Class 15 Torsion free convariant derivative

Pf: To define ∇g, we can first define ∇ on $E\otimes E$ by
$$\nabla(s_1\otimes s_2) = \nabla s_1\otimes s_2 + s_1\otimes \nabla s_2 \quad (3)$$
and then define ∇ on $(E\otimes E)^* = E^*\otimes E^*$ by defining $\nabla(s_1^*\otimes s_2^*)$
$$d\langle s_1\otimes s_2, s_1^*\otimes s_2^*\rangle = \langle \nabla(s_1\otimes s_2), s_1^*\otimes s_2^*\rangle + \langle s_1\otimes s_2, \nabla(s_1^*\otimes s_2^*)\rangle \quad (4)$$
for any $s_1\otimes s_2$ and then extend linearly over $\mathbb{R}$

Note that when multiply with a map $f: M\to\mathbb{R}$, we define
$$\nabla f(s_1^*\otimes s_2^*) = (s_1^*\otimes s_2^*)\otimes f + f\nabla(s_1^*\otimes s_2^*)$$
$$= f s_1^*\otimes f s_2^* + f\nabla(s_1^*\otimes s_2^*)$$
We need to check these two definitions are the same

i.e. (1–3) implies (4) and $(1,3,4)\Rightarrow(2)$

Use the second definition, we have
$$dg(u,v) = d\langle u\otimes v, g\rangle$$
$$= \langle \nabla(u\otimes v), g\rangle + \langle u\otimes v, \nabla g\rangle$$
$$= \langle \nabla u\otimes v + u\otimes \nabla v, g\rangle + \langle u\otimes v, \nabla g\rangle$$
$$= g(\nabla u, v) + g(u,\nabla v) + \langle u\otimes v, \nabla g\rangle \quad \square$$
Then we consider the case $E = TM$. $E^* = T^*M$.
$$\nabla: C^\infty(M; T^*M) \longrightarrow C^\infty(M; T^*M\otimes T^*M)$$
Define A to be the antisymmetrization

from $C^\infty(M; T^*M \otimes T^*M) \longrightarrow C^\infty(M; \Lambda^2 T^*M)$

by $\mathcal{A}(\omega_1 \otimes \omega_2) = \frac{1}{2}(\omega_1 \otimes \omega_2 - \omega_2 \otimes \omega_1)$

and extending linearly.

For a 1-form $\omega \in C^\infty(M; T^*M)$

we can define $d\omega \in C^\infty(M; \Lambda^2 T^*M)$

and $\mathcal{A}(\nabla \omega) = \mathcal{A}(d_\nabla \omega) \in C^\infty(M; \Lambda^2 T^*M)$

(Rem. previously we define d_∇ to be a map

$$C^\infty(M; E \otimes \Lambda^k T^*M) \to C^\infty(M; E \otimes \Lambda^{k+1} T^*M)$$

In the special case $E = T^*M$ we compose it with $\mathcal{A}$

Cliff still uses the same notation, but we write as $\mathcal{A}\nabla$

[Levi-Civita connection]

Given a Riemannian metric g on TM, a covariant derivative ∇ on TM is compatible with g if

either 1) $dg(u,v) = g(\nabla u, v) + g(u, \nabla v)$

for any vector fields u, v

or 2) $\nabla g = 0$ for the covariant derivative on $T^*M \otimes T^*M$

induced by $\nabla(s_1^* \otimes s_2^*) = \nabla s_1^* \otimes s_2^* + s_1^* \otimes \nabla s_2^*$

$$d\langle s, s^* \rangle = \langle \nabla s, s^* \rangle + \langle s, \nabla s^* \rangle$$

∇ is called torsion-free if the torsion tensor

$T_\nabla = A\nabla - d : C^\infty(M; T^*M) \to C^\infty(M; \Lambda^2 T^*M)$ vanishes,

where $A(w_1 \otimes w_2) = \frac{1}{2}(w_1 \otimes w_2 - w_2 \otimes w_1)$

Thm. Given a Riemannian metric g on TM, there is a unique metric compatible, torsion free covariant derivative ∇_{LC} called the <u>Levi-Civita connection</u>

Here we don't distinguish ∇ on TM or T^*M because then can induce each other

Pf. 1 (in the next pages, using orthonormal basis)

In the last class, we compute in a locally orthonormal basis $e_1, \dots, e_n$ Suppose

$$\nabla e_i = \sum_j e_j \otimes \alpha_i^j = \sum_j \alpha_i^j e_j = \alpha_i^j e_j \text{ for short}$$

α_i^j 1-forms, ∇ is compatible with g iff $\alpha_i^j = -\alpha_j^i$ (1)

consider the dual basis $e^i = e_i^*$

$\langle e_i, e^j \rangle = \delta_{ij} : M \to \mathbb{R}$ suppose

$$\nabla e^i = e^j \otimes \beta_j^i = e^j \otimes \beta_{jk}^i e^k = \beta_{jk}^i e^j \otimes e^k$$

β_j^i 1-form, $\beta_{jk}^i : M \to \mathbb{R}$

$$0 = d\langle e_i, e^j \rangle = \langle \nabla e_i, e^j \rangle + \langle e_i, \nabla e^j \rangle$$

$$= \langle \alpha_i^k e_k, e^j \rangle + \langle e_i, e^l \otimes \beta_l^j \rangle$$

$$= \alpha_i^j + \beta_i^j$$

$\Rightarrow \nabla$ is compatible with g iff $\beta_i^j = -\beta_j^i$ (1')

$$A \nabla e^i = A(\beta_{jk}^i e^j \otimes e^k)$$

$$= \beta_{jk}^i \left(\frac{1}{2}(e^j \otimes e^k - e^k \otimes e^j) \right)$$

$$= \frac{1}{2}(\beta_{jk}^i - \beta_{kj}^i) e^j \otimes e^k$$

Suppose $de^i = \sum_{j<k} \gamma^i_{jk} e^j \wedge e^k$

$= \sum_{j<k} \gamma^i_{jk} \frac{1}{2}(e^j \otimes e^k - e^k \otimes e^j) = \sum_{j,k} \frac{1}{2} \gamma^i_{jk} e^j \otimes e^k$

where we set $\gamma^i_{jk} = -\gamma^i_{kj}$ for $j>k$, $\gamma^i_{jj} = 0$

Then $T_\nabla e^i = \cancel{A} \nabla e^i - de^i = \frac{1}{2}(\beta^i_{jk} - \beta^i_{kj} - \gamma^i_{jk}) e^j \otimes e^k$

∇ is torsion free iff $\beta^i_{jk} - \beta^i_{kj} - \gamma^i_{jk} = 0$ (2)

Change indices, we have $\beta^j_{ki} - \beta^j_{ik} - \gamma^j_{ki} = 0$ (2')

$\beta^k_{ij} - \beta^k_{ji} - \gamma^k_{ij} = 0$ (2'')

Also from (1'), we have $\beta^j_{ik} = -\beta^i_{jk}$

(2)+(2')−(2'')

$\Longrightarrow$ $\beta^i_{jk} = \frac{1}{2}(\gamma^i_{jk} + \gamma^j_{ki} - \gamma^k_{ij})$

If there is another solution $(\beta')^i_{jk}$, suppose

$\eta^i_{jk} = \beta^i_{jk} - (\beta')^i_{jk}$, then (2) $\Rightarrow$ $\eta^i_{jk} = \eta^i_{kj}$

We have (1') $\Rightarrow$ $\eta^j_{ik} = -\eta^i_{jk}$

$\eta^j_{ik} \overset{(1')}{=} -\eta^i_{jk} \overset{(2)}{=} -\eta^i_{kj} \overset{(1')}{=} \eta^k_{ij} \overset{(2)}{=} \eta^k_{ji} \overset{(1')}{=} -\eta^j_{ki} \overset{(2)}{=} -\eta^j_{ik}$

$\Rightarrow \eta^j_{ik} = 0$, so the solution is unique.

Class 16 Covariantly constant section

Pf 2 (Using coordinate basis $\frac{\partial}{\partial x^i}$ and dx^i,)

Note that we don't have $g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) = \delta_{ij}$ in this case

But $d(dx^i) = 0$, so we don't have γ^i_{jk} terms

Suppose $\nabla \frac{\partial}{\partial x^i} = \Gamma^j_{ik} \frac{\partial}{\partial x^j} \otimes dx^k$

(We use Γ^j_{ik} because we will show this is indeed the Christoffel symbol

$$\Gamma^j_{ik} = \frac{1}{2} g^{jl} (\partial_i g_{lk} + \partial_k g_{il} - \partial_l g_{ik}) \quad)$$

From the pairing $\nabla \langle \frac{\partial}{\partial x^i}, dx^j \rangle$, we have

$$\nabla dx^i = -\Gamma^i_{jk} dx^j \otimes dx^k$$

$$T_\nabla = 0 \Rightarrow \Gamma^i_{jk} = \Gamma^i_{kj}$$

Then we need to show ∇ is compatible with g

Suppose $g = g_{ij} dx^i \otimes dx^j \quad g_{ij} = g_{ji} : U \to \mathbb{R}$

$$\{g^{ij}\} = (g_{ij})^{-1} \qquad 0 = \nabla g = \nabla (g_{ij} dx^i \otimes dx^j)$$

$$= (\nabla g_{ij}) dx^i \otimes dx^j + g_{ij} (\nabla dx^i) \otimes dx^j + g_{ij} dx^i \otimes (\nabla dx^j)$$

For simplicity, we can consider $\nabla_l g = \langle \frac{\partial}{\partial x^l}, \nabla g \rangle$

Note that $\nabla_l g_{ij} = \frac{\partial g_{ij}}{\partial x^l} = \partial_l g_{ij}$

$$\nabla_l dx^i = -\Gamma^i_{lk} dx^k$$

Hence we have

$$0=\partial_l g_{ij}\, dx^i \otimes dx^j - g_{ij}\left(\Gamma^i_{lp}\, dx^p \otimes dx^j + \Gamma^j_{lq}\, dx^i \otimes dx^q\right)$$

We can change indices and solve Γ^i_{jk} as in the first proof, and also prove the uniqueness similarly.

Here for simplicity, I just show the Christoffel symbol satisfies the equation.

Recall $\Gamma^j_{ik} = \frac{1}{2} g^{jl}\left(\partial_i g_{lk} + \partial_k g_{ik} - \partial_l g_{ik}\right)$

$$g_{ij}\Gamma^i_{lp} = \frac{1}{2} g_{ij}\, g^{ir}\left(\partial_l g_{rp} + \partial_p g_{lr} - \partial_r g_{lp}\right)$$

$$= \frac{1}{2}\, \partial_l g_{jp} + \partial_p g_{lj} - \partial_j g_{lp}$$

because $g_{ij}\, g^{ir} = \delta_{jr}$

Then $g_{ij}\left(\Gamma^i_{lp}\, dx^p \otimes dx^j + \Gamma^j_{lq}\, dx^i \otimes dx^q\right)$

$$= \frac{1}{2}\left(\partial_l g_{jp} + \partial_p g_{lj} - \partial_j g_{lp}\right) dx^p \otimes dx^j$$

$g_{ij}=g_{ji}$

$$+ \frac{1}{2}\left(\partial_l g_{iq} + \partial_q g_{li} - \partial_i g_{lq}\right) dx^i \otimes dx^q$$

$p \mapsto i$, $q \mapsto j$

$$= \frac{1}{2}\left(\partial_l g_{ji} + \cancel{\partial_i g_{lj}} \cancel{- \partial_j g_{li}} + \partial_l g_{ij} + \cancel{\partial_j g_{li}} \cancel{- \partial_i g_{lj}}\right) dx^i \otimes dx^j$$

$$= \partial_l g_{ij}\, dx^i \otimes dx^j$$

Last time, we show there is a unique covariant derivative on TM and T^*M that is compatible with a given metric g and torsion free, which is called Levi-Civita connection ∇_{LC}, we proved the existence and uniqueness using two different bases.

① orthonormal basis $e_1, \cdots, e_n$, $e^i = e_i^*$

metric compatible condition is easier, but de^i may not be zero

② coordinate basis $\frac{\partial}{\partial x^i}$, dx^i.

$d(dx^i) = 0$, so the torsion free condition is easier, but $g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$ may not be δ_{ij}

In the second proof, we notice that

$$\nabla \frac{\partial}{\partial x^i} = \Gamma^j_{ik} \frac{\partial}{\partial x^j} \otimes dx^k, \quad \nabla dx^i = -\Gamma^i_{jk} dx^j \otimes dx^k$$

where $\Gamma^j_{ik} = \frac{1}{2} g^{jl} (\partial_i g_{lk} + \partial_k g_{il} - \partial_l g_{ik})$

is the Christoffel symbol. Recall it appears in the geodesic equation (weeks 5-6)

$$\frac{d^2\gamma^i}{dt^2} + \Gamma^i_{jk} \frac{d\gamma^j}{dt} \frac{d\gamma^k}{dt} = 0 \quad \text{or} \quad \ddot{\gamma}^i + \Gamma^i_{jk} \dot{\gamma}^j \dot{\gamma}^k = 0$$

What is the relation between these two?

First, we give a definition for general v.b.

Def Let $\gamma : [0,1] \longrightarrow M$ be a smooth path

A section of $E|_\gamma = \pi^{-1}(\gamma)$ is <u>parallel</u> if

$\nabla_{\dot\gamma} S := \langle \gamma^* \frac{\partial}{\partial t}, \nabla S \rangle$ vanishes, i.e.

the covariant derivative at tangent direction of γ vanishes. For a basis e_i of E

we write $\nabla S = (ds_i + \alpha^{ij} s_j) e_i$

$$\gamma^* \frac{\partial}{\partial t} = \frac{d\gamma^k}{dt} \frac{\partial}{\partial x^k} \quad \alpha^{ij} = \alpha^{ij}_k \, dx^k \quad \alpha^{ij}_k : U \to \mathbb{R}$$

Then $$\nabla_{\dot\gamma} S = \left(\frac{d s_i \circ \gamma}{dt} + \alpha^{ij}_k \frac{d\gamma^k}{dt} s_j \circ \gamma \right) e_i \qquad (**)$$

This is a section of $E|_U$

$\nabla_{\dot\gamma} S = 0$ iff $\langle \nabla_{\dot\gamma} S, e^i \rangle = 0$ for any i

In particular if $E = TM$ $e_i = \frac{\partial}{\partial x^i}$ $S = \gamma_* \frac{\partial}{\partial t}$

$\nabla = \nabla_{LC}$ Then $s_i \circ \gamma = \frac{d\gamma^i}{dt}$

$$\nabla \frac{\partial}{\partial x^l} = \Big(\cancel{d\big(\tfrac{\partial}{\partial x^l}\big)_i}^{\,0} + \alpha^{ij} \big(\tfrac{\partial}{\partial x^l}\big)_j \Big) \frac{\partial}{\partial x^i} = \alpha^{il} \frac{\partial}{\partial x^i}$$

$$\left(\left(\frac{\partial}{\partial x^l}\right)_i = \delta_{il} \right) \Longrightarrow \alpha^{il}_k = \Gamma^l_{ik}$$

$$\nabla_{\dot\gamma} \dot\gamma = 0 \iff \frac{d^2\gamma^i}{dt^2} + \Gamma^i_{jk} \frac{d\gamma^j}{dt} \frac{d\gamma^k}{dt} = 0$$

i.e. For a geodesic γ, $\gamma_* \frac{\partial}{\partial t}$ is a parallel section with respect to ∇_{LC}