Class 1 Introduction to sutured Floer homology

Office Hour : SC505H , M 4:20–5:20 W 2:00–3:00pm

GCA : Jiakai Li SC321G Th 6:00–7:00pm

Grading 60% Homework every 2 weeks

40% Take home exam

References on webpage. Notes on slides.

Class 2 sutured manifold

Def (Gabai) A <u>sutured mfd</u> (M, γ) consists of

- M compact oriented 3-mfd <u>with boundary</u>
- $\gamma \subseteq \partial M$ a subset

Some ref assume without ∂

we always assume $\partial \neq \emptyset$

$\gamma = A(\gamma) \cup T(\gamma)$

$T(\gamma)$: disjoint union of tori

$A(\gamma)$: disjoint union of annuli

$s(\gamma)$: core of $A(\gamma)$, oriented called <u>suture</u>

$R(\gamma) = \partial M \setminus \operatorname{int} \gamma$ oriented by $s(\gamma)$

$R(\gamma) = R_+(\gamma) \cup R_-(\gamma)$

$R_+(\gamma)$ ori points out of M (compatible w/ ∂M)

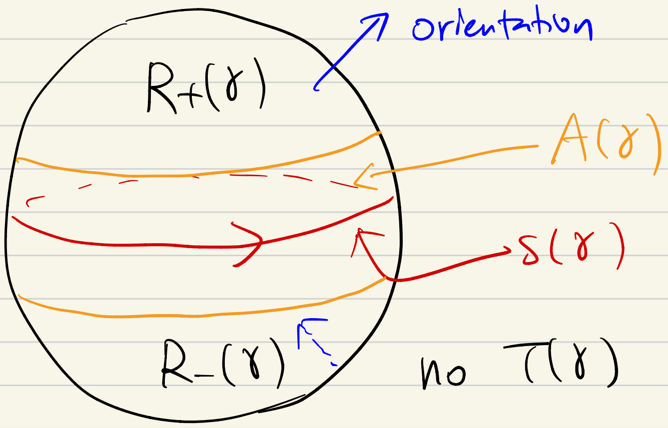
$R_-(\gamma)$ ori points in M

Rem usually just draw $s(\gamma)$ on ∂M

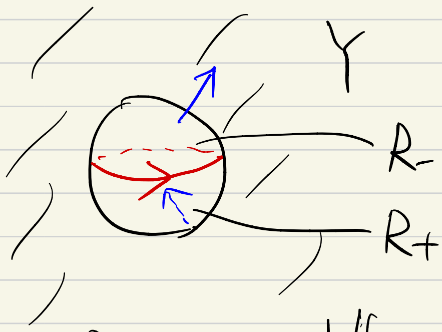
in sutured Floer homology, we only consider $T(\gamma) = \emptyset$

and write γ for $s(\gamma)$, $R_\pm$ for $R_\pm(\gamma)$

Ex • 3-ball B^3 • closed (connected, oriented) 3-mfd $Y - B^3$

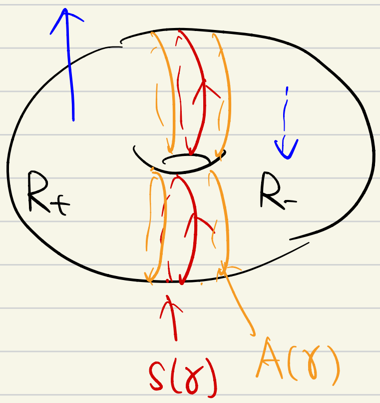


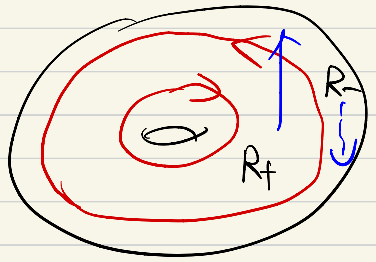
no $T(\gamma)$



Note $R_\pm(\gamma)$ are different from the left picture

• $S^1 \times D^2$





or any other two parallel curves on $\partial(S^1 \times D^2) = T^2$

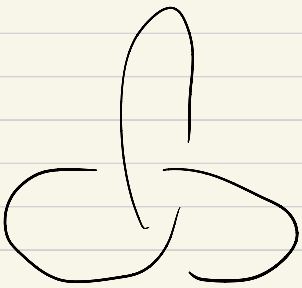
• closed 3-mfd Y, an embedding $K : S^1 \longrightarrow Y$ called a knot

$N(K)$ nbhd of K diffeo to $S^1 \times D^2$

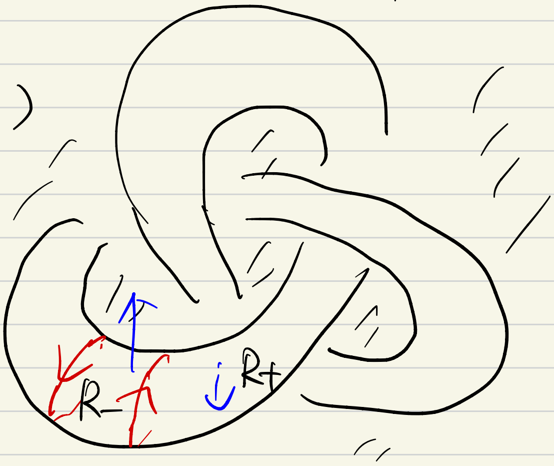
$M = Y \setminus \operatorname{int} N(K)$

γ (or $s(\gamma)$) two parallel curves with opposite orientations

K



$Y \setminus N(K)$

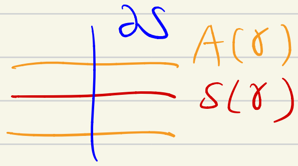


Rem An invariant of sutured mfd will induce an inv of knots or closed 3-mfds by above construction.

We can decompose sutured mfd into simpler pieces by some embedded surfaces as follows

Def A <u>decomposition surface</u> S is a properly embedded ($\partial S \subset \partial M$) oriented surface in M s.t. for any component λ of $\partial S \cap \gamma$, one of (1)–(3) holds

(1) λ is a properly embedded nonseperating arc in γ s.t. $|\lambda \cap s(\gamma)| = 1$



(2) λ is a simple closed curve in a component A of $A(\gamma)$ in the same homology class as the core $A \cap s(\gamma)$

(3) λ is a homological nontrivial curve in a component T of $T(\gamma)$ (all other components of $\partial S \cap \gamma$ in T are parallel to λ because of the embedding assuption)

Then S define a <u>sutured mfd decomposition</u>

$$(M, \gamma) \overset{S}{\rightsquigarrow} (M', \gamma') \quad \text{where}$$

- $M' = M \setminus \mathrm{int}(N(S))$ (cut M open by S)

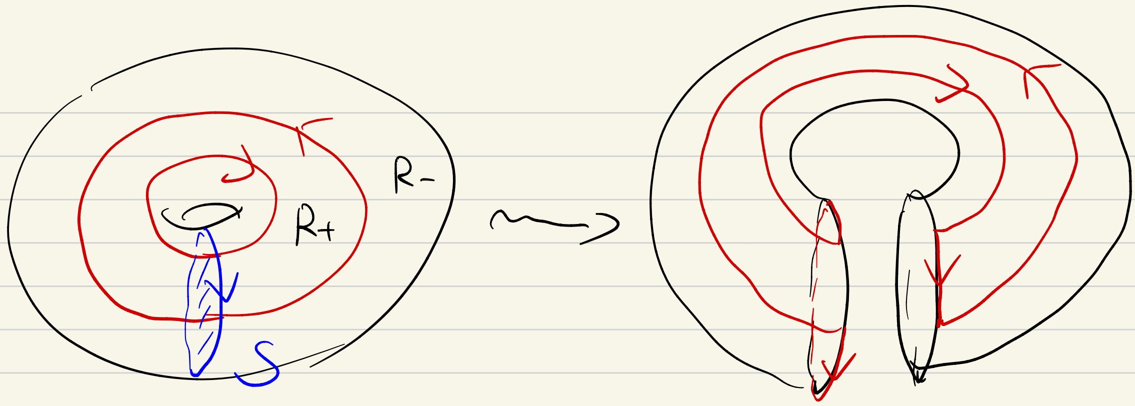
  $S'_{\pm}$ are components of $\partial N(S) \cap M'$ whose orientations out of / into M'

  $$\gamma' = (\gamma \cap M') \cup N(S'_+ \cap R_-(\gamma)) \cup N(S'_- \cap R_+(\gamma))$$

$$\left(\begin{aligned} R_+(\gamma') &= (R_+(\gamma) \cap M') \cup S'_+) \setminus \mathrm{int}\, \gamma' \\ R_-(\gamma') &= ((R_-(\gamma) \cap M') \cup S'_-) \setminus \mathrm{int}\, \gamma' \end{aligned} \right)$$

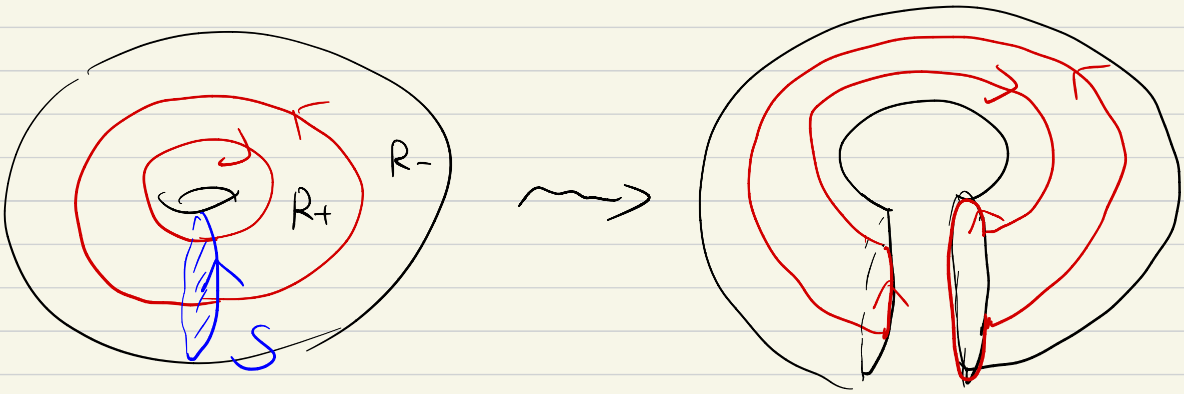
A good way to rembember γ' is by example

Ex



$S(\gamma') = S(\gamma) \cup$ a copy of ∂S with compatible ori

Note ori of $S(\gamma)$ and S are important



The main topological thm we will use in Sutured Floer homology is the following

Thm (sutured mfd hierarchy, Gabai)

Let (M,γ) be a connected <u>taut</u> sutured mfd.

Then there exists a sequence of surface decomp

$$(M,\gamma) = (M_0,\gamma_0) \overset{S_1}{\rightsquigarrow} (M_1,\gamma_1) \overset{S_2}{\rightsquigarrow} \cdots \overset{S_n}{\rightsquigarrow} (M_n,\gamma_n)$$

s.t. $(M_n,\gamma_n) \cong (R\times I, \partial R \times I)$ for surface R $\quad R_\pm = R \times \{\pm 1\}$

(called <u>product sutured mfd</u>)

Moreover, each surface S_i can be connected, and satisfies more topological conditions

The rest of these two weeks is to explain this thm.

Def. (M, γ) is taut if

- M is irreducible
- $R_\pm(\gamma)$ is incompressible
- $R_\pm(\gamma)$ is (Thurston)-norm minimizing in $H_2(M, \gamma)$

Def. A 3-mfd is called irreducible if any embedded S^2 bounds a B^3 in M

Ex. S^3 (Alexander thm), $S^1 \times D^2$, T^3, ...

Important nonexample: $S^1 \times S^2$. $pt \times S^2$ doesn't bound B^3

Prop. A 3-mfd is not irreducible if and only if it is either a connected sum of two mfds except S^3, or $S^1 \times S^2$

Pf: Let S be an embedded S^2 doesn't bound B^3

Two cases

(1) S separates M into 2 pieces $M = M_1 \cup_S M_2$

$Y_i = M_i \cup_S D^2 \quad Y_i \neq S^3 \quad M = M_1 \# M_2$

(2) S is nonseparating $M \setminus S$ is connected

a point on S induces two points in $M \setminus S$ that can be connected by a path in M

Hence $\exists$ a curve $\alpha \subset M$ s.t. $\alpha \cap S = pt$

(Ex) $N(\alpha \cup S^2) = S^1 \times S^2 \setminus B^3 \Rightarrow M = M' \# S^1 \times S^2$ $\square$

Rem. Connected sum operation doesn't depend on the base pt.

Moreover, $X \# Y \cong Y \# X$, $(X \# Y) \# Z \cong X \# (Y \# Z)$

Thurston norm.

Def. For a connected, oriented surface S of genus g with b boundary components. the Euler characteristic of S

$\chi(S) = 2-2g-b$

For properly embedded surface S (i.e. $(S,\partial S) \subset (M,\partial M)$) with components $S_1, \dots, S_n$

define $x_T(S) = \sum_i \max\{0, -\chi(S_i)\}$

For $\alpha \in H_2(M,\partial M)$ (or $H_2(M,\gamma)$)

$x_T(\alpha) = \min\{x_T(S) \mid [S,\partial S] = \alpha\}$

If S achieves the minimal, it is called norm-minimizing

Ex. a knot $K \subset S^3$ $M = S^3 \setminus \mathrm{int}\, N(K)$

$H_2(M,\partial M;\mathbb{Z}) \underset{\uparrow}{\cong} H^1(M;\mathbb{Z}) \underset{\uparrow}{\cong} \mathrm{Hom}(H_1(M;\mathbb{Z}),\mathbb{Z})$

Poincaré/Lefschetz duality universal coeff thm

$H_1(S^3\setminus \mathrm{int}\, N(K);\mathbb{Z}) \underset{\uparrow}{\cong} H^1(N(K);\mathbb{Z}) \cong H^1(S^1\times D^2;\mathbb{Z}) \cong \mathbb{Z}$

Alexander duality.

A Seifert surface S of K is a surface in $S^3\setminus K$ with $\partial S = K$. $g(K) = \min\{g(S) \mid S \text{ Seifert surface}\}$

Any Seifert surface generates $H_2(M,\partial M;\mathbb{Z}) \cong \mathbb{Z}$

$x_T([S]) = \max\{0, 2g(K)-1\}$

Rem $g(K) = 0$ K bounds a disk D $K =$ unknot $\bigcirc$

Prop (Thurston)

(1) Let $\alpha \in H_2(M, \partial M; \mathbb{Z})$. Then $\exists (S, \partial S) \subset (M, \partial M)$ s.t. $[S, \partial S] = \alpha$

(2) If $\alpha = k \cdot \beta$ for $k \in \mathbb{Z}$, then the above S satisfying $S = S_1 \cup \cdots \cup S_k$ s.t. $[S_i, \partial S_i] = \beta$

Sketch: $\alpha \in H_2(M, \partial M; \mathbb{Z}) \cong H^1(M; \mathbb{Z})$

$\Rightarrow f_\alpha : M \to S^1$

$p \in S^1$ regular value $\quad S = f_\alpha^{-1}(p)$ represents α

If S represents α, we can find f_α s.t. above holds

If $\alpha = k\beta$, then

$$\begin{tikzcd} M \arrow[r, "f_\beta"] \arrow[dr, "f_\alpha"'] & S^1 \arrow[d, "k\text{-covering}"] & p_1, \dots, p_k \arrow[d] \\ & S^1 & p \end{tikzcd}$$

$S_i = f_\beta^{-1}(p_i)$ represents β

Cor (1) $x_T(\alpha)$ is well-defined (2) $x_T(k\alpha) = k x_T(\alpha)$ for $k \geq 0$

Pf: $[S, \partial S] = \alpha \quad x_T(k\alpha) \leq x_T(kS) = k x_T(S)$

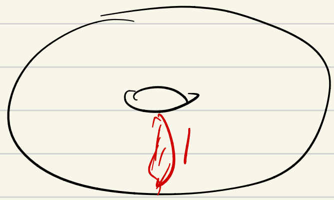
By above (2), $[S', \partial S'] = k\alpha \quad x_T(S') \geq k x_T(\alpha)$

Class 3 compression

Def a disk $D \subset M$ is called a <u>compression disk</u> of a surface S if

- $D \cap S = \partial D$
- ∂D doesn't bound a disk in ∂S

Ex $\partial(S^1 \times D^2) \cong T^2$ compressible

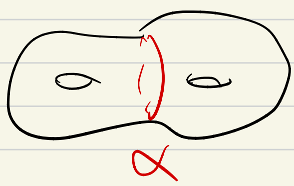


Note $[S] = [S']$ in homology gp.

$\chi(S') = \chi(S) + 2$ (replace annulus by two disks) ($\chi = 0$ for the annulus, $\chi = 1$ for the disks)

Def. A simple closed curve α in a surface S is called essential if it doesn't bound a disk in S

Ex. $S = T^2$ α essential iff $[\alpha] \neq 0 \in H_1(S)$

$S =$ 

α essential but $[\alpha] = 0 \in H_1(S)$

Lem Y closed, oriented, irreducible

S_1, S_2 both incompressible

Then we can isotope S_1 and S_2 in Y to make any component of $S_1 \cap S_2$ essential in both S_1 and S_2.

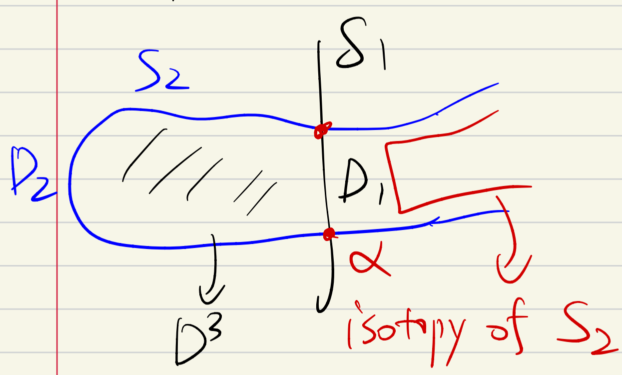
Idea of pf: simple case $S_1 \cap S_2 = \alpha$ connected.

If α is inessential in $S_1 \Rightarrow \alpha = \partial D_1$, $D_1 \subset S_1$

$$D_1 \cap S_2 = \partial D_1 = \alpha$$

S_2 incompressible $\Rightarrow \alpha$ bounds D_2 in S_2

$D_1 \cup D_2 = S \cong S^2$ $\quad Y$ irreducible $\Rightarrow S$ bounds B^3



Then after isotopy, $S_1 \cap S_2' = \phi$

General case: $S_1 \cap S_2$ has more components

Choose innermost inessential curve and do above operation

Def. $(S, \partial S) \subset (M, \partial M)$

a boundary compression disk (∂-compression) $D \subset M$ of S satisfying

$\partial D = \alpha \cup \beta$ s.t. $D \cap S = \alpha$ $\quad D \cap \partial M = \beta$

compression gives S'

$\chi(S') = \chi(S) + 1$

we can define ∂-essential arc in S similarly and prove all components of $S_1 \cap S_2$ are ∂-essential under incompression conditions for $S_1, S_2, \partial M$.

we prove more properties of Thurston norm χ_T

Prop If M is irreducible and ∂M is incompressible,

then $\chi_T(\alpha_1+\alpha_2) \le \chi_T(\alpha_1)+\chi_T(\alpha_2)$

for $\alpha_i \in H_2(M,\partial M;\mathbb{Z})$

Pf: Pick S_1, S_2 s.t. $[S_i,\partial S_i]=\alpha_i$ $\chi_T(S)=\chi_T(\alpha)$

Compress S_i if possible. to obtain S_i' $\chi(S_i') \ge \chi(S_i)$

but $[S_i']=[S_i]$ so $\chi_T(S_i')=\chi_T(S_i)$ by minimal condition

Then throw S^2 components of S_i' to obtain S_i''

S_i'' incomp, ∂-incomp, has no S^2 components

isotope S_i'' to make any curve in $S_1''\cap S_2''$ essential

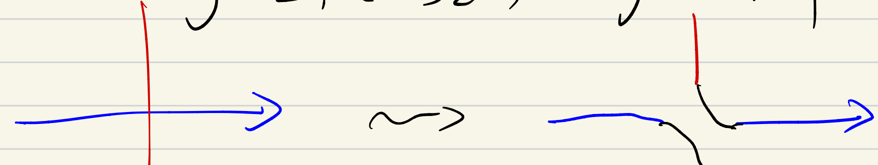
any arc in $S_1''\cap S_2''$ ∂-essential

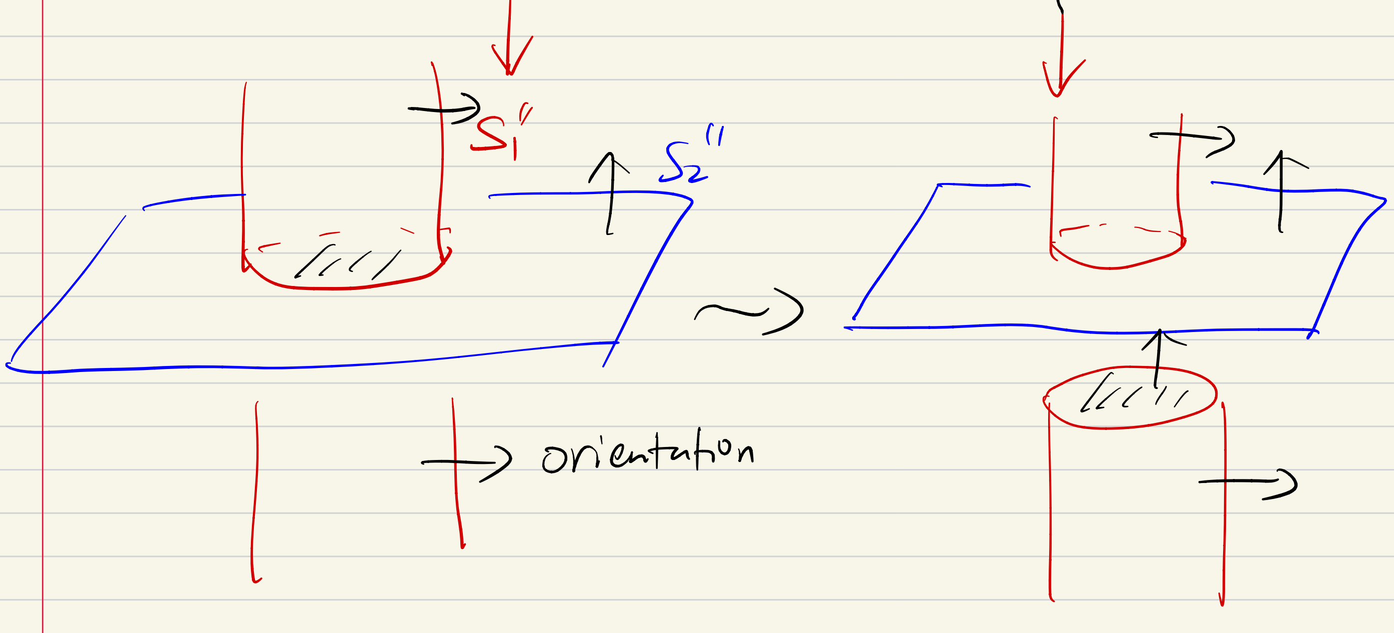
perform a cut and glue surgery on S_1'' and S_2''

(called double curve surgery by Scharlemann)

Cut open along $S_1''\cap S_2''$, reglue respect orientation

1d analog





We obtain R after surgery.

$$\chi(S_1'') + \chi(S_2'') = \chi(R)$$

$$x_T(R) = -\chi(R)\ ?$$

Need to think about disk and sphere components

Claim 1 R has no S^2 comp

Claim 2 disk comp R all come from S_1'' and S_2''

(without pf)

$$\Rightarrow \quad x_T(R) = x_T(S_1'') + x_T(S_2'')$$

$$x_T(\alpha_1+\alpha_2) \leq x_T(R) = x_T(\alpha_1) + x_T(\alpha_2)$$

We obtain the following thm

Thm: If M is incomp and ∂M is ∂-incomp,

then $x_T : H_2(M, \partial M; \mathbb{Z}) \to \mathbb{Z}_{\geq 0}$

extends to a continuous map

$$x_T : H_2(M, \partial M; \mathbb{R}) \to \mathbb{R}_{\geq 0}$$

which is a semi-norm

Rem. We can also define x_T for $H_2(M, N; \mathbb{R})$ when $N \subset \partial M$.

If M has no essential tori and annuli (i.e. atoroidal), then x_T is a norm.

Prop (Gabai, Scharlemann)

For any $\alpha, \beta \in H_2(M, \partial M)$, there exists a large number k s.t. for any number $l \geq 0$

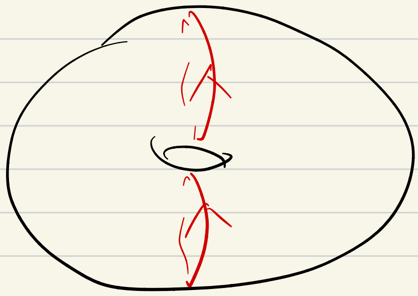
$$x_T(\alpha + (k+l)\beta) = x_T(\alpha + k\beta) + l x_T(\beta)$$

We omit the pf.

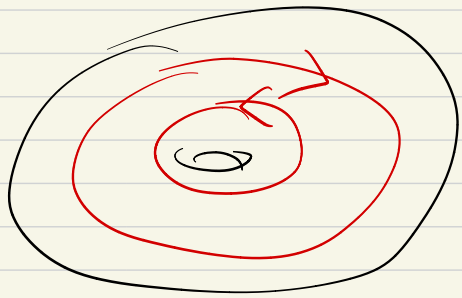
Def. (M, γ) is <u>taut</u> if

- M is <u>irreducible</u>
- $R_\pm(\gamma)$ is <u>incompressible</u>
- $R_\pm(\gamma)$ is (Thurston) <u>norm minimizing</u> in $H_2(M, \gamma)$

Ex



not taut (1),(3) ✓ (2) ✗



taut

Thm (sutured mfd hierarchy, Gabai)

Let (M, γ) be a connected <u>taut</u> sutured mfd.

Then there exists a sequence of surface decomp

$$(M, \gamma) = (M_0, \gamma_0) \overset{S_1}{\rightsquigarrow} (M_1, \gamma_1) \overset{S_2}{\rightsquigarrow} \cdots \overset{S_n}{\rightsquigarrow} (M_n, \gamma_n)$$

s.t. $(M_n, \gamma_n) \cong (R \times I, \partial R \times I)$ for surface R $R_\pm = R \times \{\pm 1\}$

(called <u>product sutured mfd</u>) $\boxed{\text{Indeed } (M_n, \gamma_n) = \coprod (B^3, S^1)}$

Moreover, each surface S_i can be connected, and satisfies more topological conditions

Idea of the pf:

Prop. If (M,γ) taut and $\alpha \neq 0 \in H_2(M,\partial M;\mathbb{Z})$ then we can find S s.t. $[S,\partial S]=\alpha$

$(M,\gamma) \overset{S}{\rightsquigarrow} (M',\gamma')$ taut. (omit pf)

In the case $H_2(M,\partial M;\mathbb{Z})=0$, we use the following prop.

Prop (half lives, half dies)

For any 3d M with ∂, the inclusion

$i_*: H_1(\partial M;\mathbb{Q}) \longrightarrow H_1(M;\mathbb{Q})$

satisfies $\dim \ker i_* = \dim \operatorname{Im} i_* = \frac{1}{2}\dim H_1(\partial M;\mathbb{Q})$

pf:

$$\begin{tikzcd}
H_2(M,\partial M) \arrow[r, "\partial_*"] \arrow[d, phantom, "\cong"] & H_1(\partial M) \arrow[r, "i_*"] \arrow[d, phantom, "\cong"] & H_1(M) \arrow[d, phantom, "\cong"] \\
H^1(M) \arrow[r, "i^*"] & H^1(\partial M) \arrow[r, "\delta^*"] & H^2(M,\partial M)
\end{tikzcd}$$

$$\begin{aligned}
\dim \ker i_* &= \dim \operatorname{coker} i^* \\
&= \dim \operatorname{coker} \partial_* \\
&= \dim H_1(\partial M) - \dim \operatorname{Im} \partial_* \\
&= \dim H_1(\partial M) - \dim \ker i_* \qquad \square
\end{aligned}$$

$H_2(M,\partial M;\mathbb{Z})=0 \Rightarrow H^1(M)=0 \quad H_1(M;\mathbb{Q})=0$

$\ker i_* = \operatorname{Im} \partial_* = 0 \quad H_1(\partial M;\mathbb{Q})=0$

$\Rightarrow \partial M = \sqcup S^2$

M irreducible $\Rightarrow M = \sqcup B^3$

Q: why finite step?

A: there is a complexity ($\in \mathbb{Z}_{\geq 0}$) that decreases every decomposition

There is also another important prop.

Prop: If $(M,\gamma) \overset{S}{\rightsquigarrow} (M',\gamma')$ is a surface decomp, and (M',γ') taut, then (M,γ) is also taut (omit pf)

Class 4 Application of hierarchy:

Thm (Property R conj. Gabai)

If $K \subset S^3$ is not the unknot $\bigcirc$.

then $S^3_0(K) \not\cong S^1 \times S^2$ (omit pf)

Dehn surgery on knot:

$\partial(S^3 \setminus \text{int}\, N(K)) \cong T^2$ has canonical basis

$m = \partial(pt \times D^2) \subset S^1 \times D^2 \cong N(K)$ called <u>meridian</u>

$l = \partial S$ for any Seifert surface called <u>longitude</u>

(It is also in the kernel of $H_1(\partial M) \hookrightarrow H_1(M)$)

Orient m, l s.t. $m \cdot l = -1$ (point into M)

Then we have

$$\mathbb{Q} \cup \{\tfrac{1}{0} = \infty\} \overset{1-1}{\longleftrightarrow} \{\text{simple closed curves on } \partial(S^3 \setminus \text{int}\, N(K))\}$$

$$p/q \qquad\qquad pm + ql$$

In particular $0 = \frac{0}{1} \mapsto l = \partial S$

$$S^3_{p/q}(K) = \left(S^3 \setminus \mathrm{int}\, N(K)\right) \cup_\phi S^1 \times D^2$$

$$\phi(pt \times \partial D^2) = pm + ql$$

(doesn't depend on $\phi(S^1 \times pt)$)

$$H_1(S^3_{p/q}(K)) = \begin{cases} \mathbb{Z}/p & p \neq 0 \\ \mathbb{Z} & p = 0 \end{cases}$$

Using sutured instanton theory, we will prove property P conj state as follows

Thm (Kronheimer–Mrowka) If $K \neq O$, then

$$S^3_{p/q}(K) \not\cong S^3$$

Rem. By homology, it suffices to prove for $p/q = \pm 1$.