Class 17 Riemann curvature

Recall in weeks 5-6, we use vector field thm to show that given any point $p = \gamma(0)$ $V_0 = TM|_{\gamma(0)}$ we can always find a unique solution of γ satisfying the geodesic equation.

Similarly, for a general vector bundle, given a path γ and and $V_0 \in E|_{\gamma(0)}$, we can find a unique parallel section s on $E|_\gamma$ (Solving (**))

Let $V_1 = s(\gamma(1))$, it is called the parallel transportation of V_0 along γ

Moreover, if $V_0^1, V_0^2, \dots, V_0^n$ a basis of $E|_{\gamma(0)}$, we obtain a basis $V_1^1, \dots V_1^n$ of $E|_{\gamma(1)}$ after parallel transportation.

i.e. ∇ and γ induces an isomorphism between $E|_{\gamma(0)}$ and $E|_{\gamma(1)}$

Question. does the isomorphism depend on γ for given $\gamma(0) = p$, $\gamma(1) = q$?

Answer: This is true when the curvature $F_\nabla = 0$.
We will come back to this topic after introducing principal bundle.

Then let's discuss curvature for metric compatible covariant derivative

Let $e_1, \dots, e_n$ be an orthonormal basis of E.

$$\nabla e_i = \alpha_i^j e_j \quad \alpha_i^j = \alpha_{ik}^j dx^k \quad \alpha = \{\alpha_i^j\}$$

$\alpha_i^j = -\alpha_j^i$ if ∇ is compatible with g

Recall $F_\nabla = d\alpha + \alpha \wedge \alpha \in C^\infty(M; \mathrm{End}(E) \otimes \Lambda^2 T^*M)$

$$F_\nabla e_i = e_j \otimes d\alpha_i^j + e_k \otimes \alpha_i^l \wedge \alpha_l^k$$

$$= e_j \otimes (d\alpha_i^j + \alpha_i^l \wedge \alpha_l^j)$$

$$(F_\nabla)_i^j = d\alpha_i^j + \alpha_i^l \wedge \alpha_l^j \qquad \alpha_j^l = -\alpha_l^j$$

$$(F_\nabla)_j^i = d\alpha_j^i + \alpha_j^l \wedge \alpha_l^i = -d\alpha_i^j + (-\alpha_l^j) \wedge (-\alpha_i^l)$$

$$= -d\alpha_i^j - \alpha_i^l \wedge \alpha_l^j = -(F_\nabla)_i^j$$

For the dual basis e^i, we have $\nabla e^i = -\alpha_j^i e^j$

$$F_\nabla e^i = e^j \otimes (-d\alpha_j^i + \alpha_l^i \wedge \alpha_j^l)$$

$$(F_\nabla)_j^i = -d\alpha_j^i + \alpha_l^i \wedge \alpha_j^l = d\alpha_i^j + \alpha_i^l \wedge \alpha_l^j$$

This is compatible with previous notation.

For $\nabla = \nabla_{LC}$, write $(F_\nabla)^i_j = \frac{1}{2} R^i_{jkl}\, e^k \wedge e^l$

$R^i_{jkl} : U \to \mathbb{R}$

$R^j_{ikl} = -R^i_{jkl} \quad R^i_{jlk} = -R^i_{jkl}$

Since ∇_{LC} is torsion free $d_\nabla \omega = d\omega$ for 1-form ω

where d_∇ denote $\mathbb{A}(\nabla \omega)$ and $\mathbb{A}$ is the antisymmetrization

We can generalize $\mathbb{A}$ for k-tensor s.t.

$\mathbb{A}(dx^1 \otimes \cdots \otimes dx^n) = dx^1 \wedge \cdots \wedge dx^n$

Define ∇ on $\Lambda^k T^*M$ by $\nabla(\omega_1 \wedge \omega_2) \doteq \nabla \omega_1 \wedge \omega_2 + (-1)^{\deg \omega_1} \omega_1 \wedge \nabla \omega_2$

$\Rightarrow \mathbb{A}\nabla(\omega_1 \wedge \omega_2) = (\mathbb{A}\nabla \omega_1) \wedge \omega_2 + (-1)^{\deg \omega_1} \omega_1 \wedge \mathbb{A}\nabla \omega_2$

Note that $d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^{\deg \omega_1} \omega_1 \wedge d_\nabla \omega_2$

So $\mathbb{A}\nabla \omega = d\omega$ for any form

We have $F_\nabla e^i = d_\nabla^2 e^i = e^j \otimes (F_\nabla)^i_j$

From $\mathbb{A}(F_\nabla e^i) = \mathbb{A} d_\nabla^2 e^i = (\mathbb{A}\nabla)^2 e^i = d^2 e^i = 0$

We have $\mathbb{A}(e^j \otimes F_{\nabla j}^{\,i}) = e^j \wedge (F_\nabla)^i_j = 0$

This means $R^i_{jkl}\, e^j \wedge e^k \wedge e^l = 0$

So $R^i_{jkl} + R^i_{klj} + R^i_{ljk} = 0$ (1)

Change indices we have

$R^j_{kli} + R^j_{lik} + R^j_{ikl} = 0$ (2) $\quad R^k_{ijl} + R^k_{jli} + R^k_{lij} = 0$ (3)

$R^l_{ijk} + R^l_{jki} + R^l_{kij} = 0$ (4) Since $R^i_{klj} = -R^k_{ilj} = R^k_{ijl}$

(1) − (2) − (3) + (4) $\Longrightarrow R^i_{jkl} = R^k_{lij}$

From the definition of R^i_{jkl},

we have $F_\nabla = \frac{1}{2} R^i_{jkl}\, e_i \otimes e^j \otimes (e^k \wedge e^l)$

Since e_i are orthonormal basis, we write

$R_{ijkl} = R^i_{jkl}$ (In $\frac{\partial}{\partial x^i}$ basis, $R_{ijkl} = g_{im} R^m_{jkl}$)

Define the Riemannian tensor

$$\mathrm{Riem} = \frac{1}{2} R_{ijkl}\, e^i \otimes e^j \otimes (e^k \wedge e^l)$$

$$= \frac{1}{4} R_{ijkl}\, (e^i \wedge e^j) \otimes (e^k \wedge e^l)$$

$$\nabla \mathrm{Riem} = \frac{1}{4} \nabla_m R_{ijkl}\, (e^i \wedge e^j) \otimes (e^k \wedge e^l) \otimes e^m$$

Since $\nabla e^i = \alpha^i_j e^j$, we have

$$\nabla_m R_{ijkl} = \partial_m R_{ijkl} + \alpha^{in} R_{njkl} + \alpha^{jn} R_{inkl} + \alpha^{kn} R_{ijnl} + \alpha^{ln} R_{ijkn}$$

Define the Ricci tensor by $\mathrm{Ric} = \mathrm{Ric}_{ik}\, e^i \otimes e^k$

$$\mathrm{Ric}_{ik} = \sum_j R_{ijkj}$$

Define the scalar curvature $R : M \to \mathbb{R}$

by trace of Ric :

$$R = \sum_i Ric_{ii} = \sum R_{ijij}$$

$$\left(Ric_{ik} - \frac{1}{2} R\,\delta_{ik}\right) e^i \otimes e^k$$ is called

the Einstein tensor (However, the Einstein equation in physics doesn't use Riemannian metric
i.e. not positive definite)

Class 18 Bianchi identity and Chern class. (Ch 14)

Last time, we compute the curvature for ∇_{LC}

Suppose $e^1, \dots, e^n$ are orthonormal basis of T^*M (locally)

Write $F_\nabla e^i = d_\nabla^2 e^i = e^j \otimes (F_\nabla)^i_j$

$(F_\nabla)^i_j = \frac{1}{2} R^i_{jkl}\, e^k \wedge e^l \quad R^i_{jkl}: U \to \mathbb{R}$

We proved the following identities

Prop. 1) $R^j_{ikl} = -R^i_{jkl}$ (from metric compatible condition)

2) $R^i_{jlk} = -R^i_{jkl}$ (from definition)

3) $R^i_{jkl} + R^i_{klj} + R^i_{ljk} = 0$ (from torsion free condition)

4) $R^i_{jkl} = R^k_{lij}$ (from 1)–3))

Today we first prove 5) $\nabla_m R^i_{jkl} + \nabla_k R^i_{jlm} + \nabla_l R^i_{jmk} = 0$

This is equivalent to $d_\nabla F_\nabla = 0$ (we omit the pf of equivalence)

Recall $F_\nabla = d\alpha + \alpha \wedge \alpha$

This means $F_\nabla s = d_\nabla^2 s = (p, (d\alpha_U + \alpha_U \wedge \alpha_U) \cdot s_U)$

$$d_\nabla(d_\nabla^2 s) = (p, d((d\alpha_u + \alpha_u \wedge \alpha_u) s_u) + \alpha_u \wedge ((d\alpha_u + \alpha_u \wedge \alpha_u) s_u)$$

$\hookrightarrow d(d\alpha_u + \alpha_u \wedge \alpha_u) s_u + (d\alpha_u + \alpha_u \wedge \alpha_u) d s_u$

$+ \alpha_u \wedge (d\alpha_u + \alpha_u \wedge \alpha_u) s_u \qquad$ Note $d_\nabla s_u = (d + \alpha_u) s_u$

$$= \Big(d(d\alpha_u + \alpha_u \wedge \alpha_u) + \alpha_u \wedge (d\alpha_u + \alpha_u \wedge \alpha_u)$$

$$- \alpha_u \wedge (d\alpha_u + \alpha_u \wedge \alpha_u) \Big) s_u + (d\alpha_u + \alpha_u \wedge \alpha_u) \wedge d_\nabla s_u$$

$$= \big(d^2 \alpha_u + d(\alpha_u \wedge \alpha_u) + \alpha_u \wedge d\alpha_u + \alpha_u \wedge \alpha_u \wedge \alpha_u$$

$$- \alpha_u \wedge d\alpha_u - \alpha_u \wedge \alpha_u \wedge \alpha_u \big) s_u$$

$$+ (d\alpha_u + \alpha_u \wedge \alpha_u) \wedge d_\nabla s_u$$

0

Then we have $d_\nabla (F_\nabla s) = F_\nabla \wedge d_\nabla s_u$

Note $d_\nabla (F_\nabla s) = (d_\nabla F_\nabla) s + (-1)^{\deg F_\nabla} F_\nabla \wedge d_\nabla s_u$

So $d_\nabla F_\nabla = 0$ $\big($Note $(d_\nabla F_\nabla) s \neq d_\nabla (F_\nabla s)\big)$

Chern classes.

For a complex vector bundle E with $E|_p \cong \mathbb{C}^n$

we can also define covariant derivative ∇

and the curvature F_∇

Define the k-th Chern class to be

$$C_k(E) = \frac{1}{(2\pi\sqrt{-1})^k} \operatorname{tr}(\underbrace{F_\nabla \wedge \cdots \wedge F_\nabla}_{k \text{ times}})$$

We define the trace. First, choose a basis

$e_1, \dots, e_n$ for E, and dual basis $e^1, \dots, e^n$

For any section S of $\overset{\operatorname{End}(E)}{=} E \otimes E^*$, we can

write $S = S_i^j e_i \otimes e^j$, define $\operatorname{tr}(S) = S_i^i$

It is independent of basis.

If $\omega \in C^\infty(\operatorname{End}(E) \otimes \Lambda^k T^*M)$,

we can check $d(\operatorname{tr}\omega) = \operatorname{tr}(d_\nabla \omega)$

If $\omega_i \in C^\infty(\operatorname{End}(E) \otimes \Lambda^{k_i} T^*M)$

define $\omega_1 \wedge \omega_2 \in C^\infty(\operatorname{End}(E) \otimes \Lambda^{k_1+k_2} T^*M)$

by $(\omega_1 \wedge \omega_2) e_1 = \omega_1 \wedge (\omega_2 e)$

Then $\mathrm{tr}(\omega_1 \wedge \omega_2) \in C^\infty(\Lambda^{k_1+k_2} T^*M)$

Note $F_\nabla \in C^\infty(\mathrm{End}(E) \otimes \Lambda^2 T^*M)$

$$\mathrm{tr}(\underbrace{F_\nabla \wedge \cdots \wedge F_\nabla}_{k}) \in C^\infty(\Lambda^{2k} T^*M)$$

$$d(\mathrm{tr}(\underbrace{F_\nabla \wedge \cdots \wedge F_\nabla}_{k})) = k\,\mathrm{tr}(d_\nabla F_\nabla \wedge \cdots \wedge F_\nabla)$$
$$= 0 \quad (\text{Bianchi identity})$$

Thm. the de Rham cohomology class $[\mathrm{tr}(\underbrace{F_\nabla \wedge \cdots \wedge F_\nabla}_{k})]$ doesn't depend on the choice of ∇, i.e. c_k only depends on E

pf. Recall any two covariant derivative ∇, ∇' are differed by $\alpha \in C^\infty(\mathrm{End}(E) \otimes T^*M)$

Consider $\nabla^t = \nabla + t\alpha$

$$F_{\nabla^t} = F_\nabla + t\, d_\nabla \alpha + t^2 \alpha \wedge \alpha$$

$$d_{\nabla^t} \alpha = d_\nabla \alpha + (t\alpha) \wedge \alpha + \alpha \wedge (t\alpha)$$

$$\frac{\partial}{\partial t} F_{\nabla^t} = d_{\nabla^t} \alpha$$

$$\frac{\partial}{\partial t} \operatorname{tr}(F_{\nabla^t} \wedge \cdots \wedge F_{\nabla^t})$$

$$= k \operatorname{tr}(d_{\nabla^t}\alpha \wedge F_{\nabla^t} \wedge \cdots \wedge F_{\nabla^t})$$

$$= k\, d \operatorname{tr}(\alpha \wedge F_{\nabla^t} \wedge \cdots \wedge F_{\nabla^t}) \quad (\text{Since } d_{\nabla^t} F_{\nabla^t} = 0)$$

Hence $C_k(F_{\nabla'}) - C_k(F_\nabla)$

$$= d \int_0^1 \frac{k}{(2\pi\sqrt{-1})^k} \operatorname{tr}(\alpha \wedge F_{\nabla^t} \wedge \cdots \wedge F_{\nabla^t})$$

Cor. If $C_k(E) \neq 0$ for some k, then $F_\nabla \neq 0$ for any ∇ on E

Note that the product bundle $M \times \mathbb{C}^n$ with $\nabla = d$ has $F_\nabla = 0$, so $C_k(M \times \mathbb{C}^n) = 0$

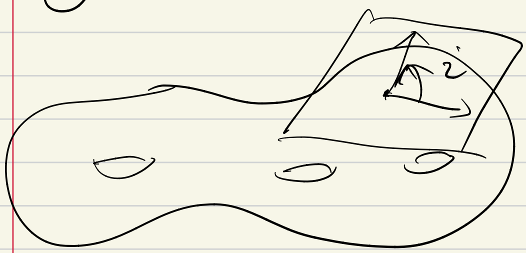
Since $C_k(E)$ is independent of the choice of ∇, we can compute $\operatorname{tr}(F_\nabla \wedge \cdots \wedge F_\nabla)$ by any simple ∇

Rem. Indeed Chern classes can be defined in $H^{2k}_{\text{singular}}(M; \mathbb{Z})$, i.e. it is an integer class,

if we intergrate the class over submanifolds, we will get integers. that's why we need $\frac{1}{(2\pi\sqrt{-1})^k}$ in the def

Ex ① $C_1(E) \neq 0$ for tautological $\mathbb{C}$-bundle over $\mathbb{CP}^n$

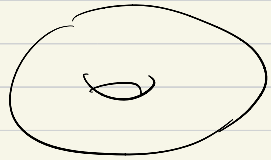
② Σ is an orientable surface in $\mathbb{R}^3$ of genus g



$T\Sigma$ can be regarded as a complex vector bundle of dim 1 if we consider multiplying by i as rotation $90°$

$$C_1(T\Sigma) = (2-2g) \cdot \text{generator in } H^2_{\text{singular}}(\Sigma; \mathbb{Z})$$

$C_1(T\Sigma) \neq 0$ unless $g=1$



The tautological bundles over complex Grassmanians have nonvanishing $C_1, \cdots, C_{\dim E}$

For real vector bundle E, we can define pontryagin class

$$p_k(E) = C_{2k}(\underbrace{E \otimes_{\mathbb{R}} \mathbb{C}}_{\text{roughly } E \oplus \overline{E}}) \in H^{4k}_{dR}(M; \mathbb{R})$$

The odd Chern class of complexification is determined by Stiefel-Whitney class of the original bundle

For more discussion of characteristic classes, see [Hatcher vector bundles and K-theory Chap 3]

Roughly speaking Char classes are obstructions of the triviality of the bundle: Chern class $C_k \in H^{2k}(M; \mathbb{Z})$, Pontryagin class $P_k \in H^{4k}(M; \mathbb{Z})$

Stiefel-Whitney $w_k \in H^k(M; \mathbb{Z}/2\mathbb{Z})$ Euler class $e \in H^?(M; \mathbb{Z})$

Property of Chern class

$f: M \to N \quad \pi: E \to N \quad c_k(f^* E) = f^* c_k(E)$

$c_k(E^*) = (-1)^k c_k(E)$

$c(E) = 1 + c_1(E) + c_2(E) + \cdots$

$c(E_1 \oplus E_2) = c(E_1) \wedge c(E_2)$