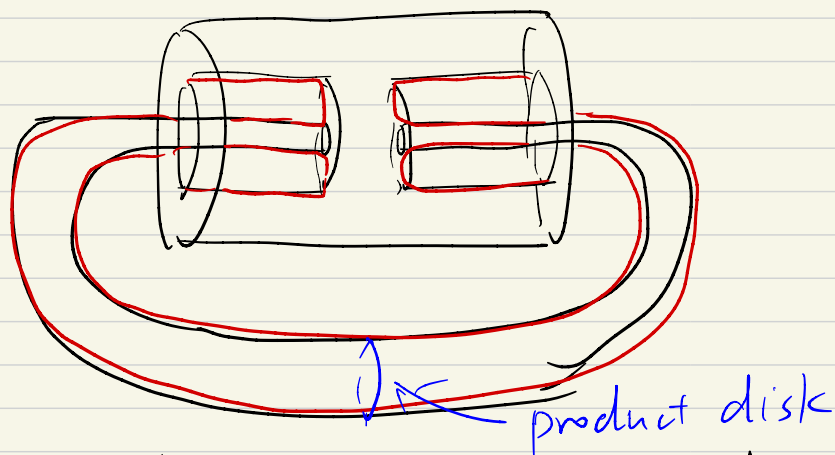
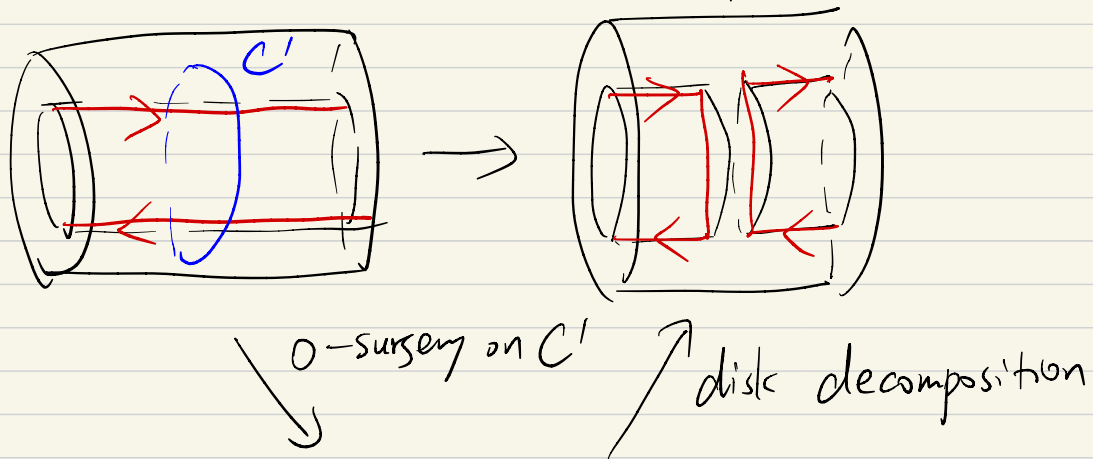
1-handle Choose two pts on γ



inverse of product disk decomposition
we can find a closure of both (M, γ) and (M, γ_1)
so H_1 is the canonical map btw the closures

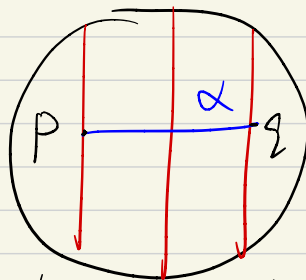
2-handle

$|C \cap \Sigma| = 2$ push C into M , get C'

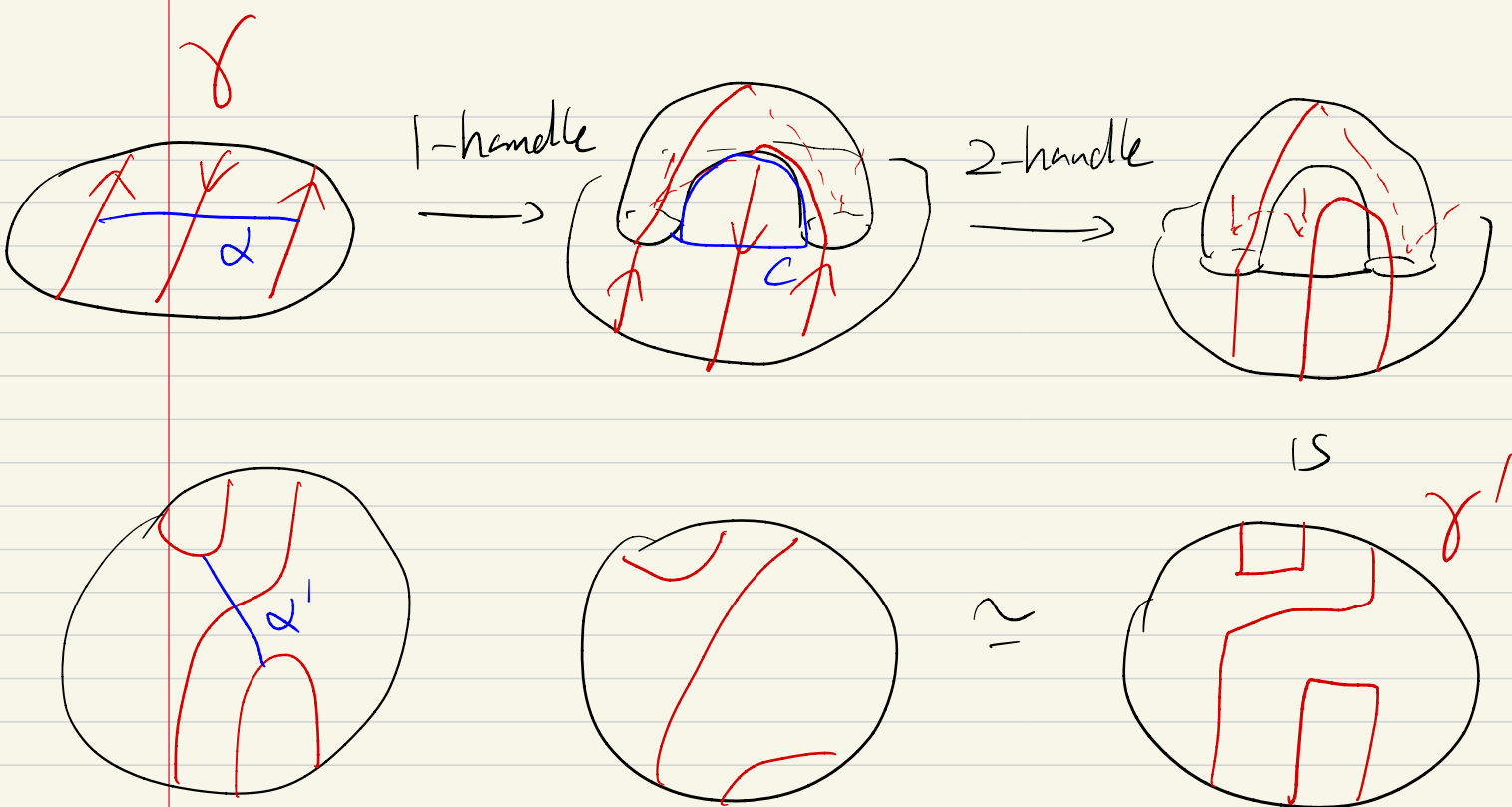


H_2 is the composition of 4d 0-surgery cobordism and the disk decomposition.

Bypass attachment. Suppose $\alpha \subset \partial M$ is an arc with endpoints p, q on Σ $|\alpha \cap \Sigma| = 3$ locally, we have



A bypass attachment along α is the composition of contact 1-handle along p, q , and contact 2-handle along $\alpha \cap \text{Core of 1 handle}$.

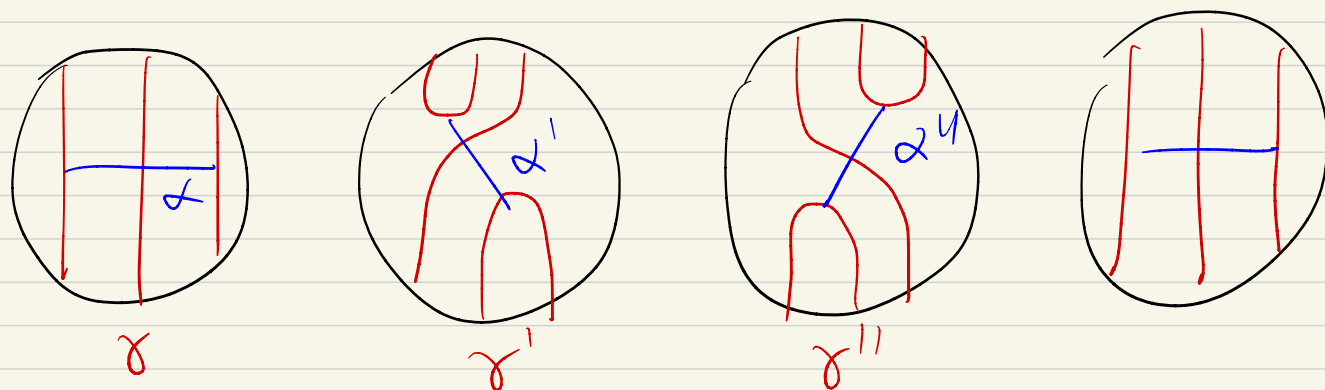


Note that the 3-mfd M is the same but γ are different.

Apply the bypass attachment again to the new arc α' ,

we obtain the third suture γ'' , and arc α'' .

Apply the bypass attachment to γ'' , α'' , we obtain the original suture γ



Thm (Baldwin - Sivek) There exists an exact triangle

$$SHI(-M, \gamma) \longrightarrow SHI(-M', -\gamma')$$

$\uparrow$

$\downarrow$

$$SHI(-M'', -\gamma'')$$

Similar for SHM

Idea of pf: In suitable closure, a bypass attaching map is just a surgery cobordism map, we can find good closure s.t. the above triangle is just the surgery exact triangle.

For instanton, we need to use some tricks to remove the extra bundle data μ

Rem Similar result holds for SFH, but since it is not defined by closure, the proof is different. Indeed, it uses bordered sutured Floer homology and local computation by Etnyre - Vela-Vick - Zarev.