Class 23 Classification of principal G-bundle

Thm: If $\psi, \phi : M \to N$ are homotopic,

i.e. $\exists\, \Phi : [0,1] \times M \to N$ $\Phi(0,-) = \psi$ $\Phi(1,-) = \phi$

For $\pi : P \to N$, $\psi^* P$ is isomorphic to $\phi^* P$

Pf: Fix a connection A on $\Phi^* P \to [0,1] \times M$

Given $p \in \psi^* P$, let $\gamma_{A,p}(t)$ be the

horizontal lift of $\gamma(t) = \Phi(t, \pi(p))$

(This path has the same image on M)

Then $\gamma_{A,p}(1) \in \phi^* P$

The map $p \mapsto \gamma_{A,p}(1)$ is an isomorphism

from $\psi^* P$ to $\phi^* P$. it is G-equ

because $\gamma_{A,pg^{-1}}(t) = \gamma_{A,p}(t)\, g^{-1}$

Cor. If M is homotopic to $\mathbb{R}^n$ (or contractible), then any principal bundle over M is product G bundle

Fact For any G, $\exists$ a universal classifying space BG (unique up to homotopy), s.t.

$$\{\text{principal } G\text{-bundle over } M\}/_{\text{iso}}$$

$$\Longleftrightarrow \{\text{maps from } X \text{ to } BG\}/_{\text{homotopy}} =: [X, BG]$$

Ex: $G = S^1 = U(1)$ $\quad BG = \mathbb{CP}^\infty$ $(\mathbb{CP}^n \hookrightarrow \mathbb{CP}^{n+1})$

$G = O(n)$ $\quad BG = \mathrm{Grass}(n, \mathbb{R}^\infty)$ ($\mathbb{R}^n$ in $\mathbb{R}^\infty$)

Another understanding of charateristic classes (for p.b. or v.b. by framed bundle)

$$H^*(\mathbb{CP}^\infty) = \mathbb{Z}[x] \quad \deg x = 2$$

first Chern class of P is the pull-back of the generator $x \in H^2(\mathbb{CP}^\infty)$ by

$$M \longrightarrow BU(1) = \mathbb{CP}^\infty$$

Yang Mills equation and ASD equation

Given g on TM, we can define the volume form $dvol$ by picking orthonormal basis $e^1, \cdots, e^n$ of T^*M and $dvol = e^1 \wedge \cdots \wedge e^n$.

There exists an operator $* : \Omega^k \to \Omega^{n-k}$, $n = \dim M$

s.t. $v \wedge * v = |v|^2 dvol$.

where $|\cdot|$ is induced by g.

Explicitly $*(e^1 \wedge \cdots \wedge e^k) = e^{k+1} \wedge \cdots \wedge e^n$

$$*(e^{i_1} \wedge \cdots \wedge e^{i_k}) = \pm e^{i_{k+1}} \wedge \cdots \wedge e^{i_n}$$

where $\{i_1, \cdots, i_k, \cdots, i_n\} = \{1, \cdots, n\}$

we have $*^2 = (-1)^{k(n-k)} : \Omega^k \to \Omega^k$

In particular, let $n=4$ $k=2$. we have $*^2 = 1$

$$\Omega^2 = \Omega^+ \oplus \Omega^- \qquad * = \pm 1 \text{ on } \Omega^\pm$$

$\Omega^\pm$ is generated by

$$e_1 \wedge e_2 \pm e_3 \wedge e_4$$
$$e_1 \wedge e_3 \pm e_4 \wedge e_2$$
$$e_1 \wedge e_4 \pm e_2 \wedge e_3$$

F_A is a matrix valued 2-form.

we also have $*F_A$.

Let $F_A^{\pm} = \frac{1}{2}(F_A \pm *F_A)$ $\quad *F_A^{\pm} = \pm F_A^{\pm}$

A is anti-self-dual if $F_A^{+} = 0$

$\Longleftrightarrow F_A = -*F_A$

This is the anti-self-dual equation

recall the space of ∇ is affine over

$C^\infty(\mathrm{End}(E) \otimes T^*M)$.

$F_A^{+} = 0$ is an equation on the space of conn

the space of conn is also affine over

$C^\infty(\mathfrak{g} \otimes T^*M)$ $\mathfrak{g}$-valued 1-form

(Note we should start with Rmfd M with g on TM)

Also, we define $d_A^* = *d_A*$

Yang-Mills equation $d_A^* F_A = 0$

Note that if $F_A^{+} = 0$ $\quad F_A = -*F_A$,

then $d_A * F_A = -d_A F_A = 0$

by Bianchi identity.

In particular $G = U(1)$ $\mathfrak{g} = \{ c \in \mathbb{C} \mid c + c^* = 0 \}$
$= i\mathbb{R}$

A is affine over $C^\infty(i\mathbb{R} \otimes T^*U)$

$d_A^* F_A = 0$ is indeed the Maxwell's equations

for electromagnetism in physics

$G = SU(2)$ $\mathfrak{g} = \{ X \in M(2, \mathbb{C}) \mid \operatorname{tr} X = 0 \ X + X^* = 0 \}$

nonabelian

$= \operatorname{span} \left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$

Pauli matrices

$d_A^* F_A = 0$ is the original one studied in physics

Class 24 Topological application of ASD

Donaldson diagonal thm: Suppose X is a closed, oriented, connected, simply-connected smooth 4-manifold,
$(\pi_1(X) = \{e\})$

Consider the cup product (wedge product on $\mathbb{Z}$ coeff)

$$H^2(X;\mathbb{Z}) \times H^2(X;\mathbb{Z}) \longrightarrow H^4(X;\mathbb{Z})$$

as a bilinear form, Q_X.

If Q_X is negative definite, then $\exists A \in GL(n;\mathbb{Z})$

$$A^T Q_X A = \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & \ddots & \\ & & & -1 \end{pmatrix}$$

(over $\mathbb{R}$, $A \in GL(n,\mathbb{R})$ also exists by linear algebra)

(over $\mathbb{Z}$, a necessary condition of diagnonalizable is that all entries on the diagnol are -1

simplest counterexample: $E8 = \begin{pmatrix} 2 & -1 & & & & & & \\ -1 & 2 & -1 & & & & & \\ & -1 & 2 & -1 & & & & \\ & & -1 & 2 & -1 & & & \\ & & & -1 & 2 & -1 & & -1 \\ & & & & -1 & 2 & -1 & \\ & & & & & -1 & 2 & \\ & & & & -1 & & & 2 \end{pmatrix}$

i.e. If Q_X is not diagonalizable. (over $\mathbb{Z}$)

there is no smooth structure on X.

Last time, we focus on $\dim M = 4$ with g on TM and introduce the Yang-Mills equation $F_A^+ = 0$.

Let's review the set-up (replace M by X)

Suppose X is closed (compact + no boundary)
and oriented (orientable + an orientation)

$\pi: P \to X$ is a principal $G = SU(2)$ bundle.

Given g on TX, define dvol by orthonormal basis

$$\mathrm{dvol} = e^1 \wedge \cdots \wedge e^n.$$

Define $* : \Omega^k \to \Omega^{4-k}$ s.t. $v \wedge * v = |v|^2 \mathrm{dvol}$

$*^2 = 1$ on Ω^2 split it into $\Omega^+ \oplus \Omega^-$

Given $\omega \in \Omega^2$, we have $\omega \pm * \omega \in \Omega^{\pm}$

For a conn A on P, we can define ∇_A, d_A, F_A

$F_A^+ = \frac{1}{2}(F_A + * F_A)$ is the ASD part of F_A.

Fact complex line bundle (or principal $U(1)$ bundle) is classified by c_1 because $\mathbb{CP}^\infty = K(\mathbb{Z}, 2)$

Def $K(\mathbb{Z}, n)$ Eilenberg–MacLane space

A topological space (unique up to weak homotop equ)

s.t. $\pi_i(K(\mathbb{Z}, n)) = \begin{cases} \mathbb{Z} & i = n \\ \{e\} & i \neq n \end{cases}$

Prop $[X, K(\mathbb{Z}, n)] \cong H^n(X; \mathbb{Z})$

$n = 2$. principal $U(1)$ bundle $\Longleftrightarrow$ $c_1 \in H^2(X; \mathbb{Z})$

Fact. Principal $SU(2)$ bundle on 4d manifold X

has $c_1(P) = c_1(P \times_{ad} \mathfrak{g}) = 0$, and is classified by

$$c_2(P) = \frac{1}{(2\pi\sqrt{-1})^2}[tr(F_A \wedge F_A)] \in H^4(X; \mathbb{Z}) \cong \mathbb{Z}$$

This is because $SU(2) \cong S^3$

$$\pi_i(BSU(2)) \cong \pi_{i-1}(SU(2)) = \pi_{i-1}(S^3)$$

$$= \begin{cases} ?? & i-1 > 3 \\ \mathbb{Z} & i-1 = 3 \\ \{e\} & i-1 < 3 \end{cases}$$

$BSU(2) \longrightarrow K(\mathbb{Z}, 4)$ inclusion, induces

$$[X, BSU(2)] \cong [X, K(\mathbb{Z}, 4)] \cong H^4(X; \mathbb{Z})$$

The orientation on X means picking a generator in $\mathbb{Z}$

Then we can write $c_2(P)$ as an integer $k \in \mathbb{Z}$

Let $\mathcal{A}_k$ be the space of all conns on P with $c_2(P) = k$. it is affine over $C^\infty(\mathrm{ad}P \otimes T^*X)$

Define $\mathrm{Ad}P = P \times_{\mathrm{Ad}} G = P \times G / (p,h) \sim (pg^{-1}, ghg^{-1})\ \forall g$

It is a bundle of Lie group $(\mathrm{ad}P = P \times_{\mathrm{ad}} \mathfrak{g})$

Let $\mathcal{G}_k = C^\infty(\mathrm{Ad}P)$ called gauge group

There is an action $\mathcal{G}_k \times \mathcal{A}_k \to \mathcal{A}_k$

$$(g, A) \mapsto g^*A = A + g^{-1} d_A g$$

This induces $F_{g^*A} = g \cdot F_A g^{-1}$

So A is ASD. $\Rightarrow$ g^*A is also ASD.

Let $M_k = \{\text{ASD conn in } \mathcal{A}_k\} / \mathcal{G}_k$

be the moduli space. the set of ASD conn and $\mathcal{G}_k$ are both infinite dimensional,

but $\dim M_k$ is finite

Note that M_k also depends on g because $*$ is.

For any g, M_k is not always manifold

To prove M_k is a smooth manifold for generic g.

(generic : roughly means dense set in the space of metrics)

we have to do completion of $\mathcal{A}_k$ and $\mathcal{G}_k$

under some norms, s.t. they become Banach manifold

(Locally <u>Banach space</u> = infinite dim space complete)

C^∞ not Banach $\quad C^r(X)$ or $L^r_k(X)$ sobolev space

Then we can apply infinite dim version of

implicit function thm and Sard thm.

For generic g, M_k is a smooth manifold with

(possibly containing singular part)

$$\dim M_k = 8k - 3(b_2^+ - b^1 + 1)$$

$\dim H^1_{dR}(X)$

$SU(2) \simeq \mathbb{R}^3$

$H^2_{dR}(X) = H^+_{dR}(X) \oplus H^-_{dR}(X)$

$b_2^+ = \dim H^+_{dR}(X)$

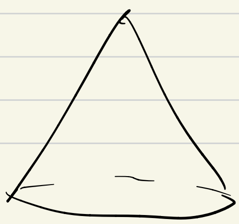
Ex. $X = S^4$ round metric $k=1$ $\quad M_k = \mathring{B}^5$

$X = \overline{\mathbb{CP}}^2$ ($\mathbb{CP}^2$ with opposite orientation)

g = Fubini-Study metric induced from $\mathbb{C}^n$

$k=1$ $\quad M_k$ = Cone of $\overline{\mathbb{CP}}^2$

$[0,1) \times \overline{\mathbb{CP}}^2 / \{0\} \times \overline{\mathbb{CP}}^2$



The cone pt is the reducible solution
(comes from $S^1 = U(1) \subset SU(2)$ connection)

when $\dim M_K < 8$, after adding boundary,
M_K is compact (>8, some bubble phenomenon)
of codim 8

In particular, if $\dim M_K = 0$, this is just
finitely many points. We can introduce orientations
on M_K, and in the case of $\dim M_K = 0$,
we count points with signs.

Fact: under some assumptions (like $b_2^+(X) > 1$),
the number of pts in M_K with $\dim M_K = 0$
is independent of g. Hence this is an invariant
that only depends on the diffeomorphism type of X

Sometimes, we also consider $SO(3)$ principal bundle
and count solutions of $F_A^+ = 0$ (note $SO(3) = SU(2)$)

$\exists$ smooth manifold X_1, X_2 s.t. $X_1 \cong_{homeo} X_2$,
but the number of solutions are different. $X_1 \not\cong_{diff} X_2$

These are called <u>exotic pair</u>

Ex For K3 surface, the signed counting of solution is 1, but $\exists X_k \cong_{homeo} K3$, with the counting $(2k+1)$ for any $k \in \mathbb{Z}$.

Proof of Donaldson's diagonalization thm.

$\dim M_1 = 8 - 3 = 5$. $\partial M_1 = X$

M_1 is orientable, and M_1 has cones on $\overline{\mathbb{CP}}^2$