

Math 230a: Differential geometry.

Instructor: Fan Ye fanye@math.harvard.edu.

Course Assistant: Keeley Hoek khoek@...

Main Reference: Differential geometry by Cliff Taubes Chap 1-16

Office Hours: Fan MW 4:20-5:20pm SC 505H

Keeley Tu 1:30-2:30 pm SC 333G

Style of this course:

0 Main difference with undergraduate DG: think object itself rather than its embedding

1. This is like a tour to a garden or a zoo,

which means you will meet many new things but can't understand them well at the first time.

Don't be afraid to discuss with classmates about basic definitions and examples, and go back to previous materials once and once again.

2. I study low-dimensional topology, so I will rearrange the materials in the book in a topologist way. That means I will focus more on the construction and motivation. The book contains many useful calculations. It's good to read and calculate by yourself.

3. Topics: Manifolds, Lie groups, vector bundles, metrics, geodesics, connections, curvatures, principal bundles, Yang-Mills equations, ...

Class 1. Introduction to manifolds (Cliff Chap 1)

Def 1. A topological manifold ^{of dimension n .} is a paracompact, Hausdorff topological space s.t. each point (pt) has a neighborhood (nbhd) that is homeomorphic to $\mathbb{R}^n$

Explanation for red lines:

- topological space: a set X with distinguished collection $\mathcal{O}$ of subsets of X called open sets s.t.

1) $\emptyset, X \in \mathcal{O}$

2) Any union of sets in $\mathcal{O}$ is still in $\mathcal{O}$

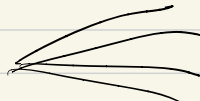
3) finite intersections of sets in $\mathcal{O}$ is still in $\mathcal{O}$

Ex: The standard topology on $\mathbb{R}^n$: $\mathcal{O}$ is the collection of sets given by unions and finite intersections of open balls

$$\{X \mid |x-p| < r \text{ for } p \in \mathbb{R}^n, r \in \mathbb{R}_+\}$$

- paracompact: every open cover has a locally finite refinement
- locally finite: each pt is in finitely many open sets.
- Hausdorff: any two pts have disjoint, open nbhds

These two conditions exclude some "wild" topo space:

1) Union of rays with irrational slopes 

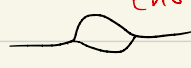
2) Hawaiian ring 

(not homeo to $\mathbb{R}^n$, still paracompact)

$$\bigcup_{n \in \mathbb{N}} \{(x, y) \mid (x - \frac{1}{n})^2 + y^2 = \frac{1}{n^2}\}$$

(infinite union)

(not Hausdorff)

3) Bug-eyed line $\mathbb{R} \cup \mathbb{R} / x \sim y \text{ if } x = y \neq 0$ 

4) The line of irrational slope in $\mathbb{R}^2 / (x, y) \sim (x+m, y+n) \text{ } m, n \in \mathbb{Z}$

- nbhd of pt: an open set containing the pt
- homeomorphism: A continuous, 1-1 map with continuous inv.
- Continuous: the preimage of an open set is an open set.

Usually we add the connected condition to a topo mfd

- Connected: it is not a disjoint union of two open sets

Fact: $\mathbb{R}^n$ is not homeo to $\mathbb{R}^m$ if $m \neq n$

So dim of a topo mfd is a definite number.

Ex of topo mfds:

1) $\mathbb{R}^n$, any open subset in $\mathbb{R}^n$

2) $S^n = \{x \in \mathbb{R}^n \mid |x| = 1\}$

3) $T^n = \mathbb{R}^n / (x_1, \dots, x_n) \sim (x_1 + m_1, \dots, x_n + m_n), m_1, \dots, m_n \in \mathbb{Z}$

4) product of manifolds. Indeed $T^n = \underbrace{S^1 \times S^1 \times \dots \times S^1}_{n \text{ copies}}$

Then we move to smooth manifolds.

Def 2. For a topo mfd X , the nbhd U of a pt with homeo

$\phi_U: U \rightarrow \mathbb{R}^n$ is called a (coordinate) chart.

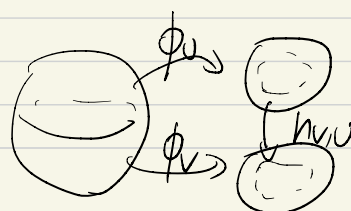
A collection of charts that covers X is

called a (coordinate) atlas.

If U, V are two charts of X . $U \cap V \neq \emptyset$, then

$h_{U,V} = \phi_V \circ \phi_U^{-1} : \underbrace{\phi_U(U \cap V)}_{\mathbb{R}^n} \rightarrow \underbrace{\phi_V(U \cap V)}_{\mathbb{R}^n}$ is also a homeo

It is called the (coordinate) transition function.



Given an atlas $\mathcal{U}$ and all trans functions $h_{V,U}$,

we can recover X by the quotient of the disjoint union

$$\coprod_{U \in \mathcal{U}} \mathbb{R}^n|_U / x \in \mathbb{R}^n|_U \sim h_{V,U}(x) \in \mathbb{R}^n|_V, \forall U, V \in \mathcal{U}$$

Idea: • Given the atlas and transition functions

we can always do computations locally

- We can understand the same manifold by different atlas

Def 3: An atlas is called a smooth structure if
all transition functions are smooth (infinitely differentiable)
or in C^∞

A topo mfd with a smooth str is called a smooth mfd

Two mfd's are diffeomorphic if $\exists$ a smooth map btw them
with a smooth inverse

Then we can do calculus on smooth manifolds

Fact. Smooth str is not unique for some topo mfd's.
does not exist for some topo mfd's.

For dimension n mfd,

If $n = 0, 1, 2, 3$, then we have existence and uniqueness
of the smooth str.

If $n \geq 4$, there exist counterexamples

Two basic examples :

$\mathbb{R}^n$: $\exists$ unique smooth str if $n \neq 4$.
uncountably many if $n = 4$.

S^n : $\exists$ unique smooth str if

$n = 1, 2, 3, 5, 6, 12, 56, 61$

non unique if $n \geq 7 \neq 56, 61$ odd

$n = 7 \quad \exists 28$ smooth str. (Milnor)

$n \geq 8 \quad < 140$ even

widely open : $n = 4$

3 : Poincaré conj Solved by Grigori Perelman (2002)

5, 6, 12 : Michel A. Kervaire and John W. Milnor (1963)

56 : Recent work of Daniel C. Isaksen (2019)

61 : Recent work of Guozhen Wang and Zhouli Xu (2017)

1, 2, 3, 4, 5, 6, 7, 8, ...

low-dimensional
topology

algebraic topology

even : complex geometry, symplectic geometry,
Kähler geometry, ...

odd : contact geometry, ...

Class 2 : Smooth manifolds (Chap 1)

Review: (some people use second countable, i.e. Countable topobasis

this is equivalent to paracompact with countably many connected components)

Def 1. A topo mfd of dim n is a paracompact

Hausdorff topo space s.t. each pt has a nbhd U

$\phi_U: U \xrightarrow{\cong} \mathbb{R}^n$ $\xleftarrow{\text{homeo}}$ (U, ϕ_U) is called a chart

The collection of charts is called an atlas.

For two charts U, V $\exists$ transition function

$$h_{V,U}: \phi_U(U \cap V) \longrightarrow \phi_V(U \cap V)$$
$$\bigcap_{\mathbb{R}^n} \qquad \qquad \qquad \bigcap_{\mathbb{R}^n}$$

If $h_{V,U}$ is smooth (infinitely differentiable),

then the atlas is called a smooth str.

A topo mfd with an smooth str is called a smooth mfd.

Rem. $\exists$ topo mfd that is not smoothable (ES mfd)

$\exists$ topo mfd that has non unique smooth str

Def 2. Suppose M and N are smooth mfd.

Let $\mathcal{U}$ and $\mathcal{V}$ be locally finite atlases corresponding to smooth str of M and N (we use the paracompact condition)

A map $h: M \rightarrow N$ is called smooth if each map $h|_{U \cap V}$

$$\{ \phi_V \circ h \circ \phi_U^{-1}: h(U) \cap V \neq \emptyset \} \quad (U, \phi_U) \in \mathcal{U}, (V, \phi_V) \in \mathcal{V}$$

is smooth

$$\begin{array}{ccc} U & \xrightarrow{h} & V \\ \phi_U \downarrow & & \downarrow \phi_V \\ \mathbb{R}^n & \xrightarrow{\phi_V \circ h \circ \phi_U^{-1}} & \mathbb{R}^n \end{array}$$

Ex: Consider the manifold $\mathbb{R}'$ with only one chart $\phi_1: \mathbb{R} \xrightarrow{x^3} \mathbb{R}$
 and also the manifold $\mathbb{R}''$ with only one chart $\phi_2: \mathbb{R}'' \xrightarrow{x} \mathbb{R}$

Define $f: \mathbb{R}' \xrightarrow{x^3} \mathbb{R}''$, then $\phi_2 \circ f \circ \phi_1^{-1} = x: \mathbb{R} \rightarrow \mathbb{R}$, f, g are smooth inverses
 $g: \mathbb{R}'' \xrightarrow{x^3} \mathbb{R}'$, $\phi_1^{-1} \circ g \circ \phi_2 = x: \mathbb{R} \rightarrow \mathbb{R}$ $\mathbb{R}'$ is diffeo to $\mathbb{R}''$

The reason to define the smooth str is to do calculus on mfd's

Thm 1 (Inverse function thm)

Let $U \subset \mathbb{R}^n$ be an open set and let $\psi: U \rightarrow \mathbb{R}^n$ be a smooth map. Choose a pt $p \in U$. Let ψ_* be the matrix $\left\{ \frac{\partial \psi^j}{\partial x^i} \right\}_{i,j \in \{1, \dots, n\}}$. It is called the Jacobian

If $(\psi_*)_p$ is invertible, then $\exists$ nbhd $V \subset \mathbb{R}^n$ of $\psi(p)$

and a smooth map $G: V \rightarrow U$ s.t. $G(\psi(p)) = p$ and

- $G \circ \psi = \text{Id}$ on nbhd $U' \subset U$ of p ($U' = \text{Im } G$)
- $\psi \circ G = \text{Id}$ on V

i.e. ψ has a local inverse around p .

Idea: The smoothness assumption reduces a non-linear question to a linear one (first derivative). We will use Taylor's thm with remainder to prove it

Pf: $\psi(x) = \psi(p) + (\psi_*)_p(x-p) + R(x-p)$ $\swarrow$ Remainder

$$|R(u)| \leq C|u|^2 \quad |R(u) - R(v)| \leq C|u-v|(|u|+|v|)$$

Set $u = x-p = (\psi_*)_p^{-1}(\psi(x) - \psi(p) - R(u))$

Set $\psi(x) = q$. We need to find a pt

$$u_q = (\psi_*)_p^{-1}(q - \psi(p) - R(u_q)) \quad \text{if } q \text{ is near } \psi(p)$$

We introduce contraction mapping thm to find u_q

Lem 1. Let $r > 0$ $\delta \in (0, 1)$, $B = \{x \in \mathbb{R}^n \mid |x| < r\}$

If $f: B \rightarrow B$ satisfies

$$1) |f(x)| < \delta r \quad \forall x \in B$$

$$2) |f(x) - f(y)| \leq \delta |x - y| \quad \forall x, y \in B$$

then $\exists$ unique $p \in B$ s.t. $f(p) = p$

Pf: Take any $x_0 \in B$. Set $x_{n+1} = f(x_n)$.

Since $|x_n - x_m| < \delta^{n-m} r$ for $n > m$, we know

$\{x_n\}_{n \geq 0}$ is a Cauchy sequence, which converges in B

Let p be the limit, then $f(p) = p$ ($\lim f(x_n) = \lim x_{n+1}$)

If q also satisfies $f(q) = q$, then

$$|p - q| = |f(p) - f(q)| \leq \delta |p - q| \Rightarrow p = q. \quad \square$$

We come back to the pf of inverse function thm.

$$\text{Take } f(u) = (\psi_x)_p'(q - \psi(p) - R(u))$$

Then f maps a small ball to itself because $|R(u)| \leq cu^2$
($|u| \leq \frac{|\psi_x(p)|}{2c}$, $|R(u)| \leq \frac{1}{4c}$, $\delta = \frac{1}{2}$) $|(\psi_x)_p|$ fixed

when $|u|, |v|$ are small

$$|f(u) - f(v)| \leq |(\psi_x)_p| |R(u) - R(v)| \leq C |(\psi_x)_p| |u - v| (|u| + |v|) < \delta |u - v|$$

We apply Lem 1 to obtain the unique element u_q s.t. $f(u_q) = u_q$

We can also use Lem 2 to analyze the function $q \mapsto u_q$.

to show it is smooth; see A.1.1 of Cliff's book.

Def 3: Let $U \subset \mathbb{R}^n$ be an open set. $\psi: U \rightarrow \mathbb{R}^m$ a smooth map
 $a \in \mathbb{R}^m$ is a regular value of ψ if the Jacobian ψ_* is surjective
 at any pt in $\psi^{-1}(a)$

Thm 2 (Sard, thm) The set of regular values has full measures

Ex: $\psi: \mathbb{R}^n \rightarrow \mathbb{R} \quad (x_1, \dots, x_n) \rightarrow \sum_{i=1}^n x_i^2$

$$\psi_* = \left(\frac{\partial \psi}{\partial x_1}, \dots, \frac{\partial \psi}{\partial x_n} \right) = (2x_1, \dots, 2x_n)$$

$$\forall r > 0 \text{ is a regular value} \quad \psi^{-1}(1) = S^{n-1}$$

Thm 3 (Implicit function thm) Let $U \subset \mathbb{R}^n$, $\psi: U \rightarrow \mathbb{R}^{n-m}$ $n-m \geq 0$
 and a is a regular value. Then $\psi^{-1}(a) \subset U$ is a smooth mfd

Pf: For $p \in \psi^{-1}(a)$, let $v_1, \dots, v_{n-m}$ span $\ker(\psi_*)_p$
 (Note $(\psi_*)_p$ is a linear map.)

let π be the projection: $\mathbb{R}^n \rightarrow \ker(\psi_*)_p \cong \mathbb{R}^m$

Define ϕ on small nbhd B of p

$$\phi(x) = (\psi(x), \pi(x)) \in \mathbb{R}^n$$

Then $\phi_*|_p$ is invertible, by inverse function thm

we get a local inverse. Thus, we get a chart for p

Def 4: A submfd M in $\mathbb{R}^n$ is a subset s.t. $\forall p \in M$
 $\exists U_p \subset \mathbb{R}^n$ and $\psi_p: U_p \rightarrow \mathbb{R}^{n-m}$ with 0 as a regular value
 and $M \cap U_p = \psi_p^{-1}(0)$

Suppose N is a smooth mfd of dim n . A subset $M \subset N$
 is a submfd if for any $p \in M$, $\exists U_p \subset N$

$\phi_p: U_p \rightarrow \mathbb{R}^n$ s.t. $\phi_p(M \cap U_p)$ is a submfd of $\mathbb{R}^n$.

Cor: Let $f: N \rightarrow M$ be a smooth map.

Let $p \in N$. If for any $q \in f^{-1}(p)$, any chart

$U_q \subset N$ with $\phi_q: U_q \rightarrow \mathbb{R}^n$, any chart

$U_p \subset M$ with $\phi_p: U_p \rightarrow \mathbb{R}^m$, the Jacobian of

$\phi_p \circ f \circ \phi_q^{-1}: \mathbb{R}^n \rightarrow \mathbb{R}^m$ at $\phi_q(q)$ is surjective,

then $f^{-1}(p)$ is a smooth submfd of N .

(Whitney embedding thm)

Fact: For any smooth mfd M , $\exists \phi: M \rightarrow \mathbb{R}^N$ $N \gg 0$

s.t. $\phi(M)$ is a submfd of $\mathbb{R}^N$,

and ϕ is a diffeomorphism.

Such ϕ is called an embedding.

The proof use a partition of unity. (see Appendix 1.2)

Class 3 Lie group (Chap 2)

Roughly speaking, a Lie group is both a group and a smooth mfd.

Def 1. A group G is a set with a multiplication law:

$m: G \times G \rightarrow G$ s.t. (we write $m(a,b)$ as ab .)

1) $a(bc) = (ab)c$

2) $\exists e \in G$ s.t. $ae = ea$ for $\forall a \in G$ called identity

3) Any element $a \in G$ has an inverse $a^{-1} \in G$ s.t.

$$aa^{-1} = a^{-1}a = e.$$

Def 2. A Lie group G is a smooth manifold with a group multiplication s.t. the maps

$m: G \times G \rightarrow G$ and $a \mapsto a^{-1}$ are smooth.

this condition can be deduced from smoothness of m .

• $M(n, \mathbb{R})$: the space of $n \times n$ matrices as a manifold, we have $M(n, \mathbb{R}) \cong \mathbb{R}^{n^2}$.

But the multiplications are different.

On $M(n, \mathbb{R})$, it is the matrix multiplication

On $\mathbb{R}^{n^2}$, we can use the addition of vectors to define a multiplication

However, $M(n, \mathbb{R})$ is not a Lie group

because the multiplication is not invertible

• $GL(n, \mathbb{R})$: general linear group

the space of invertible matrices in $M(n, \mathbb{R})$

i.e. $\det A \neq 0$ for $A \in GL(n, \mathbb{R})$

This is an open subset of $M(n, \mathbb{R})$ because

$\det : M(n, \mathbb{R}) \rightarrow \mathbb{R}$ is a smooth function.

$GL(n, \mathbb{R}) = \det^{-1}(\mathbb{R} - \{0\})$, where $\mathbb{R} - \{0\}$ is open.

So $GL(n, \mathbb{R})$ is a smooth mfd.

Moreover, the matrix multiplication is invertible on $GL(n, \mathbb{R})$.

So $GL(n, \mathbb{R})$ is a Lie group. $e = I_n = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}$

Def 3: A subgp H of a gp G is a subset containing the identity e , s.t. $m(a, b) \in H$ for $a, b \in H$
 $a^{-1} \in H$

Lem 1. A subgroup of a Lie group that is also a submanifold is a Lie group with respect to the induced smooth str.

Pf: The restrictions of smooth maps $m : G \times G \rightarrow G$

and $a \mapsto a^{-1}$ to a submfd H are also

smooth maps

□

Ex.

• $SL(n, \mathbb{R})$: special linear group

the space of matrices with $\det = 1$

Note that $\det(AB) = \det(A)\det(B)$

We can show $\det_x|_{A_0} \neq 0$

for $\det A_0 = 1$

So by Implicit function thm

$SL(n, \mathbb{R})$ is a smooth mfd, and hence a Lie group,

Since $\det$ is a map to $\mathbb{R}$, we only need to find one direction st. the partial derivative is non zero

Consider $t \mapsto tA_0$ for $t \in \mathbb{R}$ $\det(tA_0) = t^n \det(A_0)$

$$\partial_t (\det(tA_0)) = n t^{n-1} \det(A_0) \xrightarrow[t=\det(A_0)=1]{t=1} n \neq 0,$$

More examples

- $O(n)$ orthogonal group

$$\{A \in GL(n, \mathbb{R}) \mid A^{-1} = A^T\}$$

$$\text{Note } \det(A^{-1}) = (\det(A))^{-1}$$

$$\det(A^T) = \det(A)$$

$$\text{So } \det(A)^2 = 1 \Rightarrow \det(A) = \pm 1 \text{ for } A \in O(n)$$

- $SO(n)$: Special orthogonal group

$$\{A \in O(n) \mid \det(A) = 1\}$$

$A \in SO(n)$ $A = \begin{pmatrix} -1 & & \\ & 1 & \\ & & \ddots \\ & & & 1 \end{pmatrix}$ in the other component of $O(n)$

So two components of $O(n)$ are diffeomorphic if smooth.

To prove $SO(n)$ is smooth, we use implicit function thm again

$$F: GL(n, \mathbb{R}) \rightarrow \text{Sym}(n) = \{n \times n \text{ symmetric matrices}\}$$

$F(A) = AA^T$ is the Jacobian F_* surjective at $F^{-1}(\text{Id})$?

$$\text{Suppose } F(A_0) = \text{Id} \quad F_*|_{A_0}(A) = A_0 A^T + A A_0^T$$

Let S be any matrix in $\text{Sym}(n)$, set $A = \frac{1}{2} S A_0$

$$F_*|_{A_0}(A) = \frac{1}{2} (A_0 A_0^T S + S A_0 A_0^T) = S \text{ because } A_0 A_0^T = \text{Id}$$

More things are introduced in the course of.
Lie group, Lie algebra, and their representations
Complex matrix groups.

• $M(n, \mathbb{C})$ the space of $n \times n$ complex matrices

$M(n, \mathbb{C}) \cong \mathbb{R}^{2n^2}$ because a complex number

$$C = a + bi \quad a, b \in \mathbb{R}$$

• $GL(n, \mathbb{C})$ is the open set in $M(n, \mathbb{C})$
where the matrix is invertible, i.e. $\det \neq 0$

It is a Lie group

$$\text{Let } C \in GL(n, \mathbb{C}) \quad C_{jk} = a_{jk} + b_{jk}\sqrt{-1}$$

Then C can be viewed as an element M in

$$GL(2n, \mathbb{R}) \text{ by } \begin{aligned} m_{2j-1, 2k-1} &= m_{2j, 2k} = a_{jk} \\ m_{2j-1, 2k} &= -m_{2j, 2k-1} = b_{jk} \end{aligned}$$

$$\text{Ex } C = a + b\sqrt{-1} \mapsto \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \quad \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \mapsto \begin{bmatrix} a_{11} & b_{11} & a_{12} & b_{12} \\ -b_{11} & a_{11} & -b_{12} & a_{12} \\ a_{21} & b_{21} & a_{22} & b_{22} \\ -b_{21} & a_{21} & -b_{22} & a_{22} \end{bmatrix}$$

$$\det M = |\det C|^2 \neq 0$$

$$\text{Indeed, let } J = \begin{pmatrix} 0 & \mathbb{I}_n \\ \mathbb{I}_n & 0 \end{pmatrix}$$

$$GL(n, \mathbb{C}) = \{M \in GL(2n, \mathbb{R}) \mid MJ = JM\}$$

- $SL(n, \mathbb{C}) = \{A \in GL(n, \mathbb{C}) \mid \det A = 1\}$
- $U(n) = \{A \in GL(n, \mathbb{C}) \mid A^* A = Id\} \quad \begin{matrix} A^* = \overline{A}^T \\ |\det(A)| = 1 \end{matrix}$
- $SU(n) = \{A \in U(n) \mid \det(A) = 1\}$

We still prove they are smooth mfd's by implicit function thm

$SL(n, \mathbb{C})$ similar to $SL(n, \mathbb{R})$

Need to show $\det_*|_{A_0}$ is surjective. for $\det A_0 = 1$

$$t = x + yi \in \mathbb{C} \quad \partial_x \det(tA_0) = 2n t^{2n-1} \det A_0$$

$$\partial_y \det(tA_0) = 2i n t^{2n-1} \det A_0$$

(Note that $tA \in M(n, \mathbb{C})$ because $\det$ is linear from $M(n, \mathbb{C})$ to $\mathbb{C}$)

$U(n)$. Define $F: GL(n, \mathbb{C}) \rightarrow \text{Herm}(n) = \{H \in M(n, \mathbb{C}) \mid H^* = H\}$

$$F(A) = A^* A \quad \text{For } A_0 \in F^{-1}(Id)$$

$$F_*|_{A_0}(A) = A^* A_0 + A_0^* A$$

Give $H \in \text{Herm}(n)$ take $A = \frac{1}{2} A_0 H$

$$\text{Then } F_*|_{A_0}(A) = \frac{1}{2} (H A_0^* A_0 + A_0^* A_0 H) = H.$$

$SO(n)$: Since $|\det A| = 1$ for $A \in U(n)$ $\det A = e^{i\theta_0}$

Consider $\arg \det A = \theta_0$

$$\arg(\det(e^{i\theta} A)) = n\theta + \theta_0 \quad \partial_\theta \arg(\det(e^{i\theta} A)) = n$$

so the Jacobian of $\arg \det: U(n) \rightarrow S^1$ is surjective