Class 11 Exterior derivative and Lie derivative

Recall Ω^k is the space of k-form on M.

i.e. sections $M \to \Lambda^k T^*M$.

$\Omega^0 = C^\infty(M;\mathbb{R})$

Locally $\omega = \sum_{i_1<i_2<\cdots<i_k} f_{i_1\cdots i_k}\, dx^{i_1}\wedge\cdots\wedge dx^{i_k}$

$$d\omega = \sum_{i_1<\cdots<i_k}\sum_{m\notin\{i_1,\dots,i_k\}} \frac{\partial f_{i_1\cdots i_k}}{\partial x^m} dx^m\wedge dx^{i_1}\wedge\cdots\wedge dx^{i_k}$$

$$= \sum_{i_1<\cdots<i_k} df\wedge dx^{i_1}\wedge\cdots\wedge dx^{i_k}.$$

$d:\Omega^k \to \Omega^{k+1}$ is independent of charts

$d^2=0$. $\quad H^i_{dR}(M) = \dfrac{(\ker d:\Omega^i\to\Omega^{i+1})}{(\operatorname{Im} d:\Omega^{i-1}\to\Omega^i)}$

Prop 1) If $\psi: M\to N$ is a diffeomorphism. ψ^* induces an iso between $H^*_{dR}(N)$ and $H^*_{dR}(M)$ $\Rightarrow$ If $H^*_{dR}(N)\neq H^*_{dR}(M)$, then $M \not\cong_{diff} N$

2) If ψ and ϕ are homotopic map to N, i.e.

$\exists\ \Phi:[0,1]\times M\to N$ smooth, s.t. $\Phi(0,\cdot)=\phi$

and $\Phi(1,\cdot)=\psi$, then $\psi^*=\phi^*: H^*_{dR}(N)\to H^*_{dR}(M)$

Pf: 1). Since $\psi^* d = d\psi^*$. (details in next page)

(This is called a (co)chain map)

ψ^* induces a well-defined map

If $\psi: M \to N$ is a smooth map. ψ induces a map

$\psi^*: \Omega^k(N) \longrightarrow \Omega^k(M)$ by

$\psi^*\omega(v_1, \dots, v_n) = \omega(\psi_* v_1, \dots \psi_* v_n)$ $v_i \in T_pM$ ψ_* tangent map

which is well defined on $H^k_{dR}(N) \longrightarrow H^k_{dR}(M)$ because

1) For $v \in T_pM$ $df(v) = v(f) := \mathcal{L}_v f$

because $df = \frac{\partial f_0}{\partial x^i} dx^i$ $v = v^i \frac{\partial}{\partial x^i}$ $\mathcal{L}_v f = v^i \frac{\partial f_0}{\partial x^i}$

2) $\psi^* df(v) = df(\psi_* v) = (\psi_* v)(f) = v(\psi^* f) = d(\psi^* f)(v)$

$\Longrightarrow \psi^* df = d\psi^* f$

3) $d(\psi^*(dx^1 \wedge \dots \wedge dx^n)) = d(\psi^*(dx^1) \wedge \dots \wedge \psi^*(dx^n)) = 0.$

$(0 = \psi^* d(dx^i) = d(\psi^* dx^i))$

$d(w_1 + w_2) = dw_1 + dw_2$ $d(w_1 \wedge w_2) = dw_1 \wedge w_2 + (-1)^k w_1 \wedge dw_2$

4) $w = \sum_{i_1<i_2<\cdots<i_k} f_{i_1\cdots i_k}\, dx^{i_1}\wedge\cdots\wedge dx^{i_k}$

$$dw = \sum_{i_1<\cdots<i_k} \underbrace{\sum_{m\notin\{i_1,\cdots,i_k\}} \frac{\partial f_{i_1\cdots i_k}}{\partial x^m} dx^m}_{df} \wedge dx^{i_1}\wedge\cdots\wedge dx^{i_k}$$

$$\psi^* w = \sum (\psi^* f)\, \psi^*(dx^{i_1}\wedge\cdots\wedge dx^{i_k})$$

$$\psi^* dw = \sum \psi^*(df\wedge dx^{i_1}\wedge\cdots\wedge dx^{i_k})$$

$$= \sum \psi^*(df)\wedge \psi^*(dx^{i_1}\wedge\cdots\wedge dx^{i_k})$$

$$= \sum d(\psi^* f)\wedge \psi^*(\cdots\cdots)$$

$$(d\,\psi^*(dx^{i_1}\wedge\cdots\wedge dx^{i_k}) = 0)$$

$$= d\left(\sum (\psi^* f)\wedge \psi^*(\cdots\cdots)\right)$$

$$= d\psi^* w$$

$$\Longrightarrow \psi^* d = d\psi^*$$

5). If $dw=0$, then $d(\psi^* w) = \psi^* dw = 0$

If $w_1 - w_2 = dw_0$ then $\psi^*(w_1) - \psi^*(w_2)$

$$= \psi^* dw_0 = d(\psi^* w_0)$$

$\Longrightarrow \psi^*$ is well-defined on $\ker d / \operatorname{Im} d$

$$\psi^* : H^*_{dR}(N) \longrightarrow H^*_{dR}(M)$$

If ψ is a diffeo, $\exists\, \phi : N \to M$

s.t. $\psi\circ\phi = Id$ $\phi\circ\psi = Id$. Then ψ^*, ϕ^*

induce isomorphism between $H^*_{dR}(N)$ $H^*_{dR}(M)$

2) Note that $T^*_{(t,p)}([0,1]\times M) = T^*_t[0,1] \times T^*_pM$

$= \mathbb{R} \times T^*_pM$

a 1-form on $T^*([0,1]\times M)$ can be written as

$$\alpha_t dt + \beta_t \quad \alpha_t \in \Omega^0(M) = C^\infty(M;\mathbb{R})$$
$$\beta_t \in \Omega^1(M)$$

$w \in \Omega^k(N)$ $\Phi^* w \in \Omega^k([0,1]\times M)$ can be written as

$$\Phi^* w = dt \wedge \alpha_t + \beta_t \quad \alpha_t \in \Omega^{k-1}(M) \quad \beta_t \in \Omega^k(M)$$

If $dw = 0$, then $0 = d(\Phi^* w)$

$$= -dt \wedge d^\perp \alpha_t + dt \wedge \frac{\partial}{\partial t}\beta_t + d^\perp \beta_t$$

$d(w_1 \wedge w_2) = dw_1 \wedge w_2 + (-1)^k w_1 \wedge dw_2$

where $d^\perp$ is the exterior derivative on $\{t\} \times M \subset [0,1]\times M$

This implies $\frac{\partial}{\partial t}\beta_t = d^\perp \alpha_t$, $d^\perp \beta_t = 0$

by product result

($d^\perp \beta_t = 0$ means $\Phi(t,\cdot)^* w$ is closed for any t

integration on function of t)

$$\psi^* w - \phi^* w = \beta_1 - \beta_0 = \int_0^1 \frac{\partial}{\partial t}\beta_t = \int_0^1 d^\perp \alpha_t = d\int_0^1 \alpha_t$$

d on M

So $\psi^* w - \phi^* w \in \operatorname{Im} d$ $[\psi^* w] = [\phi^* w]$

since $H_{dR} = \ker d / \operatorname{Im} d$ $\square$

Cor. (Poincaré Lem) Let $U \subset M$ be a contractible open set, i.e. $\exists$ smooth map $\Phi : [0,1] \times U \to M$

s.t. $\Phi(1,\cdot)$ is the inclusion

$\Phi(0,\cdot)$ maps U to a pt $p \in M$

Then $H^k_{dR}(U) = 0$ for $k \geq 1$

i.e. any closed k-form ω ($d\omega = 0$) with $k \geq 1$

$\exists$ a $(k-1)$-form α s.t. $d\alpha = \omega$

Pf: $$\underbrace{[\Phi(1,\cdot)^* \omega]}_{\omega|_U} = \underbrace{[\Phi(0,\cdot)^* \omega]}_{0} \quad \text{forms on } U$$

Since $\Phi(0,\cdot) : U \to M$ factors through $U \to p \to M$ $\quad \Lambda^k T_p = 0$ for $k \geq 1$

$\Phi(0,\cdot)_* = 0$. $\Phi(0,\cdot)^* \omega = 0$ $\square$

Lie derivative on k-forms.

Suppose M is compact and V is a vector field

From vector field thm, we have a smooth map

$\sigma : (-\varepsilon, \varepsilon) \times M \to M$ s.t.

1) $\sigma(0, p) = p$

2) $\sigma_* \frac{\partial}{\partial t} = V$

(We use the compactness to find $\varepsilon > 0$ for any pt $p \in M$)

For $\omega \in \Omega^k(M)$, consider $\sigma^*\omega \in \Omega^k((-\varepsilon,\varepsilon)\times M)$

$$\sigma^*\omega = dt \wedge \alpha_t + \beta_t \qquad \alpha_t \in \Omega^{k-1}(M) \quad \beta_t \in \Omega^k(M)$$

Define $\mathcal{L}_V \omega = \frac{\partial}{\partial t}\beta_t \Big|_{t=0}$

Fact (Cartan formula) $\mathcal{L}_V \omega = (\iota_V d + d\iota_V)\omega$

where for a k form α, $\iota_V\alpha$ is a k-1 form defined by

$$\iota_V\alpha(V_1, \dots, V_{k-1}) = \alpha(V, V_1, \dots, V_{k-1})$$

So $\underset{\substack{\uparrow \\ k\text{-form}}}{\mathcal{L}_V\omega}(V_1, \dots, V_k) = \underset{\substack{\uparrow \\ (k+1)\text{form}}}{d\omega}(V, V_1, \dots, V_k)$

$$+ d\iota_V\omega(V_1, \dots, V_k)$$

The proof needs careful computations so we omit it.

Note that $\alpha_t|_{t=0} = \alpha_0 = \iota_V\omega$

because $\sigma^*\omega\left(\frac{\partial}{\partial t}, V_1, \dots, V_k\right)\Big|_{t=0}$

$$= \omega\left(\sigma_*\frac{\partial}{\partial t}, \sigma_* V_1, \dots, \sigma_* V_k\right)\Big|_{t=0}$$

$$= \omega(V, \sigma_* V_1, \dots, \sigma_* V_k)$$

$$= dt \wedge \alpha_t\left(\frac{\partial}{\partial t}, V_1, \dots, V_k\right)\Big|_{t=0} + \cancel{\beta_t\left(\frac{\partial}{\partial t}, V_1, \dots, V_k\right)}^{\,0}\Big|_{t=0}$$

$$= \alpha_0(V_1, \dots, V_k)$$

Recall that we define $\mathcal{L}_V$ as a derivation

$$\begin{array}{ccc} C^\infty(M;\mathbb{R}) & \longrightarrow & C^\infty(M;\mathbb{R}) \\ \parallel & & \parallel \\ \Omega^0 & & \Omega^0 \end{array}$$

The definition here $\mathcal{L}_V : \Omega^k \longrightarrow \Omega^k$ can be regarded as an extention of derivation on $\Omega^* = \bigoplus_{k=0}^{n} \Omega^k$

It satisfies the Leibniz rule

$$\mathcal{L}_V(\alpha \wedge \beta) = \mathcal{L}_V\alpha \wedge \beta + \alpha \wedge \mathcal{L}_V\beta$$

Ω^* is called the exterior algebra by wedge product $\wedge$

It is an example of superalgebra, which is a vector space with $\mathbb{Z}/2 = \{0,1\}$ grading and $\alpha\beta = (-1)^{\deg\alpha \deg\beta}\beta\alpha$ where $\deg\alpha, \deg\beta \in \{0,1\}$ for homogeneous α, β.

Note that from $dx^i \wedge dx^j = -dx^j \wedge dx^i$,

we can show for α k-form, β l-form.

$$\alpha \wedge \beta = (-1)^{kl} \beta \wedge \alpha$$

$$\Omega^* = \underbrace{\bigoplus_{k\,\text{even}} \Omega^k}_{\text{grading } 0} \oplus \underbrace{\bigoplus_{k\,\text{odd}} \Omega^k}_{\text{grading } 1}$$

Class 12 Covariant derivative (Chap 11)

Exterior derivative : $d(w_1 \wedge w_2) = dw_1 \wedge w_2 + (-1)^{\deg w_1} w_1 \wedge dw_2$

Lie derivative : $\mathcal{L}_V(w_1 \wedge w_2) = \mathcal{L}_V w_1 \wedge w_2 + w_1 \wedge \mathcal{L}_V w_2$

Ex: $w_1 = dx^1 \quad w_2 = f dx^2 \quad d(dx^1 \wedge f dx^2) = d(f dx^1 \wedge dx^2) = df \wedge dx^1 \wedge dx^2$

$d(dx^1) \wedge (f dx^2) = 0 \quad dx^1 \wedge d(f dx^2) = dx^1 \wedge df \wedge dx^2 = -df \wedge dx^1 \wedge dx^2$

This class introduces a new derivative.

Let $\pi: E \to M$ be a vector bundle $s: M \to E$ a section

we want to find some derivative of s

Ex. If $E = M \times \mathbb{R}^n$, the section $s: M \to E$ is just

$S(p) = (p, S_1(p), \dots, S_n(p)) \quad S_i : M \to \mathbb{R}$

We can define $(S_i)_* : T_pM \to T_{S_i(p)}\mathbb{R} \cong \mathbb{R}$

$(S_i)_*$ can be regarded as a section of T^*M.

Hence we can contruct $ds(p) := (p, (S_1)_*(p), \dots, (S_n)_*(p))$

as a section of $E \otimes T^*M = T^*M^{\oplus n}$

Then we extend the construction to general vector bundle

Def Let $C^\infty(M; E)$ be the space of sections $M \to E$.

A <u>covariant derivative</u> is a map

$\nabla : C^\infty(M;E) \longrightarrow C^\infty(M; E \otimes T^*M)$ obeying the

Leibniz rule $\quad \nabla(fs) = s \otimes df + f \nabla s$

for any $f \in C^\infty(M;\mathbb{R})$, $s \in C^\infty(M;E)$

Rem. The extra factor T^*M means, if we specify a vector field V, we can define $\nabla_V : C^\infty(M;E) \to C^\infty(M;E)$

Construction of ∇

Take a locally finite open cover $\mathcal{U}$ of M and a partition of unity $\{\chi_\alpha : U_\alpha \to \mathbb{R}_{\geq 0}\}$ $\quad \varphi_\alpha : E|_{U_\alpha} \to U_\alpha \times \mathbb{R}^n$

Set $\nabla s = \sum_{U_\alpha \in \mathcal{U}} \chi_\alpha \varphi_\alpha^{-1}(d\varphi_\alpha(s))$

where $\varphi_\alpha(s) : U_\alpha \to U_\alpha \times \mathbb{R}^n$

$\varphi_\alpha(s)(p) = (p, s_{\alpha 1}(p), \dots, s_{\alpha n}(p))$

and $d\varphi_\alpha(s) : U_\alpha \longrightarrow U_\alpha \times (\mathbb{R}^n \otimes \mathbb{R}^m)$ $\quad m = \dim M$, $n = \dim E_p$

$d\varphi_\alpha(s)(p) = (p, (s_{\alpha 1})_*(p), \dots (s_{\alpha n})_*(p))$

φ_α^{-1} extends to $U_\alpha \times (\mathbb{R}^n \otimes \mathbb{R}^m) \longrightarrow E \otimes T^*M|_{U_\alpha}$

To check $\nabla fs = s \otimes df + f \nabla s$

it suffices to check over U_α

$d\varphi_\alpha(fs)(p) = (p, (fs_{\alpha 1})_*(p) \dots \quad)$

$s_{\alpha 1} : U_\alpha \longrightarrow \mathbb{R}$ $\quad (s_{\alpha 1})_*|_p : T_pU_\alpha \longrightarrow T_{s_{\alpha 1}(p)}\mathbb{R}$

$f : U_\alpha \longrightarrow \mathbb{R}$

$(fs_{\alpha 1})_* = df \otimes s_{\alpha 1} + f \otimes ds_{\alpha 1}$

as section $U_\alpha \longrightarrow T^*U_\alpha$.

Another construction of ∇

Consider E as a subbundle of $M \times \mathbb{R}^N$ and write $s: M \to E$ as a section of $s: M \to M \times \mathbb{R}^N$

($E|_p$ is a linear subspace of $p \times \mathbb{R}^N$)

Let $\nabla s = \Pi_E ds$, where ds is a section of $M \times \mathbb{R}^N \otimes T^*M$ and Π_E is the orthogonal projection to $E \otimes T^*M$ by the standard inner product in $\mathbb{R}^N$

Since $\Pi_E s = s$, $\nabla fs = \Pi_E d(fs)$

$$= \Pi_E(f \otimes ds + df \otimes s)$$

$$= s \otimes df + f \nabla s$$

Rem. The covariant derivative is not unique.

Suppose ∇ and ∇' are two covariant derivatives

$$(\nabla - \nabla')(fs) = f(\nabla - \nabla')s$$

Claim $\nabla - \nabla'$ is a section of $\mathrm{End}(E) \otimes T^*M$

$$:= \mathrm{Hom}(E, E) \otimes T^*M$$

Lem. Let E, E' be two vector bundles over M

If α is a map from $C^\infty(M; E)$ to $C^\infty(M; E')$

that is linear over $C^\infty(M; \mathbb{R})$, i.e. $f\alpha = \alpha f$,

then α is a section of $\mathrm{Hom}(E, E')$

pf. Locally, fix a basis $\{e_1, \dots, e_n\}$ for E and

a basis $\{e'_1, \dots, e'_m\}$ for E'

Then we can write $\alpha e_j = \sum_i A_{ij} e'_i$

If s is a section of E $s = \sum_j s_j e_j$, $s_j : U \to \mathbb{R}$

$$\alpha s = \alpha\Big(\sum_j s_j e_j\Big) = \sum_j s_j \, \alpha(e_j) = \sum_{i,j} s_j A_{ij} e'_i$$

Then α is defined by $\{A_{ij}\}$ as a section of

$\mathrm{Hom}(E, E')$, change basis of E or E' will change

$\{A_{ij}\}$ by matrix multiplication. $\square$

we use α rather than $\mathfrak{a}$ (\mathfrak{a}) in latex just for writing simplicity.

Conversely, given $\alpha \in C^\infty(M; \mathrm{End}(E) \otimes T^*M)$,

we can construct $\nabla' = \nabla + \alpha$. This is still a

covariant derivative

Rem The space of covariant derivatives is affine over $C^\infty(M; \mathrm{End}(E) \otimes T^*M)$. This means these two spaces are isomorphic, but the isomorphism is canonical only when we pick an element ∇

Then we consider the transition of ∇ in different charts.

Locally. $\phi_U: U \to \mathbb{R}$ $\varphi_U: E|_U \to U \times \mathbb{R}^n$, $s: M \to E$

define the covariant derivative ∇^0 by

$$\nabla^0 \varphi_U s(p) = (p, ds_U(p))$$

$$s_U = \phi_U^{-1} \circ s: \mathbb{R}^n \to \mathbb{R}$$

For a general covariant derivative ∇, we know

$$\varphi_U(\nabla s)(p) = (p, ds_U + \alpha_U s_U)$$

where α_U is a section of $(\mathrm{End}(E) \otimes T^*M)|_U$

For another chart V with $U \cap V \neq \phi$

we have $\varphi_V(\nabla s)(p) = (p, ds_V + \alpha_V s_V)$

$s_V = g_{VU} s_U$ for bundle transition function

$$g_{UV}: V \cap U \to GL(n, \mathbb{R})$$

$$ds_v + \alpha_v s_v = d(g_{vu} s_u) + \alpha_v (g_{vu} s_v)$$

$$= g_{vu} ds_u + dg_{vu} s_u + \alpha_v g_{vu} s_v.$$

$$= g_{vu}(ds_u + (g_{vu}^{-1} \alpha_v g_{vu} + g_{vu}^{-1} dg_{vu}) s_u))$$

Since ∇s is a section of $E \otimes T^*M$.

this should equal to $g_{vu}(ds_u + \alpha_u s_u)$

Thus, $ds_u + (g_{vu}^{-1} \alpha_v g_{vu} + g_{vu}^{-1} dg_{vu}) s_u$

$$= ds_u + \alpha_u s_u$$

$$\Rightarrow \alpha_u = g_{vu}^{-1} \alpha_v g_{vu} + g_{vu}^{-1} d g_{vu}$$

$$\Rightarrow \alpha_v = g_{uv}^{-1} \alpha_u g_{uv} + g_{uv}^{-1} d g_{uv}$$

$$= g_{vu} \alpha_u g_{vu}^{-1} + g_{vu} d g_{vu}^{-1}$$