Class 13 Curvature of a covariant derivative (Chap 12)

Def Let $C^\infty(M;E)$ be the space of sections $M \to E$.

A <u>covariant derivative</u> is a map

$$\nabla : C^\infty(M;E) \longrightarrow C^\infty(M;E\otimes T^*M) \text{ obeying the}$$

Leibniz rule $\nabla(fs) = s\otimes df + f\nabla s$

for any $f\in C^\infty(M;\mathbb{R})$, $s\in C^\infty(M;E)$

Prop. (last class)

1) Covariant derivative exists

2) ∇, ∇' are two covariant derivatives

iff $\nabla - \nabla' = \alpha$ is a section of $\mathrm{End}(E)\otimes T^*M$

Prop $\Omega^k = C^\infty(M;\Lambda^k T^*M)$

There exists map $d: \Omega^k \to \Omega^{k+1}$ s.t. $d^2 = 0$

$d(w_1+w_2) = dw_1 + dw_2$

$d(w_1\wedge w_2) = dw_1\wedge w_2 + (-1)^{\deg w_1} w_1\wedge dw_2$

Today we combine ∇ and d together to define

exterior covariant derivative

$$d_\nabla : C^\infty(M;E\otimes\Lambda^k T^*M) \longrightarrow C^\infty(M;E\otimes\Lambda^{k+1}T^*M)$$

$k=0$ $\Lambda^0 T^*M = M\times\mathbb{R}$ $d_\nabla s = \nabla s$

$k>0$ If the section of $E\otimes\Lambda^k T^*M$ can be written as $s \otimes \omega$, then define

$$d_\nabla s \otimes \omega = \nabla s \wedge \omega + s \otimes d\omega$$

Then define $d_\nabla \left(\sum_i s_i \otimes \omega_i \right) = \sum_i d_\nabla (s_i \otimes \omega_i)$

In general, $d_\nabla^2 \neq 0$

For $s \in C^\infty(M; E)$, we write $d_\nabla^2 s = F_\nabla s$,

where F_∇ is called the <u>curvature</u>

$$F_\nabla(fs) = d_\nabla^2(fs) = d_\nabla(f d_\nabla s + s \otimes df)$$

$$= \underbrace{df \wedge d_\nabla s + f d_\nabla^2 s + d_\nabla s \wedge df}_{=0} + s \otimes \cancel{d^2 f}^{\,0}$$

$$= f d_\nabla^2 s = f F_\nabla s$$

Thus, F_∇ is a section of $\mathrm{End}(E) \otimes \Lambda^2 T^* M$

by Lem in the last class.

For $s \otimes \omega \in C^\infty(M; E \otimes \Lambda^k T^* M)$

section tensor 1-form

$$d_\nabla^2 (s \otimes \omega) = d_\nabla (d_\nabla s \wedge \omega + s \otimes d\omega)$$

$(d(\alpha \wedge \beta) = d\alpha \wedge \beta - \alpha \wedge d\beta$ for 1-form $\alpha)$

$$= d_\nabla^2 s \wedge \omega - d_\nabla s \wedge d\omega$$

$$+ d_\nabla s \wedge d\omega + s \otimes \cancel{d^2 \omega}^{\,0}$$

$$= F_\nabla s \wedge \omega$$

Local description of F_∇

$\phi_U : U \to \mathbb{R}^m \quad \varphi_U : E|_U \to U \times \mathbb{R}^n \quad \dim M = m \quad \dim E|_p = n.$

$S : M \to E. \quad S_U = \phi_U^{-1} \circ s : \mathbb{R}^n \to \mathbb{R}$

$\nabla^0 \varphi_U s(p) = (p, ds_U(p))$

$\varphi_U(\nabla s)(p) = (p, ds_U + \alpha_U s_U)$

where α_U is a section of $(\mathrm{End}(E) \otimes T^*M)|_U$

$d_\nabla^2 s = d_\nabla(\nabla s) \quad \varphi_U(d_\nabla^2 s)(p) = (p, d_\nabla(ds_U + \alpha_U))$

also have α_U part because we use ∇ rather than ∇^0.

$= (p, \cancel{d^2 s_U}^{\,0} + (d\alpha_U)s_U - \alpha_U \wedge ds_U + \alpha_U \wedge (ds_U + \alpha_U s_U))$

minus sign because α_U is 1-form.

$d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^{\deg} \omega_1 \wedge d\omega_2$

$= (p, (d\alpha_U + \alpha_U \wedge \alpha_U) \cdot s_U)$

Thus, locally $(F_\nabla)_U = d\alpha_U + \alpha_U \wedge \alpha_U$

This is a matrix-value 2-form.

i.e. a section of $(\mathrm{End}(E) \otimes \Lambda^2 T^*M)|_U$

We look more carefully on the notation $\alpha_U \wedge \alpha_U$

because α_U is a matrix-value 1-form.

Write $dx^1, \dots, dx^n$ for basis of sections on $T^*M|_U$

$\alpha_U = \sum_k \alpha_{Uk} dx^k \quad \alpha_{Uk}$ are matrices (depend on p)

$\alpha_U \wedge \alpha_U = \sum_{i,j} \underbrace{\alpha_{Ui}\alpha_{Uj}}_{\text{matrix multiplication}} dx^i \wedge dx^j$

$\left(-dx^j \wedge dx^i = dx^i \wedge dx^j\right)$

$= \sum_{i<j} (\alpha_{Ui}\alpha_{Uj} - \alpha_{Uj}\alpha_{Ui})\, dx^i \wedge dx^j$

write $[\alpha_{U_i}, \alpha_{U_j}]$ for $\alpha_{U_i}\alpha_{U_j} - \alpha_{U_j}\alpha_{U_i}$,

called the <u>commutator</u>

Also, $d\alpha_U = \sum_{i<j} (\partial_i \alpha_{U_j} - \partial_j \alpha_{U_i})\, dx^i \wedge dx^j$

$$(F_\nabla)_U = \sum_{i<j} \underbrace{\left(\partial_i \alpha_{U_j} - \partial_j \alpha_{U_i} + [\alpha_{U_i}, \alpha_{U_j}]\right)}_{\text{write as } (F_{\nabla ij})_U} dx^i \wedge dx^j$$

Check the result is independent of charts

$(F_\nabla)_V = d\alpha_V + \alpha_V \wedge \alpha_V$ From the last class

$$\alpha_V = g_{UV}^{-1} \alpha_U g_{UV} + g_{UV}^{-1} d g_{UV}$$

$$= g_{VU} \alpha_U g_{VU}^{-1} + g_{VU}\, d g_{VU}^{-1}$$

$$\Longrightarrow (F_\nabla)_U = g_{UV}^{-1} (F_\nabla)_V\, g_{UV}.$$

$$\left(\begin{aligned} d\alpha_V &= -g_{UV}^{-1} dg_{UV} \wedge g_{UV}^{-1} dg_{UV} - g_{UV}^{-1} dg_{UV} \wedge g_{UV}^{-1} \alpha_U g_{UV} \\ &\quad + g_{UV}^{-1} d\alpha_U g_{UV} - g_{UV}^{-1} \alpha_U g_{UV} \wedge g_{UV}^{-1} dg_{UV} \\ \alpha_V \wedge \alpha_V &= g_{UV}^{-1} dg_{UV} \wedge g_{UV}^{-1} dg_{UV} + g_{UV}^{-1} \alpha_U g_{UV} \wedge g_{UV}^{-1} dg_{UV} \\ &\quad + g_{UV}^{-1} dg_{UV} \wedge g_{UV}^{-1} \alpha_U g_{UV} + g_{UV}^{-1} \alpha_U \wedge \alpha_U g_{UV} \end{aligned}\right)$$

Say more

Meaning of F_∇ (for simplicity on a chart)

write $dx^1, \dots, dx^n$ for basis of sections of $T^*M|_U$.

∇s_U can be written as $\sum_k \nabla_k s_U dx^k$

$\nabla_k s_U$ is the covariant derivative of s_U along $\frac{\partial}{\partial x^k}$.

For the usual partial derivative, we have $\frac{\partial^2 f}{\partial x^i \partial x^j} = \frac{\partial^2 f}{\partial x^j \partial x^i}$

but for covariant derivative.

$$\nabla_i \nabla_j s_U - \nabla_j \nabla_i s_U = (F_{\nabla ij})_U s_U$$

where

$$(F_\nabla)_U = \sum_{i<j} (F_{\nabla ij})_U dx^i \wedge dx^j$$

$$= \frac{1}{2} \sum_{i,j} (F_{\nabla ij})_U dx^i \wedge dx^j$$

Explicitly. $\nabla_i s_U = \partial_i s_U + \alpha_i s_U$

$$\nabla_i \nabla_j s_U = \partial_i \partial_j s_U + \partial_i(\alpha_j s_U) + \alpha_i \partial_j s_U + \alpha_i \alpha_j s_U$$

$$= \partial_i \partial_j s_U + (\partial_i \alpha_j) s_U + \alpha_j \partial_i s_U + \alpha_i \partial_j s_U + \alpha_i \alpha_j s_U$$

$$(\nabla_i \nabla_j - \nabla_j \nabla_i) s_U = (\partial_i \alpha_j - \partial_j \alpha_i + \alpha_i \alpha_j - \alpha_j \alpha_i) s_U$$

Class 14 Metric compatible covariant derivative (Chap 15)

Previously, we study ① the exterior derivative

$d : \Omega^k \to \Omega^{k+1}$, where $\Omega^k = C^\infty(M; \Lambda^k T^*M)$
$= \{\text{sections in } \Lambda^k T^*M\}$.

② the covariant derivative $\nabla : C^\infty(M;E) \to C^\infty(M; E \otimes T^*M)$

③ the exterior covariant derivative

$d_\nabla : C^\infty(M; E \otimes \Lambda^k T^*M) \to C^\infty(M; E \otimes \Lambda^{k+1} T^*M)$

and

④ the curvature F_∇ of d_∇ as a section of $\mathrm{End}(E) \otimes \Lambda^2 T^*M$.

In weeks 5-6, we study metric g on E, which is a section of $E^* \otimes E^*$ s.t. at each fiber, it is symmetric and positive definite.

Using partition of unity, we show the metric and the covariant derivative exist.

(the space of the metric is convex hence contractible and the space of covariant derivative is affine over $C^\infty(M; \mathrm{End}(E) \otimes T^*M)$)

In this week, we study metric compatible covariant derivative, i.e. for any sections u, v of E,

$$dg(u,v) = g(\nabla u, v) + g(u, \nabla v)$$

(some version of Leibniz rule)

Given $u, v \in C^\infty(M; E)$ $g(u,v)$ is a map $M \to \mathbb{R}$.

$dg(u,v)$ is a 1-form (we apply exterior derivative)

$\nabla u \in C^\infty(M; E \otimes T^*M)$, $\nabla u \otimes v \in C^\infty(M; E \otimes T^*M \otimes E)$

$g(\nabla u, v)$ means we apply g to two E components.

and hence also obtain a section in T^*M. i.e a 1-form.

Following the notation in previous class, we write everything on chart U and omit subscript U for s_U, α_U

Let $e_1, \dots, e_n$ be a basis of orthonormal sections of E, where orthonormal means $g(e_i(p), e_j(p)) = \delta_{ij}$

It exists by Gram-Schmit procedure.

Then $s = \sum_{k=1}^{n} s_k e_k$ $\quad s_k : U \to \mathbb{R}$

(In weeks 5-6, we write $s_1, \dots, s_n$ for a basis of sections and $v_1, \dots, v_k$ as functions $U \to \mathbb{R}$

Now the notation follows from those in weeks 7-8)

$$\nabla s = \sum_{i,j} (ds_i + \alpha_i^j s_j) e_i$$

α_i^j is component of α in previous class. α_i^j is 1-form.

especially $\nabla e_i = \sum \alpha_i^j e_j$
$:= \alpha_i^j e_j$

(write α_i^j rather than α_{ij} because we want to omit $\sum$ later when an index appears twice, we take sum over this index

From $dg(u,v) = g(\nabla u, v) + g(u, \nabla v)$

we have $0 = dg(e_i, e_j) = g(\nabla e_i, e_j) + g(e_i, \nabla e_j)$

$= g(\alpha_i^k e_k, e_j) + g(e_i, \alpha_j^l e_l)$

$= \alpha_i^j + \alpha_j^i$

i.e. the matrix value 1-form $\alpha = \{\alpha_j^i\}_{i,j}$

satisfies $\alpha^T + \alpha = 0$

Given ∇ on E, we can define ∇ on E^* as follows.

Let s be a section of E and s^* be a section

of E^*, we have a pairing $\langle s, s^* \rangle : M \to \mathbb{R}$

$d\langle s, s^* \rangle$ is a 1-form

Then we define ∇s^* by the following equation

$$d\langle s, s^* \rangle = \langle \nabla s, s^* \rangle + \langle s, \nabla s^* \rangle \quad (1)$$

(this equation holds for any s and fixed s^*)

We can also extend ∇ to $E^* \otimes E^*$ by

$$\nabla(s_1^* \otimes s_2^*) = \nabla s_1^* \otimes s_2^* + s_1^* \otimes \nabla s_2^* \quad (2)$$

Then we can define $\nabla g \in C^\infty(M; E^* \otimes E^* \otimes T^*M)$

Lem ∇ is metric compatible iff $\nabla g = 0$