Class 21 connection and covariant derivative

Last time, we introduced Lie algebra of a Lie gp as left-invariant vector field. the space is identified with $T_e G = \mathfrak{g}$

$$L_g : G \xrightarrow{\cong} G \qquad R_g : G \xrightarrow{\cong} G$$
$$h \longmapsto gh \qquad\qquad h \longmapsto hg$$

Left inv: $(L_g)_* v = v$ (usually write v for v.f. but x for element in $\mathfrak{g}$)

Define $Ad : G \longrightarrow GL(\mathfrak{g})$

$$G \times \mathfrak{g} \longrightarrow \mathfrak{g} \qquad g \cdot x = (L_g)_* (R_{g^{-1}})_* x = g x g^{-1}$$

This is called adjoint representation of G

Recall a rep of G is a homomorphism $\rho : G \longrightarrow GL(n, \mathbb{R})$. now $\mathfrak{g}$ itself is a vector space (with a Lie bracket)

(consider tangent map, we have $ad : \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{g})$, where $\mathfrak{gl}(\mathfrak{g}) = End(\mathfrak{g})$, $ad(x)(y) = [x, y]$

$$= M(\dim \mathfrak{g}, \mathbb{R}) \qquad = xy - yx$$

This is called the adjoint representation of Lie algebra $\mathfrak{g}$)

Recall for G-principal bundle P and a rep $\rho: G \to GL(n,\mathbb{R})$, we can define an associated vector bundle $P \times_\rho \mathbb{R}^n$ by transition functions

$$g_{UV}: U \cap V \to G \xrightarrow{\rho} GL(n,\mathbb{R})$$

We can use adjoint rep to obtain $P \times_{Ad} \mathfrak{g}$

It is a vector bundle over M

we want to argue for two connections A, A' the difference $A' - A$ is a section of $P \times_{Ad} \mathfrak{g}$

Note $\mathfrak{g}$ can be regarded as a subspace of matrices. this is related to the matrix-valued 1-form in

$$\nabla' - \nabla = \alpha \in C^\infty(M; End(E) \otimes \Lambda^1 T^* M)$$

Recall the definition of connections:

we have an exact sequence of v.b. over P

$$\ker \pi_* \to TP \to \pi^* TM$$

$\ker \pi_* \cong P \times \mathfrak{g}$ (product $\mathfrak{g}$ bundle)

a connection is either ① a bundle map $TP \to \ker \pi_*$ s.t. $\ker \pi_* \hookrightarrow TP \to \ker \pi_*$ is identity.

or ② a subbundle H s.t. $H \oplus \ker \pi_* = TP$

Both satisfy some G-equivariant conditions.
H is called horizontal space, because it is tangent to the fiber G

From ①. A is a $\mathfrak{g}$-valued 1-form on P

$$\begin{aligned} \Psi_g : P &\xrightarrow{\cong} P \\ p &\longmapsto pg^{-1} \end{aligned}$$ G-equivariant means.

$$\langle \Psi_g^*(A|_{pg^{-1}}), v\rangle = g\langle A|_p, v\rangle g^{-1} \quad \forall g \in G, v \in TP|_p$$

② G-equivariant of H means $(\Psi_g)_* H_p = H_{pg^{-1}}$

$H_A = \ker A \qquad H \cong \pi^* TM$

The difference $A' - A : TP \longrightarrow \ker \pi_*$

s.t. $\ker \pi_* \hookrightarrow TP \xrightarrow{A'-A} \ker \pi_*$ is zero

So it induces a map $\pi^* TM = TP/\ker\pi_* \longrightarrow \underset{\underset{P\times\mathfrak{g}}{\parallel}}{\ker \pi_*}$

that is G-equivariant, so this gives a section

in $P \times_{ad} \mathfrak{g} = \{(p, x) \in P \times \mathfrak{g}\} / (p,x) \sim (pg^{-1}, gxg^{-1})$

from G-equivariant condition

We show a general argument about associated bundle

Let $V = \mathbb{R}^n$ or $\mathbb{C}^n$ $\quad End(V) = GL(n,\mathbb{R})$ or $GL(n,\mathbb{C})$

Let $\rho : G \to End(V)$ be an representation

$$E = P \times_\rho V = \{(p,v) \in P \times V\} / (p,g) \sim (pg^{-1}, \rho(g)v)$$

$$= P \times V / \text{action of } G$$

Then a section $s : M \to E$ is the same as

G-equ map $\$: P \to V$

Given a connection A (or H_A) on P, we can define

cov der ∇ on E as follows.

First, take $d\$: TP \to V$, regarded as

a section $\underline{V} \otimes T^*P$, where $\underline{V}$ denote the bundle

$P \times V$. We have $\langle (r_g)^* (d\$|_{pg^{-1}}), w \rangle = \rho(g) \langle d\$|_p, w \rangle$

for $w \in TP|_p$

For $v|_p \in \pi^* TM|_p$ (or TM), we pick the unique horizontal

lift $v_A|_p \in H_p \subset T_pP$ s.t. $\pi_* v_A|_p = v|_p$

we have $(r_g)_* v_A|_p = v_A|_{pg^{-1}}$

Let $\nabla \tilde{s}$ be the G-equ map from $\pi^* TM$ to V

by $\langle \nabla \tilde{s}, v \rangle = \langle d\tilde{s}, v_A \rangle$

we have

$$\langle \nabla((\pi^* f)\tilde{s}), v \rangle = \langle d((\pi^* f)\tilde{s}), v_A \rangle$$

$$= \tilde{s} \langle \pi^* df, v_A \rangle + \pi^* f \langle d\tilde{s}, v_A \rangle$$

$$= \tilde{s} \langle df, v \rangle + \pi^* f \langle \nabla \tilde{s}, v \rangle$$

Hence, $\nabla \tilde{s}$ induces a covariant derivative ∇s

Locally, we have $\varphi_u : P|_u \to U \times G$

a connection A on P can be written explicit

$$A = \varphi_u^* (g^{-1} dg + g^{-1} \alpha_u g)$$

($\mathfrak{g}$-valued 1-form on $U \times G$, not just U)

α_u is a $\mathfrak{g}$-valued 1-form on U

$g^{-1} dg$ comes from $T^* G$ part g^{-1} identify $T_g G$ with $T_e G$

$g^{-1} \alpha_u g$ comes from G-equivariant.

Class 22 Horizontal lift

$\pi: P \to M$ principal bundle with fiber G

we have $\ker \pi_* \hookrightarrow TP \twoheadrightarrow \pi^* TM$ exact

a connection A is a G-equivariant map

$TP \to \ker\pi_*$ s.t. $\ker\pi_* \hookrightarrow TP \xrightarrow{A} \ker\pi_*$

is identity. take horizontal space H_A as $\ker A$

we have $\pi_*: H_A \subset TP \longrightarrow TM$

is isomorphism.

Given $v \in TM$, we get $v \in \pi^* TM$,

pick $v_A \in H_A$ s.t. $\pi_* v_A = v$

v_A is called horizontal lift of v.

On associated bundle $E = P \times_\rho V$, we can define

∇s by lift $s: M \to E$ to

G-equ $\mathfrak{s}: P \to V$. and $d\mathfrak{s}: TP \to V$

$\langle \nabla s, v \rangle = \langle d\mathfrak{s}, v_A \rangle$

Given a path $\gamma: [0,1] \to M$, we want to

lift γ to $\gamma_A: [0,1] \to P$ horizontally.

Prop. Given $t_0 \in [0,1]$, $p \in P|_{\gamma(t_0)}$ $\exists$ unique

$\gamma_A : [0,1] \to P$ satisfying

1) $\gamma_A(t_0) = p$

2) $\pi(\gamma_A) = \gamma$

3) $\dot{\gamma}_A \subset H_A$

Moreover, this lift is G-equ. i.e.

the path $\gamma_A' : t \mapsto \gamma_A(t) g^{-1}$ is the horizontal lift

of γ with $\gamma_A'(t_0) = pg^{-1}$

Recall in the case of E, ∇.

a section s is parallel along γ

if $\nabla_{\dot{\gamma}} s = 0$

If $E = P \times_\rho V$ is the associated bundle

a section $s : M \to E$ corresponds to G-equ. $\mathcal{S} : P \to V$

$s(\gamma(t_0)) \in E|_{\gamma(t_0)}$ $\langle \nabla s, \dot{\gamma} \rangle = 0$

$\Rightarrow \langle d\mathcal{S}, \dot{\gamma}_A \rangle = 0$

$\Rightarrow$ partial derivative along $\dot{\gamma}_A$ vanishes

To prove the existence of horizontal lift.
We need local model of the Connection

Locally, we have $\varphi_U : P|_U \to U \times G$

a connection A on P can be written explicit

$$A = \varphi_U^*(g^{-1}dg + g^{-1}\alpha_U g)$$

($\mathfrak{g}$-valued 1-form on $U \times G$, not just U)

α_U is a $\mathfrak{g}$-valued 1-form on U

This α_U is the same one as in $\nabla s_U = ds_U + \alpha_U s_U$

In another chart $V \times G$, we have

$$(x, g_V) = (x, g_{VU}(x) g_U)$$

$$g_V^{-1} dg_V + g_V^{-1} \alpha_V g_V$$

$$= g_U^{-1} dg_U + g_U^{-1} (g_{VU}^{-1} \alpha_V g_{VU} + g_{VU}^{-1} dg_{VU}) g_U$$

$$\Rightarrow \alpha_U = g_{VU}^{-1} \alpha_V g_{VU} + g_{VU}^{-1} dg_{VU}$$

transition function is similar to those in cov deri

In chart $U \times G$, we have

$$\gamma_A(t) = (\gamma(t), g(t))$$

$$\dot\gamma_A(t) = (\dot\gamma(t), g^{-1}(t)\,\dot g(t))$$ identify T_gG with T_eG by g^{-1}

$$\dot\gamma_A \subset H_A \qquad (\dot g(t) = \langle dg(t), \dot\gamma(t)\rangle)$$

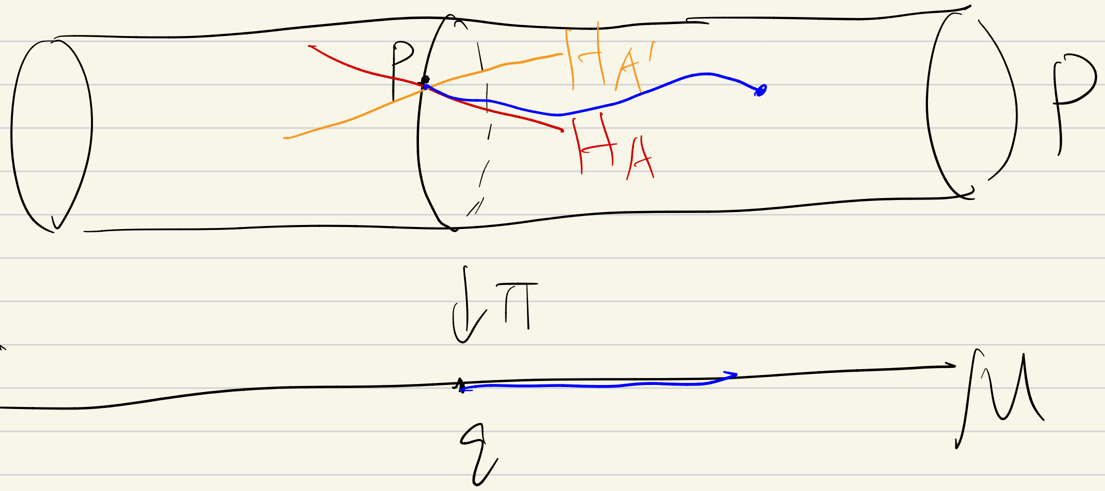
$$\Rightarrow g^{-1}(t)\,\dot g(t) + g^{-1}(t)\langle \alpha_U(\gamma(t)), \dot\gamma(t)\rangle g(t) = 0$$

$$\Rightarrow \dot g(t) + \langle \alpha_U(\gamma(t)), \dot\gamma(t)\rangle g(t) = 0$$

This is an ODE, given $g(t_0)$, $\exists$ unique solution

Ex

$G = S^1$



Curvature of conn

We can consider the exterior covariant derivative

$$d_A : C^\infty(\mathrm{ad}P \otimes \Lambda^k T^*M) \longrightarrow C^\infty(\mathrm{ad}P \otimes \Lambda^{k+1} T^*M)$$

and $F_A s = d_A^2 s$ for $s \in C^\infty(\mathrm{ad}P)$

Locally. $F_A = d\alpha_u + \alpha_u \wedge \alpha_u \in C^\infty(\mathrm{ad}P \otimes \Lambda^2 T^*M)$

matrix valued 2-form.

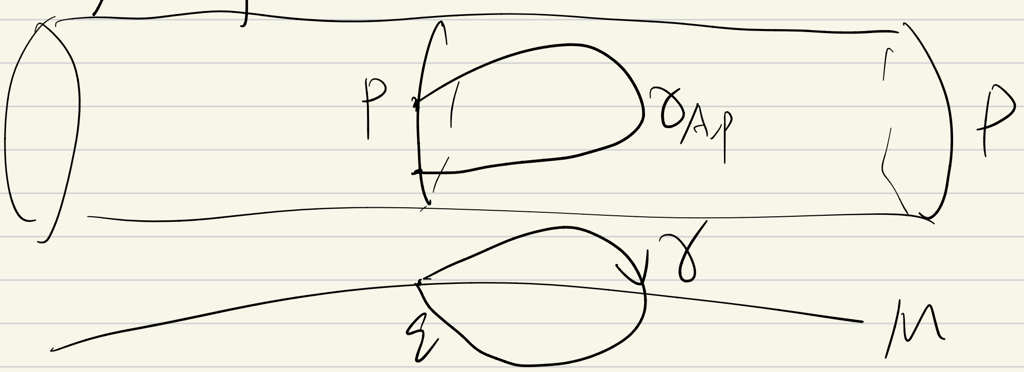
If $F_A = 0$ A is called a <u>flat connection</u>

In such case, the horizontal lift $\gamma_{A,p}(t)$ of γ

with $\gamma(0)=\gamma(1)$, $\gamma_{A,p}(0)=p$ $\dot{\gamma}_{A,p}(t) \subset H_A$ satisfies that

$\gamma_{A,p}(1)$ only depends on homotopy class of γ

(by computation, we omit details



Define $\mathrm{hol}_{A,p} : \pi_1(M,q) \longrightarrow G \cong P|_q$ holonomy map

$$[\gamma] \longmapsto \gamma_{A,p}(1)$$

where $\pi_1(M,q) = \dfrac{\{\gamma : [0,1] \to M \ \ \gamma(0)=\gamma(1)=q\}}{\text{homotopy}}$

we have $hol_{A,pg^{-1}}([\gamma]) = g\, hol_{A,p}([\gamma])\, g^{-1}$

$$hol_{A,p}([\gamma_1]\cdot[\gamma_2]) = hol_{A,p}([\gamma_1])\cdot hol_{A,p}([\gamma_2])$$

$$[\gamma_1]\cdot[\gamma_2] = [\gamma]$$

s.t. $\gamma(t) = \begin{cases} \gamma_1(2t) & t\in[0,\frac{1}{2}] \\ \gamma_2(2t-1) & t\in[\frac{1}{2},1] \end{cases}$

So flat connection determines a representation of $\pi_1(M)$ up to conjugation

Conversely, the universal cover $\tilde{M}$ $(\pi_1(\tilde{M}) = \{e\})$ is a principal $\pi_1(M)$ bundle over M

a representation $\rho: \pi_1(M) \longrightarrow G$

induces a principal G-bundle

$$M \times_\rho G = \tilde{M} \times G \,/\, (q,g) \sim (\gamma\cdot q, \rho(\gamma)\, g\, \rho(\gamma)^{-1})$$

Let $(\tilde{\gamma}, e)$ be the horizontal lift of $\gamma = \pi(\tilde{\gamma})$

This defines a flat connection.

$$\{\text{flat conn}\}/_{\text{iso}} \longleftrightarrow \{\pi_1 \text{ rep}\}/_{\text{conj}}$$