Class 19 Principal bundle (Chap 10)

Def Principal bundle P over $\underset{\text{mfd.}}{\overset{\uparrow}{M}}$ with fiber $\underset{\text{Lie gp}}{\overset{\uparrow}{G}}$

has the following structures

1) a smooth map $m : G \times P \to P$ with

   ① $m(e,p) = p$ e is identity in G

   ② $m(h, m(g,p)) = m(hg, p)$

   usually write $m(g,p)$ by pg^{-1}

2) a smooth map $\pi : P \to M$ surjective with

   $\pi(pg^{-1}) = \pi(p)$. π is called projection

3) Any pt in M has nbhd U with

   $\varphi_U : P|_U = \pi^{-1}(U) \longrightarrow U \times G$ diffeo

   s.t if $\varphi_U(p) = (\pi(p), h(p))$

   then $\varphi_U(pg^{-1}) = (\pi(p), h(p)g^{-1})$

Def 2 Given a locally finite open cover $\mathcal{U}$ of M

and $g_{VU} : U \cap V \longrightarrow G$ satisfying

$$g_{VU} = g_{UV}^{-1} \quad g_{VU}\, g_{UW}\, g_{WV} = Id$$

$$P = \coprod_{U \in \mathcal{U}} U \times G \Big/ (p,g) \in U \times G \sim (p, g_{VU} g) \in V \times G$$

The G action $m(h, (p,g)) = (p, gh^{-1})$

Rem generalization of product pricipal bundle $M \times G$

Def A bundle homomorphism $f : P \longrightarrow P'$ over M

satisfies $\pi(f(p)) = \pi'(p)$

$$f(pg^{-1}) = f(p)\, g^{-1}$$

P and P' are iso if f has an inverse

Def: A fiber bundle E is a smooth mfd with

① $\pi : E \to B$. B called base manifold

② $\forall p \in B$. $\exists U \subset B$ containing p

s.t. $\exists \varphi_U : \pi^{-1}(U) \longrightarrow U \times F$ diffeo

F mfd. called fiber (Global view)

Rem Can also be defined by bundle transition function

$$g_{VU} : U \cap V \to \underbrace{\mathrm{Diff}(F)}_{\text{diffeo group}}$$

(Local version)

A vector bundle is a fiber bundle with fiber $\mathbb{R}^n, \mathbb{C}^n$

+ scalar multiplication

A principal bundle is a fiber bundle with fiber G

+ group multiplication.

Note that $GL(n,\mathbb{R})$ contains linear transformation of $\mathbb{R}^n$

$$\begin{tikzcd}
\text{vector bundle } E \arrow[r, shift left, "\text{framed bundle}"] & \text{principal bundle} \arrow[l, shift left, "\text{associated vector bundle}"] \\
\text{fiber } \mathbb{R}^n & \text{fiber } GL(n,\mathbb{R})
\end{tikzcd}$$

Framed bundle of E: recall the transition function of E is $g_{VU} : U \cap V \to GL(n,\mathbb{R})$

$$E = \coprod_{U \in \mathcal{U}} U \times \mathbb{R}^n \Big/ (p,u) \in U \times \mathbb{R}^n \sim (p, g_{VU}u) \in V \times \mathbb{R}^n$$

$$P = \coprod_{U \in \mathcal{U}} U \times G \Big/ (p,g) \in U \times G \sim (p, g_{VU}g) \in V \times G$$

Let $G = GL(n,\mathbb{R})$ $P = P_{GL(E)}$

Associated vector bundle: a representation of G is a homomorphism $\rho : G \longrightarrow GL(n,\mathbb{R})$

i.e. $\rho(e) = Id$ $\rho(gh) = \rho(g)\rho(h)$ $\rho(g^{-1}) = \rho(g)^{-1}$

Given $g_{VU} : U \cap V \to G$ for P

construct vector bundle E by

$$g_{VU} : U \cap V \to G \xrightarrow{\rho} GL(n,\mathbb{R})$$

$$E = P \times_{\rho} \mathbb{R}^n$$

The existence of metric can help us choose an orthonormal basis $e_1, \cdots, e_n$ locally (only on U) which is different from the coordinate basis $\frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^n}$ of $TM|_U$ $dx^1, \cdots, dx^n$ of $T^*M|_U$

Hence we can choose transition function

$g_{VU} : U \cap V \to O(n)$

If further more E is orientable, i.e. $\Lambda^{\dim E} E \cong M \times \mathbb{R}$

then we have $\det g_{VU} > 0$. $g_{VU} : U \cap V \to SO(n)$

Then we can construct principal bundle $P_{O(n)}$ or $P_{SO(n)}$ just by g_{VU}.

Conversely, if we construct an associate bundle from some principal bundle with fiber $SO(n)$, then the vector bundle is orientable

	mfd M	v.b. E	p.b. P
chart (Global	$\phi_U : U \to \mathbb{R}^m$	$\varphi_U : E\|_U \to U \times \mathbb{R}^n$	$\varphi_U : P\|_U \to U \times G$
transition (Local)	$h_{VU} : \phi_U(U\cap V) \subset \mathbb{R}^m$ $\downarrow$ $\phi_V(U\cap V) \subset \mathbb{R}^m$	$g_{VU} : U \cap V \to$ $GL(n,\mathbb{R})$	$g_{VU} : U\cap V \to$ G
action	/	scalar action $\mathbb{R}\times E \to E$ $\mathbb{C}\times E \to E$	group action $G \times E \to E$

Lie algebra

Class 20 Connection on principal bundle (Chap 11.4)

Suppose $\pi : P \to M$ is a principal bundle with fiber G.

First, we study the tangent space of G at e, denoted by

$\mathfrak{g} = T_eG$, called Lie algebra of G (lie(G) in Cliff's book)

Since G is a Lie group, for any fixed $g \in G$, define

$L_g : G \to G$, $R_g : G \to G$ they are diffeomorphism.

$h \mapsto gh$ $\qquad$ $h \mapsto hg$

$(L_g)_* : T_eG \to T_gG$ is a vector space isomorphism.

$TG \cong G \times T_eG$ by the following map.

For $(g, v) \in G \times T_eG$, define $f(g,v) = (L_g)_* v \in TG$

Def A vector field $v : G \to TG$ is left-invariant if

$$v(gh) = (L_g)_* v(h)$$

Note that left-invariant v.f. is 1-1 cor to vector in T_eG

Let $\mathfrak{g} = \mathrm{Lie}(G)$ be the set of all left-inv v.f.
\mathfrak{g}

Recall that there is a 1-1 correspondence btw vector field v and derivative $L_v : C^\infty(M;\mathbb{R}) \to C^\infty(M;\mathbb{R})$

Given v, w v.f. we can define $[v,w]$ by

$$\mathcal{L}_{[v,w]}(f) = \mathcal{L}_v \mathcal{L}_w(f) - \mathcal{L}_w \mathcal{L}_v(f)$$

Question: If $v = v^i \frac{\partial}{\partial x^i}$ $\quad w = w^j \frac{\partial}{\partial x^j}$ $\quad [v,w] = u^k \frac{\partial}{\partial x^k}$
what is u_k by v^i and w^j ?

Then ① $[av+bw, u] = a[v,u] + b[w,u]$

$[v, aw+bu] = a[v,w] + b[v,u]$

$a, b \in \mathbb{R}$ v, w, u v.f.

② $[v,v]=0 \Rightarrow [v,w]=-[w,v]$

③ $[v,[w,u]] + [u,[v,w]] + [w,[u,v]] = 0$

(Jacobi identity)

$[-,-]$ is called <u>Lie bracket</u> (of vector fields)

In the case of $\mathfrak{g}$, we can check for left inv v, w,
$[v,w]$ is still left inv, so we have a map
$[-,-] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$,
$\mathfrak{g}$ is called Lie algebra associated to G,
as a vector space, it is just T_eG

Ex.

G	$\mathfrak{g}$
$GL(n,\mathbb{R})$ ($\det \neq 0$) $GL(n,\mathbb{C})$	$\mathfrak{gl}(n,\mathbb{R}) = M(n,\mathbb{R})$ $\mathfrak{gl}(n,\mathbb{C}) = M(n,\mathbb{C})$ $[X,Y] = XY - YX$
$SL(n,\mathbb{R})$ ($\det = 1$) $SL(n,\mathbb{C})$	$\mathfrak{sl}(n,\mathbb{R}) \subset M(n,\mathbb{R})$ $\operatorname{tr} X = 0$ (traceless)
$O(n)$ ($AA^T = Id$) $U(n)$ ($AA^* = Id$)	$X + X^T = 0$ (skew-symmetric) $X + X^* = 0$
$SO(n)$ $\begin{pmatrix} AA^T = Id \\ \det A = 1 \end{pmatrix}$	$\mathfrak{so}(n)$ $\operatorname{tr} X = 0$ $X + X^T = 0$ $\begin{pmatrix} 0&0&0\\0&0&1\\0&-1&0 \end{pmatrix}$ $\mathfrak{su}(2) = \mathfrak{so}(3)$ $\begin{pmatrix} 0&1&0\\-1&0&0\\0&0&0 \end{pmatrix}$ $\begin{pmatrix} 0&0&1\\0&0&0\\-1&0&0 \end{pmatrix}$
$SU(n)$ $\begin{pmatrix} AA^* = Id \\ \det A = 1 \end{pmatrix}$	$\mathfrak{su}(n)$ $\operatorname{tr} X = 0$ $X + X^* = 0$ Pauli matrices $\begin{pmatrix} 0&-1\\1&0 \end{pmatrix}$ $\begin{pmatrix} 0&i\\i&0 \end{pmatrix}$ $\begin{pmatrix} i&0\\0&-i \end{pmatrix}$

$\pi: P \to M$ is a smooth (surjective map), we can

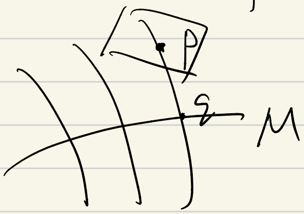
Consider the tangent map $\pi_*: T_pP \to T_{\pi(p)}M$

$\ker \pi_*$ is a sub-(vector)-bundle of TP

s.t. sections in $\ker \pi_*$ are sent to zero sections of TM

So $(\ker \pi_*)$ over $P|_q = \pi^{-1}(q) \cong G$ for $q \in M$

is isomorphic to $TP|_q \cong T_eG = \mathfrak{g}$



also, we consider $\pi^* TM$ as a bundle over P.

We have a sequence of vector bundles

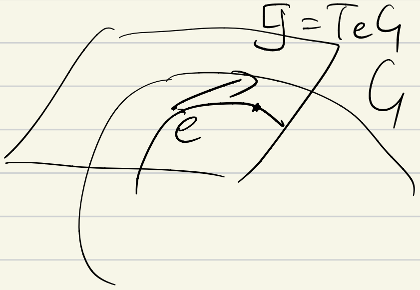
$$\ker \pi_* \to TP \to \pi^* TM$$

s.t. at each fiber, it is exact $\left(\ker(TP \to \pi^* TM) = \mathrm{Im}(\ker \pi_* \to TP) \right)$

Indeed $\ker \pi_* \cong P \times \mathfrak{g}$

For $(p, x) \in P \times \mathfrak{g}$, consider the map $t \mapsto p\exp(tx)$

where $\exp: \mathfrak{g} \to G$ is the exponential map



define $f(p, x) \in \ker \pi_*$

to be the tangent vector at $t = 0$

Any $g \in G$ define $\psi_g: P \xrightarrow{\cong} P$ by $p \mapsto pg^{-1}$

It induces $(\psi_g)_*\ T_pP \to T_{pg^{-1}}P$

Then we have $f(pg^{-1}, \underbrace{(L_g)_*(R_{g^{-1}})_* x}_{gxg^{-1}}) = (\psi_g)_*(f(p,x))$

Def a connection A on P is one of the following:

1) A section $P \longrightarrow T^*P \otimes \ker \pi_*$.

(or $\mathrm{Hom}(TP, \mathfrak{g})$ where $\mathfrak{g}$ is the product v.b. $P \times \mathfrak{g}$)

Satisfies ① $\langle A|_p, f(p,x) \rangle = x \in \mathfrak{g}$

② $\langle \psi_g^*(A|_{pg^{-1}}), v \rangle = g \langle A|_p, v \rangle g^{-1}$ $\forall g \in G, v \in TP|_p$

2) Horizontal subspaces $H \subset TP$ tangent to fibers of π i.e.

$TP = H \oplus \ker \pi_*$

(A choice of splitting in $\ker \pi_* \longrightarrow TP \to \pi^*TM$)

H is isomorphic to π^*TM, but not canonically.

Also. H need to be preserved under $(\psi_g)_*$

1) ⇒ 2). Let H_A be kernel of $A : TP \to \ker \pi_*$

2) ⇒ 1) Use H to define A by projection

If A and A' are two connections, then $A - A$ is a map from π^*TM to $\ker \pi_* = P \times \mathfrak{g}$ which is compatible with the action of g