Class 4 Vector bundle (Chap 3)

Manifold + group = Lie group

Manifold + vector space = vector bundle

Def 1: Let $M$ be a smooth mfd of dim $m$.

A smooth mfd $E$ is called a <u>(real) vector bundle</u> over $M$ with fiber dimension $n$ if we have the following

1) There is a smooth map $\pi : E \to M$

for each $p \in M$, $\exists\, U_p \subset M$ $\lambda_U : \pi^{-1}(U_p) \to \mathbb{R}^n$

s.t. for each $x \in U_p$. $\lambda_U : \pi^{-1}(x) \to \mathbb{R}^n$

is a diffeomorphism

2) There is a smooth map $\hat{o} : M \to E$ s.t. $\pi \circ \hat{o} = Id$

3) There is a smooth map $\mu : \mathbb{R} \times E \to E$ s.t.

a) $\pi(\mu(r,v)) = \pi(v)$

b) $\mu(r, \mu(r',v)) = \mu(rr',v)$

c) $\mu(1,v) = v$

d) $\mu(r,v) = v$ for $r \neq 1$ iff $v \in \operatorname{Im} \hat{o}$

4) $\lambda_U(\mu(r,v)) = r\lambda_U(v)$ for any $U$ in 1)

Ex. The product bundle (also called the trivial bundle)

$E = M \times \mathbb{R}^n$

Rem 1) $\pi$ is called the (bundle) <u>projection map</u>

For $W \subset M$, write $E|_W = \pi^{-1}(W)$

For $p \in M$, $E|_p$ is called a <u>fiber</u> over $p$

$\lambda_U$ defines a diffeo $\varphi_U : E|_U \to U \times \mathbb{R}^n$ by

$(\pi(v), \lambda_U(v))$ $\varphi_U$ is called <u>local trivialization</u> of $E$

(Remember a topo mfd can have multiple smooth strs, here we specify the smooth str on $E$ by " $E|_U \to U \times \mathbb{R}^n \xrightarrow{\phi_U \times Id} \mathbb{R}^m \times \mathbb{R}^n$ ")

2) $\hat{0}$ or its image is called the zero section

3) $\mu$ corresponds to the scalar multiplication in the vector space

we usually write $\mu(r, v)$ as $rv$

Def 2: Let $E, E'$ be two vector bundles over $M$.

A <u>section</u> of $E$ is a smooth map $s: M \to E$ s.t. $\pi \circ s = Id$

A <u>homomorphism</u> $\phi: E \to E'$ is a smooth map s.t.

1) $\pi'(\phi(v)) = \pi(v)$ $(\hat{\pi}: E \to M$ $\hat{0}: M \to E)$

2) $\phi(rv) = r\phi(v)$ (Hence $\phi(\hat{0}(p)) = \hat{0}'(p)$)

A <u>bundle isomorphism</u> is a homomorphism with an inverse $g: E' \to E$

Question: Is any bundle isomorphic to the trivial bundle?

Counterexample: Möbius bundle over $S^1$

$$\left\{(\theta, v) \in S^1 \times \mathbb{R}^2 : \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} v = v \right\}$$

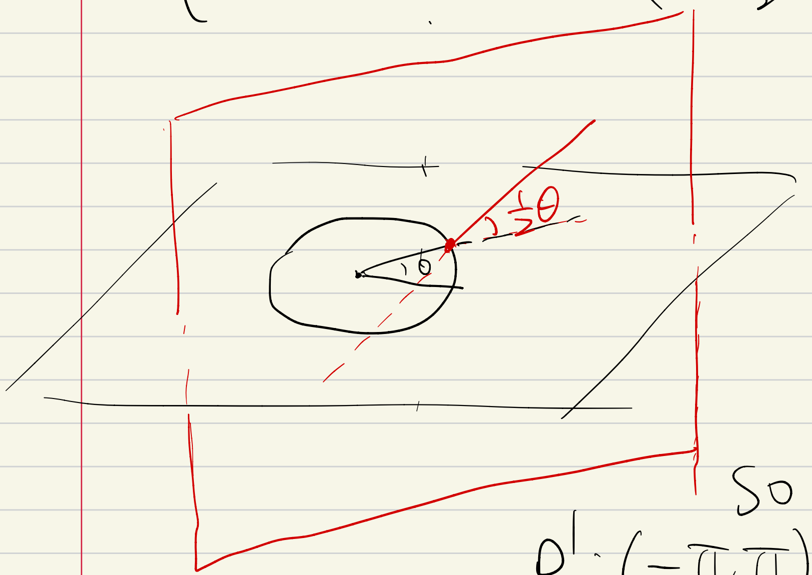
We have a smooth map

$$\rho : (0, 2\pi) \times \mathbb{R} \longrightarrow E$$

$$(\theta, t) \longmapsto (\theta, t\cos(\tfrac{1}{2}\theta), t\sin(\tfrac{1}{2}\theta))$$

$$\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} \begin{pmatrix} \cos\frac{1}{2}\theta \\ \sin\frac{1}{2}\theta \end{pmatrix} = \begin{pmatrix} \cos\theta\cos\frac{1}{2}\theta + \sin\theta\sin\frac{1}{2}\theta \\ \sin\theta\cos\frac{1}{2}\theta - \cos\theta\sin\frac{1}{2}\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos\frac{1}{2}\theta \\ \sin\frac{1}{2}\theta \end{pmatrix}$$

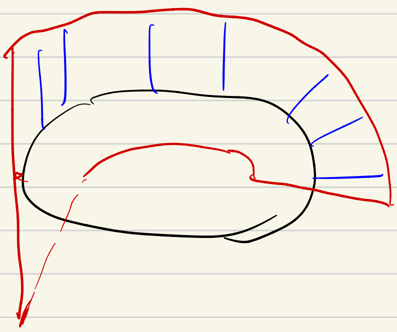


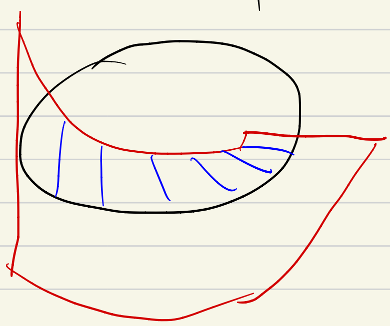

$\rho$ is not continuous at $\theta = 0 = 2\pi$

So we need the second chart

$$\rho' : (-\pi, \pi) \times \mathbb{R} \longrightarrow E$$

using the same map

$\theta = \pi$  $\theta = 0$

 $\theta = 2\pi$

We can find a nonzero section of $S^1 \times \mathbb{R} = E'$

(just use $s : S^1 \longrightarrow E'$ $s(\theta) = (\theta, 1)$)

But we cannot find a nonzero section of Möbius bundle

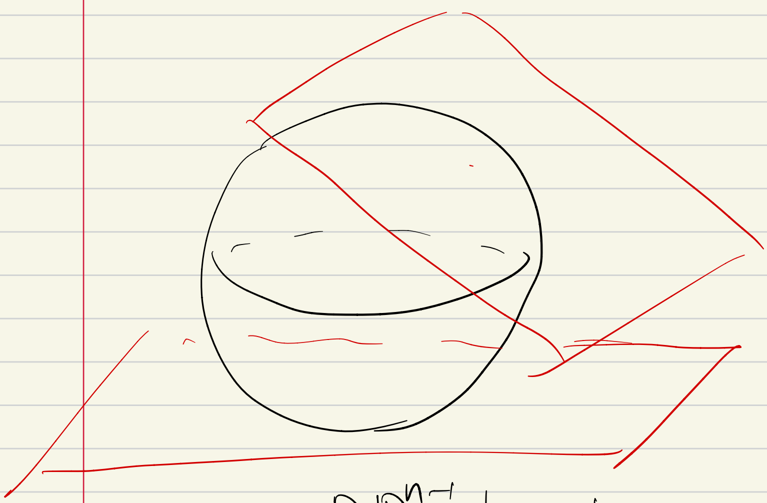
(Intuitively, when we move $s(0)$ smoothly to $s(2\pi)$, we will have $s(0) \neq s(2\pi)$)

More examples of vector bundles.

- Let $S^2 = \{x \in \mathbb{R}^3 \mid |x| = 1\}$ $E = \{(x,v) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid \substack{|x|=1 \\ x \cdot v = 0}\}$

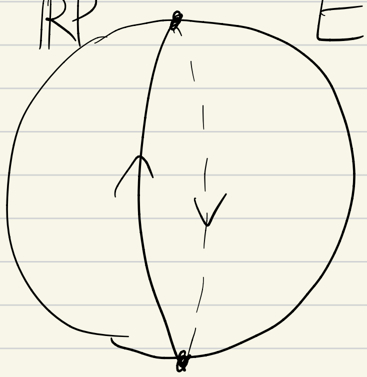
$\pi : E \to S^2$ $\quad \pi(x,v) = x \quad r(x,v) = (x, rv)$ $r \in \mathbb{R}$

$\hat{0} : S^2 \to E \quad \hat{0}(x) = (x, 0)$



- Let $\mathbb{R}P^{n-1}$ be the space of lines through $0 \in \mathbb{R}^n$ (a pt is a line) It can also be regarded as $S^{n-1}/\{\pm 1\}$ where $-1$ acts on $x$ by $-x$.

$\mathbb{R}P^2$



$E = \{(\pm x, v) \in \mathbb{R}P^{n-1} \times \mathbb{R}^n \mid v = r \cdot x \text{ for } r \in \mathbb{R}\}$

tautological bundle

More properties about sections:

Prop 1: 1) $\phi: E \to E'$ is a homomorphism. $s: M \to E$ is a section. Then $s': x \mapsto \phi(s(x))$ is a section of $E'$ called the <u>pushforward</u> of $s$

2) $s_1, s_2: M \to E$ sections. Then $s_1 + s_2 : x \mapsto s_1(x) \underset{\text{sum in fiber } \mathbb{R}^n}{+} s_2(x)$ is also a section. Also $rs_1$ is a section. for $r \in \mathbb{R}$

Hence the space of sections is a vector space.

3) $f: M \to \mathbb{R}$, $s: M \to E$. Then $fs$ is also a section.

4) On a nbhd $U$ of any $p \in M$. we have a diffeo

$\varphi_U: E|_U = \pi^{-1}(U) \longrightarrow U \times \mathbb{R}^n$

Suppose $e_1, \cdots, e_n$ are basis of $\mathbb{R}^n$. Then we can construct sections $s_1, \cdots, s_n : U \longrightarrow E|_U$ $\quad s_i(x) = \varphi_U^{-1}(x, e_i)$

i.e. locally, the vector bundle is always generated by sections.

Rem Globally (on $M$), we may not have everywhere nonzero sections (e.g. Möbius bundle)

5) If we have sections $s_1, \cdots, s_n : M \to E$ so that in any nbhd $U$. $s_i|_U$ forms a basis

Then we can use them to trivialize $E$:

i.e. $\exists$ a bundle isomorphism $\phi: M \times \mathbb{R}^n \longrightarrow E$

$$(x, \textstyle\sum r_i e_i) \longmapsto \textstyle\sum_i r_i s_i(x)$$

We have an alternative way to define $E$

Def 2. Fix a locally finite open cover $\mathcal{U}$ of $M$

For $U, V \in \mathcal{U}$ s.t. $U \cap V \neq \phi$. choose a function

$g_{VU} : U \cap V \longrightarrow GL(n, \mathbb{R})$ called <u>bundle transition function</u>

$$E = \bigcup_{U \in \mathcal{U}} U \times \mathbb{R}^n \Big/ (p, u) \in U \times \mathbb{R}^n \sim (p, g_{VU} \cdot u) \in V \times \mathbb{R}^n \quad p \in U \cap V$$

$\{g_{VU}\}_{U,V \in \mathcal{U}}$ need to satisfy the following condition

1) $g_{VU} = g_{UV}^{-1} \quad U \cap V \neq \phi$

2) $g_{VU} \circ g_{UW} \circ g_{WU} = Id \quad U \cap V \cap W \neq \phi$

called <u>cocycle condition</u>

The reason to introduce $g_{VU}$ is the following

From Def 1. for any $p \in M$. we have a nbhd $U \subset M$

s.t. $E|_U = \pi^{-1}(U) \xrightarrow{\cong} U \times \mathbb{R}^n$

$v \longmapsto (\pi(v), \lambda_U(v))$

If $v_1, v_2 \in E|_p$, then we can define

$$v_1 + v_2 = (\lambda_U|_p)^{-1} (\underbrace{\lambda_U|_p(v_1)}_{\mathbb{R}^n} + \underbrace{\lambda_U|_p(v_2)}_{\mathbb{R}^n}) \in E|_p$$

The definition should be independent of the choice of $U$

Because $(L_s f)_U = \sum_{k=1}^{n} V_k \frac{\partial f_U}{\partial x^k}$. we may write $S|_U = \sum_{k=1}^{n} V_k \frac{\partial}{\partial x^k}$ and write $\{\frac{\partial}{\partial x^k}\}$ as a basis of sections for $T\mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n$

This notation is important becauce in the future we usually do calculation locally and write a vector field as $\sum_{k=1}^{n} V_k \frac{\partial}{\partial x^k}$

For another chart $V$. we write $\frac{\partial}{\partial y^k}$ as basis and $S|_V = \sum V_k' \frac{\partial}{\partial y^k}$, we have

$$\frac{\partial(f\circ\phi_V^{-1})}{\partial y^k} = \frac{\partial(f\circ\phi_U^{-1}\circ\phi_U\circ\phi_V^{-1})}{\partial y^k} = \sum_l \frac{\partial(f\circ\phi_U^{-1})}{\partial x^l} \frac{\partial(\chi_l\circ\phi_U\circ\phi_V^{-1})}{\partial y^k}$$

$$\phi_U\circ\phi_V^{-1} : \mathbb{R}^n \text{ (with basis } y^k) \longrightarrow \mathbb{R}^n \text{ (with basis } x^k)$$

So $\frac{\partial}{\partial y^k} = \sum_l \frac{\partial(\chi_l\circ\phi_U\circ\phi_V^{-1})}{\partial y^k} \frac{\partial}{\partial x^l}$ ($\chi_l : \mathbb{R}^n \to \mathbb{R}$ projection to $l$-th coordinate)

$(\frac{\partial}{\partial y^1}, \dots, \frac{\partial}{\partial y^n}) = (\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n})(h_{UV})_*$ another reason to write as $\frac{\partial}{\partial x^k}$

Also $\underbrace{\sum_l V_l \frac{\partial}{\partial x_l}}_{= S|_U} = \underbrace{\sum_k V_k' \frac{\partial}{\partial y^k}}_{= S|_V} = \sum_k V_k' \left(\sum_l \frac{\partial(\chi_l\circ\phi_U\circ\phi_V^{-1})}{\partial y^k} \frac{\partial}{\partial x_l}\right)$

take coefficients of $\frac{\partial}{\partial x^l}$

$$\Longrightarrow V_l = \sum_k V_k' \frac{\partial(\chi_l\circ\phi_U\circ\phi_V^{-1})}{\partial y^k}$$

$$\Longrightarrow \begin{pmatrix} V_1 \\ \vdots \\ V_n \end{pmatrix} = (h_{UV})_* \begin{pmatrix} V_1' \\ \vdots \\ V_n' \end{pmatrix} \Rightarrow \begin{matrix} (L_s f)_U = \sum V_k \frac{\partial f_U}{\partial x^k} \\ (L_s f)_V = \sum V_k' \frac{\partial f_V}{\partial y^k} \end{matrix} \text{ equal}$$

$$\left(\frac{\partial f}{\partial x^1}, \dots, \frac{\partial f}{\partial x^n}\right)\begin{pmatrix} V_1 \\ \vdots \\ V_n \end{pmatrix} = (L_s f)_U$$

Conversely, given a derivation $\mathcal{L}$. We construct a vector field as follows

Let $V_K : U \to \mathbb{R}$ be defined by

$\mathcal{L}(x_K \circ \phi_U)$ where $x_K : \mathbb{R}^n \to \mathbb{R}$ is the projection

Let $\frac{\partial}{\partial x^k} : U \to TM|_U \cong U \times \mathbb{R}^n$ be the section

Correspondly to basis of $\mathbb{R}^n$ set $S = \sum_K V_K \frac{\partial}{\partial x^k}$

Exercise: for any $f : M \to \mathbb{R}$

$$(Lf)|_U = (L_S f)|_U = \sum_K V_K \frac{\partial (f \circ \phi_U^{-1})}{\partial x^k}$$

A function $f : M \to \mathbb{R}$ define a vector field $\nabla f$ as follows.

for a chart $U \subset M$, $\phi_U : U \to \mathbb{R}^n$

Define $f_U = f \circ \phi_U^{-1} : \mathbb{R}^n \to \mathbb{R}$ Consider $(f_U)_* = \left(\frac{\partial f_U}{\partial x_1}, \ldots \frac{\partial f_U}{\partial x_n}\right)$

Let $(\nabla f)|_U = \sum_K \frac{\partial f_U}{\partial x_K} \frac{\partial}{\partial x_K}$.