Class 4 Vector bundle (Chap 3)

Manifold + group = Lie group

Manifold + vector space = vector bundle

Def 1: Let M be a smooth mfd of dim m.

A smooth mfd E is called a <u>(real) vector bundle</u> over M with fiber dimension n if we have the following

1) There is a smooth map $\pi: E \to M$

for each $p \in M$, $\exists\, U_p \subset M$ $\lambda_U : \pi^{-1}(U_p) \to \mathbb{R}^n$

s.t. for each $x \in U_p$. $\lambda_U : \pi^{-1}(x) \to \mathbb{R}^n$

is a diffeomorphism

2) There is a smooth map $\hat{o} : M \to E$ s.t. $\pi \circ \hat{o} = Id$

3) There is a smooth map $\mu : \mathbb{R} \times E \to E$ s.t.

a) $\pi(\mu(r,v)) = \pi(v)$

b) $\mu(r, \mu(r',v)) = \mu(rr',v)$

c) $\mu(1,v) = v$

d) $\mu(r,v) = v$ for $r \neq 1$ iff $v \in \mathrm{Im}\, \hat{o}$

4) $\lambda_U(\mu(r,v)) = r \lambda_U(v)$ for any U in 1)

Ex. The product bundle (also called the trivial bundle)

$E = M \times \mathbb{R}^n$

Rem 1) π is called the (bundle) projection map

For $W \subset M$, write $E|_W = \pi^{-1}(W)$

For $p \in M$, $E|_p$ is called a fiber over p

λ_U defines a diffeo $\varphi_U : E|_U \to U \times \mathbb{R}^n$ by

$(\pi(v), \lambda_U(v))$ φ_U is called local trivialization of E

(Remember a topo mfd can have multiple smooth strs, here we specify the smooth str on E by $E|_U \to U \times \mathbb{R}^n \xrightarrow{\varphi_U \times Id} \mathbb{R}^m \times \mathbb{R}^n$)

2) $\hat{0}$ or its image is called the zero section

3) μ corresponds to the scalar multiplication in the vector space

we usually write $\mu(r, v)$ as rv

Def 2: Let E, E' be two vector bundles over M.

A section of E is a smooth map $s: M \to E$ s.t. $\pi \circ s = Id$

A homomorphism $\phi : E \to E'$ is a smooth map s.t.

1) $\pi'(\phi(v)) = \pi(v)$ $(\pi' : E' \to M \quad \hat{0}' : M \to E')$

2) $\phi(rv) = r\phi(v)$ (Hence $\phi(\hat{0}(p)) = \hat{0}'(p)$)

A bundle isomorphism is a homomorphism with an inverse $g: E' \to E$

Question: Is any bundle isomorphic to the trivial bundle?

Counterexample: Möbius bundle over S^1

$$\left\{(\theta, v) \in S^1 \times \mathbb{R}^2 : \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} v = v \right\}$$

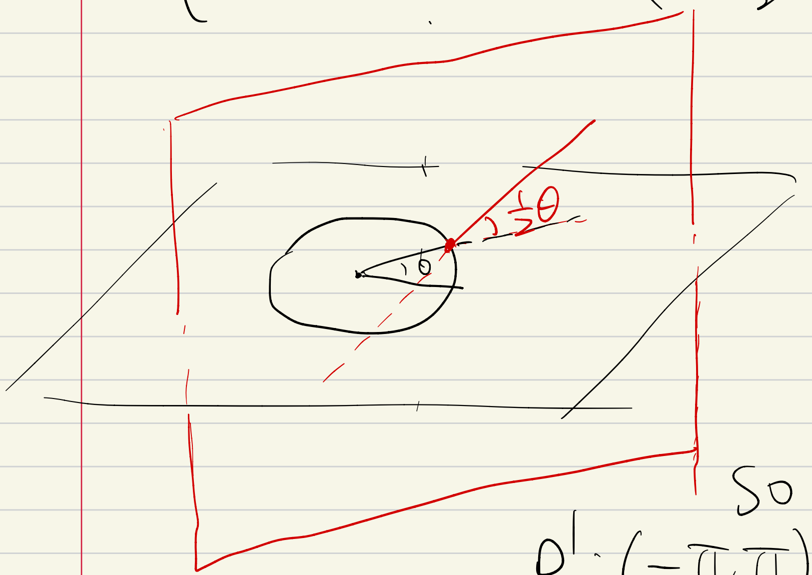
We have a smooth map

$$p : (0, 2\pi) \times \mathbb{R} \longrightarrow E$$

$$(\theta, t) \longmapsto (\theta, t\cos(\tfrac{1}{2}\theta), t\sin(\tfrac{1}{2}\theta))$$

$$\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} \begin{pmatrix} \cos\frac{1}{2}\theta \\ \sin\frac{1}{2}\theta \end{pmatrix} = \begin{pmatrix} \cos\theta\cos\frac{1}{2}\theta + \sin\theta\sin\frac{1}{2}\theta \\ \sin\theta\cos\frac{1}{2}\theta - \cos\theta\sin\frac{1}{2}\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos\frac{1}{2}\theta \\ \sin\frac{1}{2}\theta \end{pmatrix}$$



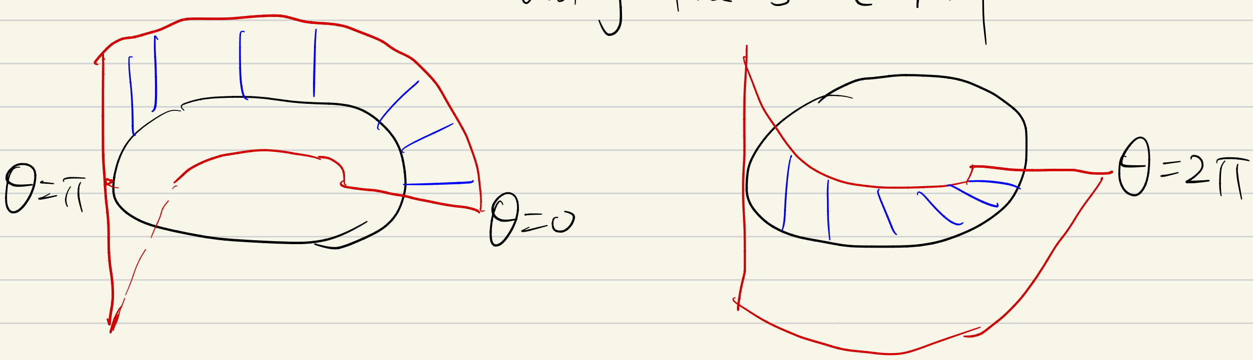
p is not continuous at

$\theta = 0 = 2\pi$

So we need the second chart

$$p' : (-\pi, \pi) \times \mathbb{R} \longrightarrow E$$

using the same map



We can find a nonzero section of $S^1 \times \mathbb{R} = E'$

(just use $s : S^1 \longrightarrow E'$ $\quad s(\theta) = (\theta, 1)$)

But we cannot find a nonzero section of Möbius bundle

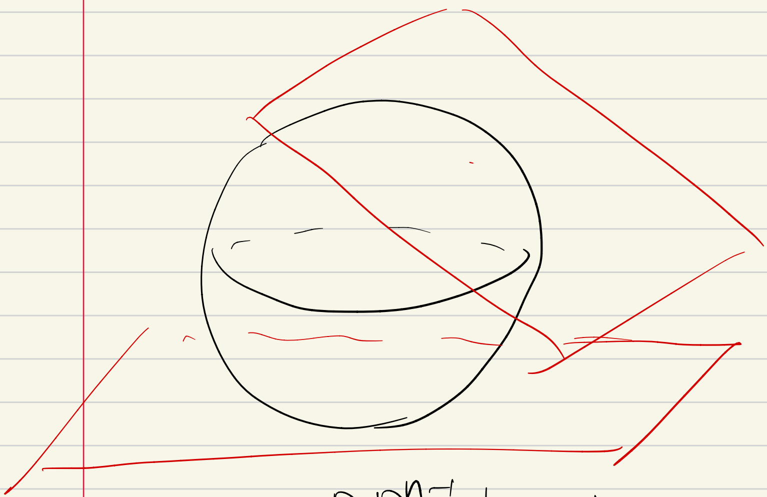
(Intuitively, when we move $s(0)$ smoothly to $s(2\pi)$,
we will have $s(0) \neq s(2\pi)$)

More examples of vector bundles.

- Let $S^2 = \{x \in \mathbb{R}^3 \mid |x| = 1\}$ $E = \{(x,v) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid \substack{|x|=1 \\ x \cdot v = 0}\}$

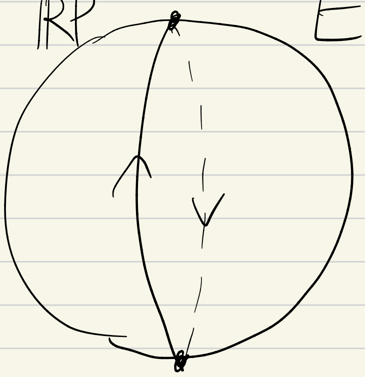
$\pi : E \to S^2$ $\quad \pi(x,v) = x$ $\quad r(x,v) = (x, rv)$ $r \in \mathbb{R}$

$\hat{0} : S^2 \to E$ $\quad \hat{0}(x) = (x, 0)$



- Let $\mathbb{R}P^{n-1}$ be the space of lines through $0 \in \mathbb{R}^n$ (a pt is a line) It can also be regarded as $S^{n-1}/\{\pm 1\}$ where -1 acts on x by $-x$.

$\mathbb{R}P^2$



$E = \{(\pm x, v) \in \mathbb{R}P^{n-1} \times \mathbb{R}^n \mid v = r \cdot x \text{ for } r \in \mathbb{R}\}$

tautological bundle

More properties about sections:

Prop 1: 1) $\phi: E \to E'$ is a homomorphism. $s: M \to E$ is a section. Then $s': x \mapsto \phi(s(x))$ is a section of E' called the <u>pushforward</u> of s

2) $s_1, s_2: M \to E$ sections. Then $s_1 + s_2 : x \mapsto \underbrace{s_1(x) + s_2(x)}_{\text{sum in fiber } \mathbb{R}^n}$ is also a section. Also rs_1 is a section for $r \in \mathbb{R}$. Hence the space of sections is a vector space.

3) $f: M \to \mathbb{R}$, $s: M \to E$. Then fs is also a section.

4) On a nbhd U of any $p \in M$. we have a diffeo
$$\varphi_U : E|_U = \pi^{-1}(U) \longrightarrow U \times \mathbb{R}^n$$
Suppose $e_1, \dots, e_n$ are basis of $\mathbb{R}^n$. Then we can construct sections $s_1, \dots, s_n : U \longrightarrow E|_U$ $\quad s_i(x) = \varphi_U^{-1}(x, e_i)$

i.e. locally, the vector bundle is always generated by sections.

Rem Globally (on M), we may not have everywhere nonzero sections (e.g. Möbius bundle)

5) If we have sections $s_1, \dots, s_n : M \to E$ so that in any nbhd U. $s_i|_U$ forms a basis

Then we can use them to trivialize E:

i.e. $\exists$ a bundle isomorphism $\phi: M \times \mathbb{R}^n \longrightarrow E$
$$(x, \sum r_i e_i) \longmapsto \sum_i r_i s_i(x)$$

We have an alternative way to define E

Def 2. Fix a locally finite open cover $\mathcal{U}$ of M

For $U, V \in \mathcal{U}$ s.t. $U \cap V \neq \phi$. choose a function

$g_{VU} : U \cap V \longrightarrow GL(n, \mathbb{R})$ called <u>bundle transition function</u>

$$E = \bigcup_{U \in \mathcal{U}} U \times \mathbb{R}^n \Big/ (p, u) \in U \times \mathbb{R}^n \sim (p, g_{VU} \cdot u) \in V \times \mathbb{R}^n \quad p \in U \cap V$$

$\{g_{VU}\}_{U,V \in \mathcal{U}}$ need to satisfy the following condition

1) $g_{VU} = g_{UV}^{-1} \quad U \cap V \neq \phi$

2) $g_{V,U} \circ g_{UW} \circ g_{WU} = Id \quad U \cap V \cap W \neq \phi$

called <u>cocycle condition</u>

The reason to introduce g_{VU} is the following

From Def 1. for any $p \in M$. we have a nbhd $U \subset M$

s.t. $E|_U = \pi^{-1}(U) \xrightarrow{\cong} U \times \mathbb{R}^n$

$$v \longmapsto (\pi(v), \lambda_U(v))$$

If $v_1, v_2 \in E|_p$. then we can define

$$v_1 + v_2 = (\lambda_U|_p)^{-1} (\underbrace{\lambda_U|_p(v_1)}_{\mathbb{R}^n} + \underbrace{\lambda_U|_p(v_2)}_{\mathbb{R}^n}) \in E|_p$$

The definition should be independent of the choice of U

if p is also in V, we set $e_i = \lambda U|_p(v_i)$ $i=1,2$

$$v_1+v_2 = (\lambda U|_p)^{-1}(e_1+e_2) = (\lambda V|_p)^{-1}(\lambda V|_p \circ (\lambda U|_p)^{-1}(e_1)$$
$$+ \lambda V|_p \circ (\lambda U|_p)^{-1}(e_2))$$

Let $\Psi_{VU} := \lambda_V \circ (\lambda_U)^{-1} : U \cap V \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$

Then we need to show

$$\Psi_{VU}|_p(e_1+e_2) = \Psi_{VU}|_p(e_1) + \Psi_{VU}|_p(e_2)$$

Since $e_1, e_2 \in \mathbb{R}^n$ can be arbitary,

we need to show $\Psi_{VU}|_p$ is linear.

We have $\Psi_{VU}|_p(rw) = r\Psi_{VU}|_p(w)$

by $\lambda_U(\mu(r,v)) = r\lambda_U(v)$, $(\lambda_U(v)=w, \mu(r,v)=rv)$

$\lambda_V(\mu(r,v)) = r\lambda_V(v)$

Lem A map $\Psi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ satisfying

$\Psi(rw) = r\Psi(w)$ for any r is linear

Pf: By chain rule, $\frac{\partial}{\partial t}\Psi(tw) = \Psi_*|_{tw}(w)$

Since $\Psi(tw) = t\Psi(w)$, $\frac{\partial}{\partial t}\Psi(tw) = \Psi(w)$

Hence $\Psi(w) = \Psi_*|_{tw}(w)$ for any t

Especially $\Psi(w) = \Psi_*|_0(w)$ is linear

Hence we have a map $g_{VU}: U \cap V \to GL(n, \mathbb{R})$

$g_{VU}(p) = \Psi_{VU}|_p$ this map is smooth because Ψ_{VU} is.

$g_{VU} \circ g_{UV} = Id$ $\quad g_{VU} \circ g_{UW} \circ g_{WV} = Id$

by construction. Conversely. Given $\{g_{VU}\}$. we can

also recover the vector bundle in Def 1.

Class 5 Tangent bundle

The second def is useful to construct the tangent bundle:

Def M is a smooth mfd of dim n. The <u>tangent bundle</u> of M is a vector bundle TM constructed as follows.

Fix a locally finite coordinate atlas $\mathcal{U}$ for M

For $U, V \in \mathcal{U}$ with $U \cap V \neq \phi$, we have diffeo

$$\phi_U : U \to \mathbb{R}^n \quad \phi_V : V \to \mathbb{R}^n$$

$$h_{VU} = \phi_V^{-1} \circ \phi_U : \mathbb{R}^n \to \mathbb{R}^n \quad \text{is the transition function}$$

Let $g_{VU} = \underset{\substack{\uparrow \\ \text{the Jacobian of } h_{VU}}}{(h_{VU})_*} \circ \phi_U : U \cap V \to GL(n, \mathbb{R})$

$$h_{VU} \circ h_{UV} = Id \xRightarrow{\text{chain rule}} g_{VU} \circ g_{UV} = Id$$

$$h_{VU} \circ h_{UW} \circ h_{WV} = Id \Rightarrow g_{VU} \circ g_{UW} \circ g_{WV} = Id$$

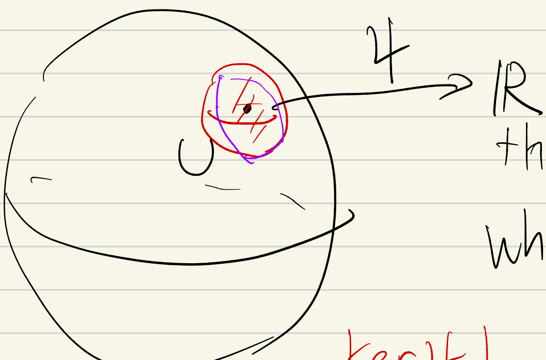
Rem The construction is indeed independent of $\mathcal{U}$

(see Cliff's book 3.4)

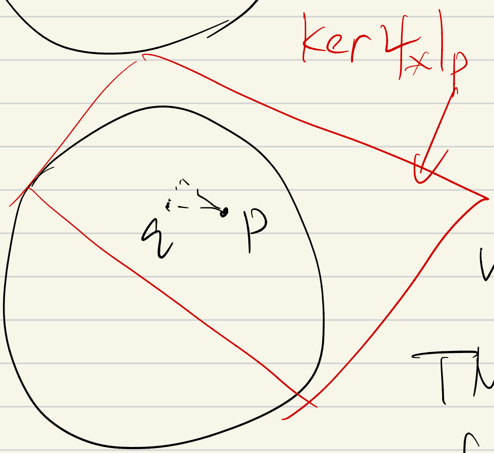
As a result. for any chart $U \subset M$. we have $TM|_U \xrightarrow{\cong} U \times \mathbb{R}^n$, we denote the diffeo by $(\phi_U)_*$

Prop If $M \subset \mathbb{R}^N$ is a submfd of dim n. by definition, for each $p \in M$. $\exists$ nbhd $U \subset \mathbb{R}^N$, $\psi: U \to \mathbb{R}^{N-n}$ with 0 as the regular value i.e. the Jacobian ψ_* is surjective at each pt in $\psi^{-1}(0)$, and $M \cap U = \psi^{-1}(0)$

$N=3 \quad n=2$

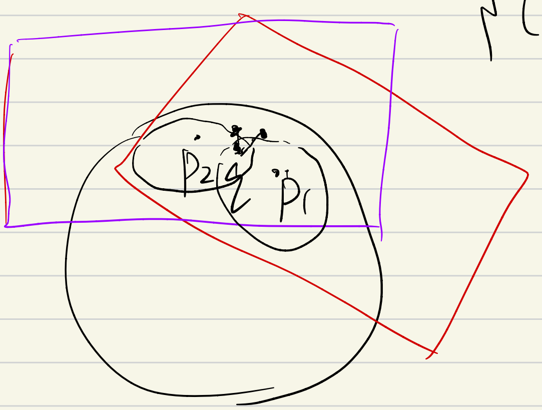


$\ker \psi_*|_p$ is identified with the subspace $\mathbb{R}^n$ tangent to M at p. which is independent of the choice of U



For a pt $q \in M$ near p. the projection of $q-p$ to $\ker \psi_*|_p$ is a diffeomorphism to a ball $\subset \ker \psi_*|_p$ which gives a chart for M

TM can be regarded as subset $\{(p,v) \in M \times \mathbb{R}^N \mid p \cdot v = 0\}$



Def A section $s: M \to TM$ is called a vector field

Let $C^\infty(M;\mathbb{R})$ be the space of smooth functions $M \to \mathbb{R}$ with addition and multiplication

A derivation is a map $\mathcal{L}$ from $C^\infty(M;\mathbb{R})$ to itself

s.t. 1) $\mathcal{L}(f+g) = \mathcal{L}(f) + \mathcal{L}(g)$ 2) $\mathcal{L}(r) = 0$ for constant function r

3) $\mathcal{L}(fg) = (\mathcal{L}f)g + f(\mathcal{L}g)$ (Leibniz rule)

We can identify a derivation with a vector field

Given a vector field $s: M \to TM$. we construct a derivation $\mathcal{L}_s$ called Lie derivative

Let $U \subset M$ be a chart $\phi_U : U \to \mathbb{R}^n$

Write $\phi_{U*} : TM|_U \xrightarrow{\cong} U \times \mathbb{R}^n$

doesn't mean Jacobian.

For $f: M \to \mathbb{R}$, define $f_U = f \circ \phi_U^{-1} : \mathbb{R}^n \to \mathbb{R}$

Suppose $\phi_{U*} \circ s|_U = (Id_U, V_1, V_2, \cdots V_n)$

$V_k : U \to \mathbb{R}$

Define $(L_s f)_U = \sum_{k=1}^{n} V_k \frac{\partial f_U}{\partial x^k} : U \to \mathbb{R}$

Goal : check $L_s f : M \to \mathbb{R}$ is independent of the choice of U by the bundle transition function of TM

Because $(L_S f)_U = \sum_{k=1}^{n} V_k \frac{\partial f_U}{\partial x^k}$, we may write $S|_U = \sum_{k=1}^{n} V_k \frac{\partial}{\partial x^k}$ and write $\{\frac{\partial}{\partial x^k}\}$ as a basis of sections for $T\mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n$

This notation is important becauce in the future we usually do calculation locally and write a vector field as $\sum_{k=1}^{n} V_k \frac{\partial}{\partial x^k}$

For another chart V. we write $\frac{\partial}{\partial y^k}$ as basis and $S|_V = \sum V_k' \frac{\partial}{\partial y^k}$, we have

$$\frac{\partial(f\circ\phi_V^{-1})}{\partial y^k} = \frac{\partial(f\circ\phi_U^{-1}\circ\phi_U\circ\phi_V^{-1})}{\partial y^k} = \sum_l \frac{\partial(f\circ\phi_U^{-1})}{\partial x^l} \frac{\partial(\chi_l\circ\phi_U\circ\phi_V^{-1})}{\partial y^k}$$

$\phi_U\circ\phi_V^{-1} : \mathbb{R}^n$ (with basis y^k) $\longrightarrow \mathbb{R}^n$ (with basis x^k)

So $$\frac{\partial}{\partial y^k} = \sum_l \frac{\partial(\chi_l\circ\phi_U\circ\phi_V^{-1})}{\partial y^k}\frac{\partial}{\partial x^l}$$ ($\chi_l : \mathbb{R}^n \to \mathbb{R}$ projection to l-th coordinate)

$(\frac{\partial}{\partial y^1}, \cdots, \frac{\partial}{\partial y^n}) = (\frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^n})(h_{UV})_*$ another reason to write as $\frac{\partial}{\partial x^k}$

Also $$\underbrace{\sum_l V_l \frac{\partial}{\partial x_l}}_{=S|_U} = \underbrace{\sum_k V_k' \frac{\partial}{\partial y^k}}_{=S|_V} = \sum_k V_k'\left(\sum_l \frac{\partial(\chi_l\circ\phi_U\circ\phi_V^{-1})}{\partial y^k}\frac{\partial}{\partial x_l}\right)$$

take coefficients of $\frac{\partial}{\partial x^l}$

$$\Longrightarrow V_l = \sum_k V_k' \frac{\partial(\chi_l\circ\phi_U\circ\phi_V^{-1})}{\partial y^k}$$

$$\Longrightarrow \begin{pmatrix} V_1 \\ \vdots \\ V_n \end{pmatrix} = (h_{UV})_* \begin{pmatrix} V_1' \\ \vdots \\ V_n' \end{pmatrix} \Rightarrow (L_S f)_U = \sum V_k \frac{\partial f_U}{\partial x^k} \quad \text{equal} \quad (L_S f)_V = \sum V_k' \frac{\partial f_V}{\partial y^k}$$

$$(\frac{\partial f}{\partial x^1}, \cdots, \frac{\partial f}{\partial x^n})\begin{pmatrix} V_1 \\ \vdots \\ V_n \end{pmatrix} = (L_S f)_U$$

Conversely, given a derivation $\mathcal{L}$. We construct a vector field as follows

Let $V_K : U \to \mathbb{R}$ be defined by

$\mathcal{L}(x_K \circ \phi_U)$ where $x_K : \mathbb{R}^n \to \mathbb{R}$ is the projection

Let $\frac{\partial}{\partial x^k} : U \to TM|_U \cong U \times \mathbb{R}^n$ be the section

Correspondly to basis of $\mathbb{R}^n$ set $S = \sum_K V_K \frac{\partial}{\partial x^k}$

Exercise : for any $f : M \to \mathbb{R}$

$$(Lf)|_U = (L_S f)|_U = \sum_K V_K \frac{\partial (f \circ \phi_U^{-1})}{\partial x^k}$$

A function $f : M \to \mathbb{R}$ define a vector field ∇f as follows.

for a chart $U \subset M$, $\phi_U : U \to \mathbb{R}^n$

Define $f_U = f \circ \phi_U^{-1} : \mathbb{R}^n \to \mathbb{R}$ Consider $(f_U)_* = \left(\frac{\partial f_U}{\partial x_1}, \ldots, \frac{\partial f_U}{\partial x_n}\right)$

Let $(\nabla f)|_U = \sum_K \frac{\partial f_U}{\partial x_K} \frac{\partial}{\partial x_K}$.

Class 6 Cotangent bundle and bundle algebra

Def The <u>cotangent bundle</u> T^*M is defined by $g_{VU} = ((h_{VU})_*^{-1})^T$

A section $s: M \to T^*M$ is called a <u>1-form</u>

Similar to the definition of $\frac{\partial}{\partial x^k}$

Given $U \subset M$. we define $\{dx^k\}$ to be the sections corresponding to basis of $\mathbb{R}^n$.

$\Phi_{U_*} : T^*M|_U \to U \times \mathbb{R}^n$

Then for $f: M \to \mathbb{R}$, define a 1-form df by

$$df = \sum \frac{\partial f_U}{\partial x^k} dx^k$$

Def Given a vector bundle $E \to M$ with bundle transition functions $\{g_{VU}\}$. Then define the <u>dual</u> bundle $E^* \to M$ by $\{(g_{VU}^{-1})^T\}$

Rem T^*M is the dual bundle of TM.

$$E^*|_p \cong \mathrm{Hom}(E|_p, \mathbb{R}) \quad (E^*)^* = E$$

The direct sum of two bundles $E_1, E_2 \to M$ is given by $E_1 \oplus E_2 = \{(v_1, v_2) \in E_1 \times E_2 \mid \pi_1(v_1) = \pi_2(v_2)\}$

Rem. $(E_1 \oplus E_2)|_p = E_1|_p \oplus E_2|_p$

$$g_{VU} = g_{VU,1} \oplus g_{VU,2}$$

The bundle $\mathrm{Hom}(E_1, E_2)$ is defined by

$$\mathrm{Hom}(E_1, E_2)|_p = \mathrm{Hom}(E_1|_p, E_2|_p)$$

If $\dim E_1|_p = n$ $\dim E_2|_p = k$

then $\dim \mathrm{Hom}(E_1, E_2)|_p = nk$

$$\mathrm{Hom}(E_1, E_2)|_U \cong U \times M(k, n)$$

$M(k,n)$ $k \times n$ matrices

the bundle transition function sends $m_U \in M(k,n)$

to $m_V = \underbrace{g_{VU,2}}_{k\times k}\, \underbrace{m_U}_{k\times n}\, (\underbrace{g^{-1}_{VU,1}}_{n\times n})^T$

The bundle $E_1 \otimes E_2$ can be defined by $\mathrm{Hom}(E_1^*, E_2)$

More constructions

Def.

- $E_1 \otimes E_2 \to M$ by $\mathrm{Hom}(E_1^*, E_2)$
- E_1 is a <u>subbundle</u> of E_2 if E_1 is a submfd of E_2 and $E_1|_p$ is a subspace of $E_2|_p$

Fact: every vector bundle is a subbundle of a product bundle

equivalently. for any E, $\exists$ E' s.t. $E_1 \oplus E_2$ is isomorphic to a product bundle

- If E_1 is a subbundle of E_2, then define the <u>quotient bundle</u> E_2/E_1 by $(E_2/E_1)|_p = E_2|_p / E_1|_p$.

Def. We introduce some operations on a vector space V and then generalize them to vector bundles

V^* is the dual of V : $V^* = \mathrm{Hom}(V, \mathbb{R})$

$V^* \otimes \cdots \otimes V^*$ is the set of multilinear functions $f: V \times V \times \cdots \times V \to \mathbb{R}$

i.e. when all variables except one are fixed. it is a linear function.

$\mathrm{Sym}^k(V^*)$ is the set of symmetric, multilinear map

i.e. $f(\cdots, v_i, \cdots, v_j, \cdots) = f(\cdots, v_j, \cdots, v_i, \cdots)$

$\Lambda^k(V^*)$ is the set of anti-sym mul map

i.e. $f(\cdots, v_i, \cdots, v_j, \cdots) = -f(\cdots, v_j, \cdots, v_i, \cdots)$

$\dim (V^*)^{\otimes k} = (\dim V^*)^k = n^k$

$\dim \mathrm{Sym}^k(V^*) = \binom{n+k-1}{k}$ $\quad \dim \Lambda^k V^* = \begin{cases} \binom{n}{k} & 0 \le k \le n \\ 0 & k > n \end{cases}$

$\dim \Lambda^n V^* = 1 \qquad \Lambda^n V^* = \mathbb{R}$

We can also construct $\mathrm{Sym}^k E^*$, $\Lambda^k E^*$ (also $\mathrm{Sym}^k E$, $\Lambda^k E$)

If $\{g_{VU}\}$ is for E, then $\{\det(g_{VU})\}$ is for $\Lambda^n E$

An <u>orientation</u> of E is a nowhere zero section of $\Lambda^n E$.
which induces a bundle isomorphism $\Lambda^n E \cong M \times \mathbb{R}$

An orientation of M is an orientation of TM.

E (or M) is <u>orientable</u> if there is an orientation

E is orientable iff E^* is

Ex. The Mobius bundle is not orientable. TS^n is orientable.

Any two orientations $s_1, s_2 : M \to \Lambda^n E$ satisfy $s_1 = f s_2$
for $f : M \to \mathbb{R}$, $f(p) \neq 0$ for any $p \in M$.

If f is always positive, we say two orientations are the same

If f is always negative, we say two orientations are opposite

A section of $\Lambda^k T^* M$ is called a <u>k-form</u>.
(we will go back to study $\Lambda^k T^* M$ later in de Rham cohomology)

Start here

Def. $f: M \to N$ is a smooth map. $\pi: E \to N$ is a v.b.

Define $f^*E = \{(p,v) \in M \times E \mid f(p) = \pi(v)\}$

This is called the pull-back of E, $(f^*E)|_p = E|_{f(p)}$.

However, we DON'T have the "push-forward" construction for general bundle. For tangent bundle, we can define the tangent map $f_*: TM|_p \to TN|_{f(p)}$ $(= f^*TN|_p)$ as follows. For $U \subset M$, $V \subset N$, we have

$\phi_{U*}: TM|_U \to U \times \mathbb{R}^m$ $\quad \phi_{V*}: TN|_V \to V \times \mathbb{R}^n$

f is smooth $\Rightarrow \phi_V \circ f \circ \phi_U^{-1}: \mathbb{R}^m \to \mathbb{R}^n$ is smooth

For $v \in TM|_U$, define $f_* v = \phi_{V*}^{-1} \circ (\phi_V \circ f \circ \phi_U^{-1})_* \circ \phi_{U*} v$

We can show it is independent of U, V because the bundle transition functions of TM, TN

Class 7 Complex vector bundles (Chap 6) and deRham cohomology (Chap 12)

Def. A <u>complex v.b.</u> E over M satisfies the following

1) There is a smooth map $\pi : E \longrightarrow M$

for each $p \in M$, $\exists\, U_p \subset M$ $\lambda_U : \pi^{-1}(U_p) \longrightarrow \mathbb{C}^n$

s.t. for each $x \in U_p$. $\lambda_U : \pi^{-1}(x) \longrightarrow \mathbb{C}^n = \mathbb{R}^{2n}$

is an diffeomorphism

2) There is a smooth map $\hat{o} : M \rightarrow E$ s.t. $\pi \circ \hat{o} = Id$

3) There is a smooth map $\mu : \mathbb{C} \times E \longrightarrow E$ s.t.

a) $\pi(\mu(r, v)) = \pi(v)$

b) $\mu(r, \mu(r', v)) = \mu(rr', v)$

c) $\mu(1, v) = v$

d) $\mu(r, v) = v$ for $r \neq 1$ iff $v \in Im\, \hat{o}$

4) $\lambda_U(\mu(r, v)) = r\lambda_U(v)$ for any U in 1)

Or we use the bundle transition function

$g_{VU} : U \cap V \longrightarrow GL(n, \mathbb{C})$ called <u>bundle transition function</u>

$$E = \bigcup_{U \in \mathcal{U}} U \times \mathbb{C}^n / (p, u) \in U \times \mathbb{C}^n \sim (p, g_{VU} \cdot u) \in V \times \mathbb{C}^n \quad p \in U \cap V$$

$\{g_{VU}\}_{U,V \in \mathcal{U}}$ need to satisfy the following condition

1) $g_{VU} = g_{UV}^{-1}$ $\quad U \cap V \neq \phi$

2) $g_{VU} \circ g_{UW} \circ g_{WU} = Id$ $\quad U \cap V \cap W \neq \phi$

called <u>cocycle condition</u>

Rem. In the class of complex geometry, there is a def for complex manifold. We can get some natural complex v.b. similar to the natural real v.b. for real mfd (e.g. tangent bundle. cotangent bundle)

But we do have complex v.b. over <u>real</u> mfd

Ex. (Complexification): Let $E_{\mathbb{R}} \to M$ be a real v.b.

Let $E_{\mathbb{C}} = (E_{\mathbb{R}} \times \mathbb{C}) / (rv, c) \sim (v, rc) \quad r \in \mathbb{R}$

This is a cpx v.b. over M

We also have dual, $\oplus$, Hom, $\otimes$, Sym^k, Λ^k subbundle, quotient bundle, pullback for complex v.b.

de Rham cohomology. (Chap 12)

Basic idea of homology/cohomology:

Given a graded abelian group $C = \bigoplus_i C_i$ and an endomorphism

$d: C \to C$ s.t. $d(C_i) \subset C_{i+1}$ (for cohomology)

$\subset C_{i-1}$ (for homology)

and $d^2 = 0$

Then $\operatorname{Im} d = \{ c \in C \mid \exists\, a \in C,\ da = c \}$

$\subset \ker d = \{ c \in C \mid dc = 0 \}$

$$H^i(C,d) = \ker(d: C_i \to C_{i+1}) / \operatorname{Im}(d: C_{i-1} \to C_i)$$

(Notation $H_i(C,d)$ for homology

$H^*(C,d) = \bigoplus_i H^i(C,d)$ $\quad H_*(C,d) = \bigoplus_i H_i(C,d)$)

Def. (C,d) is called a (co)chain complex. $H_*(C,d)$ or $H^*(C,d)$ is called the (co)homology group of (C,d)

elements in $\operatorname{Im} d$ are called <u>(co) boundaries</u>

elements in $\ker d$ are called <u>(co) cycles</u>

For an element $c \in \ker d$, the corresponding element in H^* is denoted by $[c]$

Rem

① The chain complex may not be an invariant (of manifold or bundle), but the homology and its rank are invariants (up to isomorphisms)

② When taking $\ker d / \operatorname{Im} d$, we may lose some information in (C,d), but that makes H^* simpler to study.

③ Sometimes, we don't need the grading, just start with (C, d)

with $d^2=0$ and take $H^*(C,d) = \ker d / \mathrm{Im}\, d$.

If C is a vector space, d can be understood as a matrix

Then let's define the de Rham cohomology

Let M be a smooth manifold of dim n

For $k=0,1,\cdots,n$, let $\Omega^k = \Omega^k(M)$ be the space of

sections $\omega : M \longrightarrow \Lambda^k T^* M$

(At each point $p \in M$, we know ω_p is an antisymmetric multilinear map

$$\omega_p : \underbrace{T_pM \times T_pM \times \cdots \times T_pM}_{k} \longrightarrow \mathbb{R}$$

Antisym means $\omega_p(\cdots v_i, \cdots, v_j, \cdots) = -\omega_p(\cdots, v_j, \cdots, v_i, \cdots)$

$\dim (\Lambda^k T^* M)\big|_p = \binom{n}{k}$)

Since sections are called k-form, Ω^k is the space of k-forms

on M. In particular $\Omega^0 = C^\infty(M, \mathbb{R})$ is the space of

smooth functions on M $(\Lambda^0 T_p^* M = \mathbb{R})$

Previously, for a smooth function (a 0-form) $f : M \to \mathbb{R}$

we construct a 1-form df by local chart $df = \sum_{i=1}^{n} \frac{\partial f_U}{\partial x^i} dx^i$

d is a map : $\Omega^0 \longrightarrow \Omega^1$

Goal: extend d to $\Omega^k \longrightarrow \Omega^{k+1}$ s.t. $d^2 = 0$

This map is called the exterior derivative

($\Lambda^k V$ is called the exterior algebra for V)

Fact: Locally, any k-form can be written as

$$\sum_{i_1<i_2<\cdots<i_k} f_{i_1\cdots i_k}\, dx^{i_1}\wedge\cdots\wedge dx^{i_k} \qquad f_{i_1\cdots i_k}: U\to\mathbb{R}$$

($dx^{i_1}\wedge\cdots\wedge dx^{i_k}$ are basis of sections on $\Lambda^k\mathbb{R}^n$ and we use $\phi_U: U\to\mathbb{R}$ to pull-back sections)

Define

$$d(\quad) = \sum_{i_1<\cdots<i_k}\ \sum_{m\notin\{i_1,\cdots,i_k\}} \frac{\partial f_{i_1\cdots i_k}}{\partial x^m}\, dx^m\wedge dx^{i_1}\wedge\cdots\wedge dx^{i_k}$$

$$= \sum_{i_1<\cdots<i_k<i_{k+1}} \left(\frac{\partial f_{i_2 i_3\cdots i_{k+1}}}{\partial x^{i_1}} - \frac{\partial f_{i_1 i_3\cdots i_{k+1}}}{\partial x^{i_2}} + \cdots\right) dx^{i_1}\wedge\cdots\wedge dx^{i_{k+1}}$$

Ex $\omega = y\,dx$ on $\mathbb{R}^2$

$d\omega = dy\wedge dx$.

sign comes from anti-sym.

This generalizes $df = \sum_m \frac{\partial f}{\partial x^m} dx^m$ (Need to check it is independent of charts)

$$d(df) = d\left(\sum_m \frac{\partial f}{\partial x^m} dx^m\right) = \sum_{m,l} \frac{\partial^2 f}{\partial x^m \partial x^l} dx^m\wedge dx^l$$

Since $\frac{\partial^2 f}{\partial x^m \partial x^l} = \frac{\partial^2 f}{\partial x^l \partial x^m}$ and $dx^m\wedge dx^l = -dx^l\wedge dx^m$

$d(df)=0$. In general, we can check $d^2=0: \Omega^k\to\Omega^{k+2}$

Prop 1) $d(\omega_1+\omega_2) = d\omega_1 + d\omega_2$

2) ω_1, ω_2 are k, l forms

$$d(\omega_1\wedge\omega_2) = d\omega_1\wedge\omega_2 + (-1)^k \omega_1\wedge d\omega_2$$

Def. ω is <u>closed</u> if $d\omega=0$ (cocycle)

ω is <u>exact</u> if $\omega = d\alpha$ for some α. (coboundary)

The de Rham cohomology $H^*_{dR}(M) = \ker d / \operatorname{Im} d$

It is a vector space of $\mathbb{R}$.

(If you know singular cohomology $H^*(M)$ for a topological space M, then we have $H^*_{dR}(M) \cong H^*(M;\mathbb{R})$

in $\mathbb{R}$ coefficient

In general $H^*(M)$ can be defined over $\mathbb{Z}$.

$$\text{e.g. } H^i(\mathbb{R}P^3;\mathbb{Z}) = \begin{cases} \mathbb{Z} & i=0 \\ 0 & i=1 \\ \mathbb{Z}/2 & i=2 \\ \mathbb{Z} & i=3 \end{cases}$$

$$H^i_{dR}(\mathbb{R}P^3) = \begin{cases} \mathbb{R} & i=0 \\ 0 & i=1 \\ 0 & i=2 \\ \mathbb{R}. & i=3 \end{cases}$$

If $\varphi: M \to N$ is a smooth map. φ induces a map

$$\varphi^*: \Omega^k(N) \longrightarrow \Omega^k(M) \text{ by}$$

$\varphi^*\omega(v_1,\dots,v_k) = \omega(\varphi_* v_1, \dots \varphi_* v_k)$ $v_i \in T_pM$ φ_* tangent map

which is well defined on $H^k_{dR}(N) \longrightarrow H^k_{dR}(M)$ because

1) For $v \in T_pM$ $df(v) = v(f) := \mathcal{L}_v f$

because $df = \frac{\partial f_0}{\partial x^i}dx^i$ $v = v^i\frac{\partial}{\partial x^i}$ $\mathcal{L}_v f = v^i \frac{\partial f_0}{\partial x^i}$

2) $\varphi^* df(v) = df(\varphi_* v) = (\varphi_* v)(f) = v(\varphi^* f) = d(\varphi^* f)(v)$

$\Longrightarrow \varphi^* df = d\varphi^* f$

3) $d(\varphi^*(dx^1 \wedge \dots \wedge dx^n)) = d(\varphi^*(dx^1) \wedge \dots \wedge \varphi^*(dx^n)) = 0.$

$(0 = \varphi^* d(dx^i) = d(\varphi^* dx^i))$

$d(w_1 + w_2) = dw_1 + dw_2$ $d(w_1 \wedge w_2) = dw_1 \wedge w_2 + (-1)^k w_1 \wedge dw_2$

4) $\omega = \sum_{i_1<i_2<\cdots<i_k} f_{i_1\cdots i_k} dx^{i_1}\wedge\cdots\wedge dx^{i_k}$

$$d\omega = \sum_{i_1<\cdots<i_k} \underbrace{\sum_{m\notin\{i_1,\cdots,i_k\}} \frac{\partial f_{i_1\cdots i_k}}{\partial x^m} dx^m}_{df} \wedge dx^{i_1}\wedge\cdots\wedge dx^{i_k}$$

$$\psi^*\omega = \sum (\psi^* f)\,\psi^*(dx^{i_1}\wedge\cdots\wedge dx^{i_k})$$

$$\psi^* d\omega = \sum \psi^*(df\wedge dx^{i_1}\wedge\cdots\wedge dx^{i_k})$$

$$= \sum \psi^*(df)\wedge\psi^*(dx^{i_1}\wedge\cdots\wedge dx^{i_k})$$

$$= \sum d(\psi^* f)\wedge\psi^*(\cdots\cdots)$$

$$(d\,\psi^*(dx^{i_1}\wedge\cdots\wedge dx^{i_k}) = 0)$$

$$= d\left(\sum(\psi^* f)\wedge\psi^*(\cdots\cdots)\right)$$

$$= d\psi^*\omega$$

$\Longrightarrow \psi^* d = d\psi^*$

5). If $d\omega=0$, then $d(\psi^*\omega) = \psi^* d\omega = 0$

If $\omega_1-\omega_2 = d\omega_0$ then $\psi^*(\omega_1)-\psi^*(\omega_2)$

$= \psi^* d\omega_0 = d(\psi^*\omega_0)$

$\Longrightarrow \psi^*$ is well-defined on $\ker d/\operatorname{Im} d$

Prop If ψ is a diffeomorphism. ψ^* induces an iso between $H^*_{dR}(N)$ and $H^*_{dR}(M)$ $\Longrightarrow$ If $H^*_{dR}(N)\neq H^*_{dR}(M)$, then $M \not\cong_{diff} N$

If ψ and ϕ are homotopic map to N, i.e.

$\exists\ \Phi : [0,1]\times M \longrightarrow N$ smooth, s.t. $\Phi(0,\cdot)=\phi$

and $\Phi(1,\cdot)=\psi$, then $\psi^* = \phi^* : H^*_{dR}(N)\longrightarrow H^*_{dR}(M)$