Class 8 Metrics on vector bundles. (Chap 7)

Basic idea: For a real/complex vector bundle $E \to M$.

The fiber at each $pt \in M$ is $\mathbb{R}^n$ or $\mathbb{C}^n$

with vector space structures.

A metric on E is a positive definite symmetric bilinear form for each fiber

Def 1 A <u>metric</u> g on a real vector bundle $E \to M$ is a section $M \longrightarrow E^* \otimes E^*$. s.t. for each $p \in M$

1) $g|_p(v,v) > 0 \quad v \neq 0 \in E|_p$ (positive definite)

2) $g|_p(v,w) = g|_p(w,v)$ (Symmetric)

$v, w \in E$.

Rem By 2), we can choose a basis of $E|_p \cong \mathbb{R}^n$ s.t. the matrix representing $g|_p$ is diagonal.

i.e. $g|_p(v,w) = w^T \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} v = \sum_{i=1}^n \lambda_i v_i w_i$

By 1). $\lambda_i > 0$

Def 2 An <u>Hermitian metric</u> g on a complex vector bundle $E \to M$ is a section $M \to \overline{E}^* \otimes E^*$ s.t. for $p \in M$

1) $g(v,v) \in (0,\infty) \quad v \neq 0 \in E|_p$

2) $g(v, cw) = c\, g(v,w)$

3) $g(cv, w) = \bar{c}\, g(v,w)$

4) $\overline{g(v,w)} = g(w,v)$

Rem Similarly, at each pt, we can choose a basis s.t.

$g|_p$ is diagonal $g|_p(v,w) = \sum_{i=1}^{n} \lambda_i \bar{v}_i w_i \quad \lambda_i \in (0,\infty)$

Prop: Any real / cpx v.b. admits an (Hermitian) metric

We use partition of unity to construct it.

pf. For a chart $U \subset M$. $\lambda_U : \pi^{-1}(U) \to \mathbb{R}^n \, (\mathbb{C}^n)$

Given a map $g_U : U \to U \times (\mathbb{R}^* \otimes \mathbb{R}^*)$ satisfying

the positive definite, symmetric condition on each fiber.

we obtain a metric on U by

$$g_U|_p(\lambda_U v, \lambda_U w) \qquad v, w \in E|_p$$

We can choose a locally finite atlas $\mathcal{U}$ of M

and construct the metric on each $U \in \mathcal{U}$

To put them together, we notice that if g_1, g_2 are

two metrics, then $s g_1 + t g_2$ is still a metric

for $s,t \geq 0$. s,t are not both 0.

(Just check Condition 1) & 2))

Lem. Suppose $\mathcal{U}$ is a locally finite atlas of M. Then there exists a set of smooth functions $\{\chi_\alpha : M \to [0,\mathbb{R}]\}_{U_\alpha \in \mathcal{U}}$ called a <u>partition of unity</u>, satisfying the following conditions

1) χ_α has support in U_α (i.e. $\{p \in M \mid \chi_\alpha(p) \neq 0\} \subset U_\alpha$)

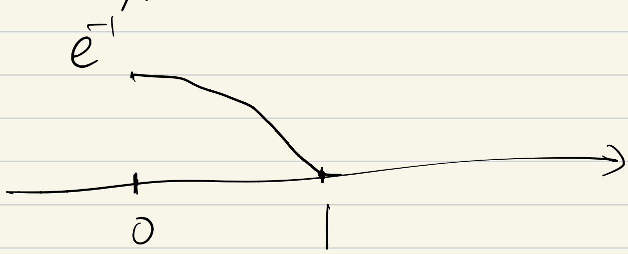
2) $\sum_\alpha \chi_\alpha(p) = 1$ for any $p \in M$.

Note that the locally finite condition + 1) implies at each p only finitely many χ_α is nonzero.

Pf: (See Cliff's book Appendix 1.2)

First we construct a function $\chi : [0,\infty) \to \mathbb{R}$

$$\chi(t) = \begin{cases} e^{\frac{1}{t-1}} & t < 1 \\ 0 & t \geq 1 \end{cases}$$



This function is smooth.

$$\lim_{t \to 1^-} \frac{d\left(e^{\frac{1}{t-1}}\right)}{dt} = \lim_{t \to 1^-} e^{\frac{1}{t-1}} \cdot \left(-\frac{1}{(t-1)^2}\right) = 0$$

Similar for higher derivatives

Then for $\phi_{U_\alpha} : U_\alpha \to \mathbb{R}^n$. we construct $G_\alpha : M \to \mathbb{R}$

$$G_\alpha(p) = \begin{cases} \chi\left(\frac{|\phi_{U_\alpha}(p)|}{r_{U_\alpha}}\right) & p \in U_\alpha \\ 0 & p \notin U_\alpha \end{cases}$$

We can choose r_{U_α} s.t. $\{\varphi_{U_\alpha}^{-1}(B(r_{U_\alpha}))\}_\alpha$ covers M.

Then $\sum_\alpha \sigma_\alpha$ is nowhere zero

We define $\chi_\alpha = \left(\sum_\alpha \sigma_\alpha\right)^{-1} \sigma_\alpha$ □

Using χ_α, we can define $g = \sum_\alpha \chi_\alpha g_{U_\alpha}$ as a metric on M □

Recall a v.b. can be constructed by bundle transition functions $\{g_{VU} : U \cap V \to GL(n, \mathbb{R})\}$ with some conditions

If we have a metric then we can choose $\operatorname{Im} g_{VU} \subset O(n) \subset GL(n, \mathbb{R})$ as follows

First, locally we have $\varphi_U : \pi^{-1}(U) \to U \times \mathbb{R}^n$ and a basis of sections $\{s_i : U \to \pi^{-1}(U)\}$

Since we have a metric on each fiber, we can use the Gram-Schmid procedure to obtain another basis of sections $\{s_i'\}$ which are orthogonal with respect to the metric at each fiber. Then we define a new diffeo $\varphi_U' : \pi^{-1}(U) \to U \times \mathbb{R}^n$ by setting

$$(\varphi_U')^{-1}(p, (v_1, \cdots, v_n)) = \sum_{i=1}^{n} v_i s_i'(p)$$

Then the bundle transition function $g_{VU} = \varphi'_V \circ (\varphi'_U)^{-1}$

sends orthogonal basis to orthogonal basis

so $g_{VU} : U \cap V \to O(n)$

If furthermore a real v.b. E is orientable, i.e.

$\Lambda^n E$ is isomorphic to $M \times \mathbb{R}$, then we have

$\det g_{VU}(p) > 0$. so $g_{VU} : U \cap V \to SO(n)$

(The groups $GL(n,\mathbb{R})$, $O(n)$, $SO(n)$ for g_{VU}

are called <u>structure groups</u> of the bundles)

The existence of metric $\Longleftrightarrow$ str gp can be reduced to $O(n)$

- - metric + orientation $\Longleftrightarrow$ str. gp - - - $SO(n)$

- - Hermitian metric for complex v.b. $\Longleftrightarrow$ $U(n) \subset GL(n,\mathbb{C})$

(almost complex str) $\subset GL(2n,\mathbb{R})$

Hermitian metric + orientation $\Longleftrightarrow$ $SU(n)$

- - - symplectic str. $\Longleftrightarrow$ Some str gps.

More facts: 1) a metric induces an isomorphism $E \xrightarrow{\cong} E^*$

2) an Hermitian metric induces a $\mathbb{C}$ anti-linear isomorphism

$$E \xrightarrow{\cong} E^*$$

3) an metric on E induces a metric on E^*, $E^{\otimes k}$

$\Lambda^k E$, $\mathrm{Sym}^k E$. metrics on E_1, E_2 induce

a metric on $E_1 \otimes E_2$ $\mathrm{Hom}(E_1, E_2)$. $E_1 \oplus E_2$

4) For a smooth map $f: M \to N$ and a pull-back

$f^*E \longrightarrow M$ of $E \to N$, the metric on E

induces a metric on f^*E.

Def. A metric on TM is called a <u>Riemannian metric</u>.

also called a metric of M.

Rem. Metrics on the submfds can be induced by the

metric of the ambient space.

Any M can be embedded into $\mathbb{R}^N$; the metric

of $\mathbb{R}^N$ also induces a metric on M

Class 9. Riemannian metric.

A metric on a (real) v.b. associates a positive definite symmetric bilinear form on each fiber. i.e.
a section $M \to E^* \otimes E^*$ satisfies some conditions.

Basic properties

1) metrics alway exist.

2) g_1, g_2 are metrics, then $sg_1 + tg_2$ are also metris
$s, t \geq 0 \quad s^2+t^2 \neq 0$
This implies the space of metrics is contractible because any metrics can be connected linearly.
by $tg_1 + (1-t)g_2$

3) The existence of metric reduces the str gp of the bundle transition functions from $GL(n, \mathbb{R})$ to $O(n)$ (Hermitian metric reduces $GL(n, \mathbb{C})$ to $U(n)$)

Def. A metric on TM is called a <u>Riemannian metric</u>.
also called a metric of M.

Recall that we write coordinates of

$$T\mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n \text{ as } x_1, \cdots, x_n, \frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^n}$$

and $T^*\mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n$ as $x_1, \cdots, x_n, dx_1, \cdots, dx_n$

Locally, a vector field (a section $M \to TM$)

is written as $\sum_{k=1}^{n} V_k \frac{\partial}{\partial x^k}$ for $V_k : U \to \mathbb{R}$

(Here we omit the function $\varphi_U : \pi^{-1}(U) \to U \times \mathbb{R}^n$)

a 1-form (a section $M \to T^*M$)

is written as $\sum_{k=1}^{n} V_k \, dx^k$

Hence a metric (a section of $T^*M \otimes T^*M$)

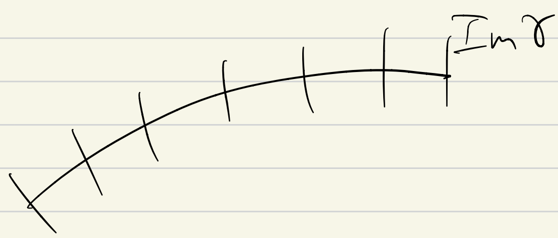
Can be written as $\sum_{i,j=1}^{n} g_{ij} \, dx^i \otimes dx^j$

with $\{g_{ij}(p)\}$ a sym, posi definite matric

$$g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = g_{ij}$$

Intuition of length of a path in $\mathbb{R}^n$

Let $\gamma : I = [0,1] \to \mathbb{R}^n$



Step 1. Break I into N segments

i.e. consider $t_i = \frac{i}{N}$ and $\gamma(t_i)$

Approximate $\operatorname{Im}\gamma\big|_{[t_i, t_{i+1}]}$ by $|\gamma(t_{i+1}) - \gamma(t_i)|$

$\Delta t = \frac{1}{N}$

Step 2. Use Taylor's thm to write

$$\frac{|\gamma(t_{i+1}) - \gamma(t_i)|}{\Delta t} = \left|\frac{d}{dt}\gamma\right|_{t=t_i} + O(\Delta t)$$

$$\sum_{i=0}^{N-1} |\gamma(t_{i+1}) - \gamma(t_i)| = \left(\sum \left|\frac{d}{dt}\gamma\right|_{t=t_i}\right)\Delta t + O(\Delta t)$$

Step 3 Take the limit $N \to \infty$, $\Delta t \to 0$

we get the length of γ

$$L_\gamma := \int_I \left|\frac{d}{dt}\gamma\right| dt \quad \text{(This is a definition)}$$

Def. Given a Riemannian metric g on M and a path $\gamma : I \to M$ we define <u>the length</u> of γ as $L_\gamma := \int_I \left(g\left(\frac{d}{dt}\gamma, \frac{d}{dt}\gamma\right)\right)^{\frac{1}{2}} dt$

Rem. $\frac{d}{dt}\gamma$ means the pushforward of (tangent map)

$\frac{\partial}{\partial t} \in T_t I$ to $\gamma_* \frac{\partial}{\partial t} \in T_{\gamma(t)} M$

Explicitly $U \subset I \quad V \subset M$

$\phi_U : U \longrightarrow \mathbb{R} \quad \phi_V : V \longrightarrow \mathbb{R}^n$

$\phi_{U*} \; TI|_U \longrightarrow U \times \mathbb{R} \quad \phi_{U*} : TM|_V \longrightarrow V \times \mathbb{R}^n$

$\phi_V \circ \gamma \circ \phi_U^{-1} : \mathbb{R} \longrightarrow \mathbb{R}^n$

$$\gamma_* \frac{\partial}{\partial t} = \phi_{V*}^{-1} \circ (\phi_V \circ \gamma \circ \phi_U^{-1})_* \circ \phi_{U*} \frac{\partial}{\partial t}$$

Since the notation $\frac{\partial}{\partial t}$ already means $\phi_{U*}^{-1}\left(\frac{\partial}{\partial t}\right)$,
and $\phi_U = Id$,

we have

$$\frac{d}{dt}\gamma\Big|_p = \phi_{V*}^{-1} \circ (\phi_V \circ \gamma)_* \frac{\partial}{\partial t}\Big|_p$$

$$= \phi_{V*}^{-1}\left(p, \frac{d(x_1 \circ \phi_V \circ \gamma)}{dt}, \ldots, \frac{d(x_n \circ \phi_V \circ \gamma)}{dt}\right)$$

$x_i : \mathbb{R}^n \longrightarrow \mathbb{R}$ coordinate map

Def The distance between $p, q \in M$ is defined by

$d(p,q) := \inf_\gamma L\gamma \quad \gamma$ is a path with $\gamma(0) = p$
$\gamma(1) = q$

Prop 1) $d(p,p)=0$ $d(p,q)\neq 0$ for $p\neq q$

2) $d(p,q) = d(q,p)$

3) $d(p,q) \leq d(p,r) + d(r,q)$ $r \in M$

Rem. 1). if $p \neq q$. we have a nbhd of p that is disjoint from q (Haursdorff). the path must intersects the boundary of the nbhd and in the nbhd the length is nonzero

Thm (Geodesic thm)

1) There is a smooth path γ from p to q with $L_\gamma = d(p,q)$, such path is called <u>geodesic</u>
(or its image)

2) Any geodesic is an embedded, 1-dim submfd that can be reparameterized s.t. $g(\gamma_*\frac{\partial}{\partial t}, \gamma_*\frac{\partial}{\partial t})=1$

This is called a <u>constant speed</u> geodesic

(Repara means we get a new path by composition

$I \xrightarrow{\psi} I \xrightarrow{\gamma} M$)

3) A constant speed geodesic is characterized as follows

Let $U \subset M$, $\phi_U : U \to \mathbb{R}^n$ $g|_U = \phi_U^*(\sum g_{ij}\, dx^i \otimes dx^j)$

We write $\gamma_U : \gamma^{-1}(U) \cap I \longrightarrow \mathbb{R}^n$

$$t \longmapsto \phi_U \circ \gamma(t)$$
$$= (\gamma_U^1(t), \gamma_U^2(t), \dots, \gamma_U^n(t))$$

Then
$$(*) \quad \frac{d^2}{dt^2}\gamma_U^i + \sum_{j,k=1}^n \Gamma^i_{jk} \frac{d}{dt}\gamma_U^j \frac{d}{dt}\gamma_U^k = 0$$

for any i where

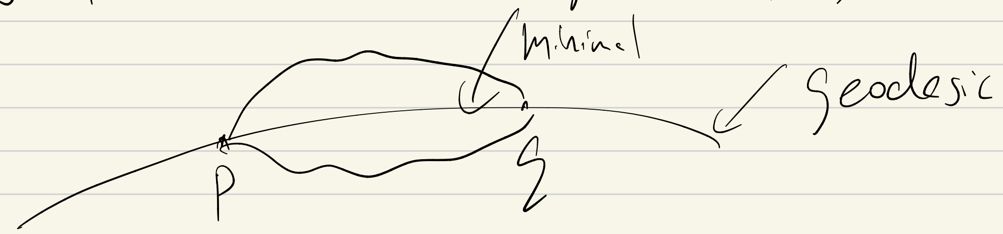
$$\Gamma^i_{jk} = \frac{1}{2}\sum_{l=1}^n g^{il}(\partial_j g_{lk} + \partial_k g_{jl} - \partial_l g_{jk})$$

$\{g^{ij}\}$ is the inverse matrix of $\{g_{ij}\}$

Γ^i_{jk} is called Christoffel symbols

4) Any path satisfying $(*)$ is locally length minimizing.

This means $\exists\, c_0 < 1$ s.t. when p, q on the curves with $d(p,q) \le c_0$, then $\exists$ a segment of path connecting p and q with length $d(p,q)$



Class 10 Geodesic (Chap 8)

Recall a path in M is a smooth map

$\gamma : I = [0,1] \to M$. the length of γ is

$$L_\gamma := \int_I \left| \frac{d\gamma}{dt} \right| dt = \int_I g\left(\gamma_* \frac{\partial}{\partial t}, \gamma_* \frac{\partial}{\partial t}\right)^{\frac{1}{2}} dt$$

in $\mathbb{R}^n$ in M

where $\frac{\partial}{\partial t}$ is the second coordinate of $TI \cong I \times \mathbb{R}$

and $\gamma_* : TI \to TM$ is the tangent map

(push forward) The distance $d(p,q) = \inf_{\substack{\gamma(0)=p \\ \gamma(1)=q}} L_\gamma$

Thm (Geodesic thm) M compact manifold

1) $\exists\ \gamma$ achieves $d(p,q)$. called geodesic

2) A geodesic is embedded, and can be reparameterized

Ex. so that $g(\gamma_* \frac{\partial}{\partial t}, \gamma_* \frac{\partial}{\partial t}) = 1$

3) Let $U \subset M$, $\phi_U : U \to \mathbb{R}^n$ $\quad g|_U = \phi_U^*(\sum g_{ij}\, dx^i \otimes dx^j)$

We write $\gamma_U : \gamma^{-1}(U) \cap I \to \mathbb{R}^n$

$$t \longmapsto \phi_U \circ \gamma(t)$$
$$= (\gamma_U^1(t), \gamma_U^2(t), \dots, \gamma_U^n(t))$$

Then
$$(*) \quad \frac{d^2}{dt^2}\gamma_U^i + \sum_{j,k=1}^n \Gamma^i_{jk} \frac{d}{dt}\gamma_U^j \frac{d}{dt}\gamma_U^k = 0$$

geodesic equation

for any i where

$$\Gamma^i_{jk} = \frac{1}{2}\sum_{l=1}^n g^{il}(\partial_j g_{lk} + \partial_k g_{jl} - \partial_l g_{jk})$$

$\{g^{ij}\}$ is the inverse matrix of $\{g_{ij}\}$

Γ^{i}_{jk} is called Christoffel symbols

4) Any path satisfying $(*)$ is locally length minimizing.

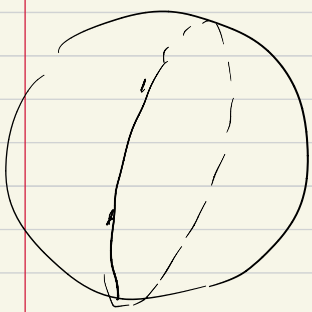
Ex

1) Geodesic in $\mathbb{R}^n$ with respect to the standard metric:

$$\partial_i g = 0. \quad (*) \Rightarrow \frac{d^2 \gamma_i}{dt^2} = 0 \Rightarrow \gamma_i(t) = at + b$$

i.e. geodesics in $\mathbb{R}^n$ are straight lines

2) Geodesic in S^2 with the induced metric from $\mathbb{R}^3$



the short segment of the great circle.

(see Cliff's book §4.2 for computations)

If p, q are two pole pts. then geodesics are not unique

Pf of Thm.

Step 1. $\mathbb{R}^n$ with the standard metric g

Consider the balls centered at the origin

Then $g = dr \otimes dr + r^2 g_S$

g_S is the induced metric on S^{n-1} with radius 1.

Let $\gamma : I \to \mathbb{R}^n$ be a path in a ball with radius R.

$\gamma(0) = 0 \in \mathbb{R}^n$ $\gamma(1)$ is on the boundary

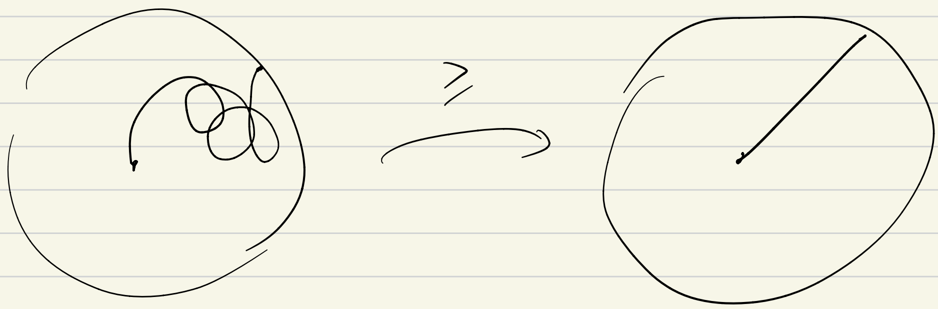
Write $\gamma(t)$ as $(r(t), \theta(t))$ $r: I \to \mathbb{R}_{\geq 0}$ $\theta: I \to S^{n-1}$

$$g(\gamma_* \tfrac{\partial}{\partial t}, \gamma_* \tfrac{\partial}{\partial t}) = \left|\frac{\partial r}{\partial t}\right|^2 + r^2 g_S(\theta_* \tfrac{\partial}{\partial t}, \theta_* \tfrac{\partial}{\partial t})$$

$$\geq \left|\frac{\partial r}{\partial t}\right|^2$$ the equation holds iff θ is constant

$$l_\gamma \geq \int_I \left|\frac{\partial r}{\partial t}\right| dt \geq \int_I \frac{\partial r}{dt} dt = R$$

the equation holds when γ is a straight line.



Moreover, we can reparameterize γ s.t. $g(\gamma_* \frac{\partial}{\partial t}, \gamma_* \frac{\partial}{\partial t}) = 1$

Step 2.

Lem (Special coordinates) If M is compact, then $\exists R > 0$ s.t. for each $p \in M$, $\exists$ chart $\phi_U : U \to \mathbb{R}^n$ s.t. $\phi_U(p) = 0$ and on a ball B_R of $\mathbb{R}^n$, the metric g on $\phi_U^{-1}(B_R)$ is $\phi_U^*(dr \otimes dr + r^2 g_r)$, g_r metric on S^{n-1}

Assuming the Lemma, we prove that for a fixed $p \in M$ and any $q \in M$ there exists a length minimizing curve from p to q. (1) in Thm

a) $d(p,q) \leq \frac{R}{2}$, such curve exists because q is in B_R for p

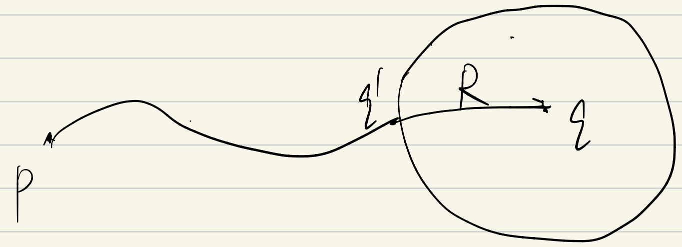
b) Suppose we already show the existence for q with $d(p,q) \leq \frac{NR}{2}$, we prove the existence for $\frac{NR}{2} < d(p,q) \leq \frac{(N+1)R}{2}$

There is a path γ_0 from p, q s.t. $l_{\gamma_0} < d(p,q) + \frac{R}{2}$

Suppose $q' = \mathrm{Im}\,\gamma_0 \cap \partial B_R(q)$

Then $d(p,q') \leq l_{\gamma_0} - R < d(p,q) - \frac{R}{2} \leq \frac{NR}{2}$

use $d(p,q) \leq \frac{(N+1)R}{2}$



Let $q'' \in \partial B_R(q)$ s.t. $d(p,q'')$ is minimal.

we know $d(p,q'') \leq d(p,q') < \frac{NR}{2}$

Then $\exists$ geodesics from p to q'', q'' to q.

the union of them is the geodesic from p to q

because it can't be shorten inside or outside the $B_R(q)$

We can reperaterize it so that it has constant speed

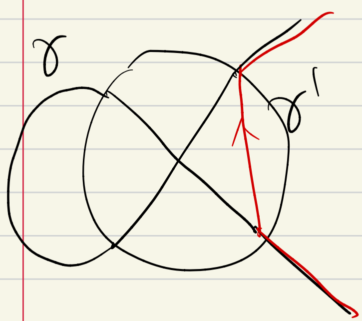
(2) in Thm

Step 3. Nbhd of each pt on the geodesic is a straight line

some B_R, so γ is an immersion (locally injective)

If $\mathrm{Im}\,\gamma$ has self-intersection, then locally it is

two straight lines in B_R, if we replace γ by γ',



then $L_{\gamma'} < L_\gamma$. which contradicts the minimal

length condition. Hence γ is an embedded curve

(locally injective and no self-intersection)

Note that the existence of the solution of the geodesic equation only for $t \in (-\varepsilon, \varepsilon)$. But we may extend the solution to a larger interval.

Def If all geodesics are defined on $\mathbb{R}$, the manifold is called geodesically complete.

Fact all compact manifolds are geodesically compact

Ex. a manifold that is not geodesically compact

$\mathbb{R}^2 - \{0\}$

Understand the geodesic equation by Euler-Lagrange equation (variation principal)

Consider a functional:

$$S : \left\{ \begin{array}{l} \text{the space of paths } \gamma \subset \mathbb{R}^n \\ \text{with } \gamma(0) = p \ \ \gamma(1) = q \end{array} \right\} \longrightarrow \mathbb{R}$$

$$\gamma \longmapsto \int_I g\left(\frac{d\gamma}{dt}, \frac{d\gamma}{dt}\right)^{\frac{1}{2}} dt$$

For $s \in (-\varepsilon, \varepsilon)$, consider the path

$$\gamma_s : I \longrightarrow \mathbb{R}^n$$
$$t \longmapsto \gamma(t) + s\eta(t) \qquad \eta : I \to \mathbb{R}^n \text{ any path}$$
$$\eta(0) = \eta(1) = 0$$

$$\left.\frac{dS(\gamma_s)}{ds}\right|_{s=0} = \int_I \frac{d\, g\left(\frac{d\gamma_s}{dt}, \frac{d\gamma_s}{dt}\right)^{\frac{1}{2}}}{ds} dt \bigg|_{s=0}$$

$$= \int_I \frac{d\left(g_{ij}(\gamma_s)\dot{\gamma}_s^i \dot{\gamma}_s^j\right)^{\frac{1}{2}}}{ds} dt \bigg|_{s=0} \quad \left(\begin{array}{l} \frac{d\gamma_s}{ds} = \eta(t) \\ \frac{d^2\gamma_s}{ds\,dt} = \dot{\eta}(t) \end{array}\right)$$

$$= \int_I \frac{1}{2} \cdot \frac{1}{\sqrt{g_{ij}(\gamma)\dot{\gamma}^i\dot{\gamma}^j}} \left(\partial_k g_{ij} \cdot \eta^k \dot{\gamma}^i \dot{\gamma}^j + 2 g_{ij} \dot{\eta}^i \dot{\gamma}^j\right) dt$$

integrating by parts on the second term. the boundary terms vanish because $\eta(0) = \eta(1) = 0$

$$= \frac{1}{2}\int_I \left(\frac{\partial_k g_{ij}\dot{\gamma}^i\dot{\gamma}^j}{\sqrt{g_{ij}(\gamma)\dot{\gamma}^i\dot{\gamma}^j}} - 2\frac{d}{dt}\left(\frac{g_{kj}\dot{\gamma}^j}{\sqrt{g_{ij}(\gamma)\dot{\gamma}^i\dot{\gamma}^j}}\right)\right)\eta^k dt$$

To obtain a minimal length path. we need

$$\left.\frac{dS(\gamma_s)}{ds}\right|_{s=0} = 0 \quad \text{for any } \eta(t).$$

So $$\frac{\partial_k g_{ij}\dot{\gamma}^i\dot{\gamma}^j}{\sqrt{g_{ij}\dot{\gamma}^i\dot{\gamma}^j}} - 2\frac{d}{dt}\left(\frac{g_{kj}\dot{\gamma}^j}{\sqrt{g_{ij}\dot{\gamma}^i\dot{\gamma}^j}}\right)$$ must vanish

Let $d\tau = dt\sqrt{g_{ij}\dot{\gamma}^i\dot{\gamma}^j}$

Then we have

$$\partial_k g_{ij}\dot{\gamma}^i\dot{\gamma}^j - 2\frac{d}{d\tau}\left(g_{kj}\dot{\gamma}^j\right) = 0 \quad \left(\text{Now } \dot{\gamma}^i = \frac{d\gamma^i}{d\tau}\right).$$

$2 g_{kl} \ddot{\gamma}^l$ (replace j by l)

$$\Rightarrow \partial_k g_{ij} \dot{\gamma}^i \dot{\gamma}^j - 2 \partial_i g_{kj} \dot{\gamma}^i \dot{\gamma}^j - \cancel{2 g_{kj} \ddot{\gamma}^j} = 0$$

$$\Rightarrow \ddot{\gamma}^l + \frac{1}{2} g^{kl} \Big(\underbrace{2\partial_i g_{kj}}_{\partial_i g_{kj} + \partial_j g_{ik}} - \partial_k g_{ij} \Big) \dot{\gamma}^i \dot{\gamma}^j = 0$$

$\partial_i g_{kj} + \partial_j g_{ik}$ after times $\dot{\gamma}^i \dot{\gamma}^j$

$$\Rightarrow \ddot{\gamma}^l + \Gamma^l_{ij} \dot{\gamma}^i \dot{\gamma}^j = 0$$

$$\Gamma^i_{jk} = \frac{1}{2} g^{il} (\partial_j g_{lk} + \partial_k g_{jl} - \partial_l g_{jk})$$

Appendix

Lem1 (Special coordinates) If M is compact, then $\exists R>0$ s.t. for each $p\in M$, $\exists$ chart $\phi_U: U\to\mathbb{R}^n$ s.t. $\phi_U(p)=0$ and on a ball B_R of $\mathbb{R}^n$, the metric g on $\phi_U^{-1}(B_R)$ is $\phi_U^*(dr\otimes dr + r^2 g_r)$

Last time we use Lem2 to prove 1), 2) in Thm 1

Today we use the following thm to prove Lem 1

Thm2 (vector field thm) Let $s: X\to TX$ be a vector field. For any $q\in X$, $\exists\ \varepsilon>0$ and a unique path $\sigma_q: (-\varepsilon,\varepsilon)\to X$ s.t.

1) $\sigma_q(0)=q$ 2) $\sigma_{q*}\frac{\partial}{\partial t} = s(\sigma_q(t))$

We take $X=TM$ locally $q=(p,v)\in X$

Define $s(p,v) = (v_i, -\Gamma^i_{jk} v_j v_k)$

$$\Gamma^i_{jk} = \frac{1}{2}\sum_{l=1}^{n} g^{il}(\partial_j g_{lk} + \partial_k g_{jl} - \partial_l g_{jk})$$

$$g|_U = \phi_U^*(\Sigma g_{ij}\, dx^i\otimes dx^j)$$

$\{g^{ij}\}$ is the inverse matrix of $\{g_{ij}\}$

Γ^j_{km} is called Christoffel symbols

Then we obtain a unique path

$G_q : (-\varepsilon, \varepsilon) \longrightarrow TM$ s.t. $G_q(0) = q = (p, v)$

$G_{q*}\frac{\partial}{\partial t} = S(G_q(t))$

This implies a unique path $\gamma_{(p,v)} : (-\varepsilon, \varepsilon) \to M$

s.t. $\gamma(0) = p$ $\gamma_* \frac{\partial}{\partial t}\Big|_{t=0} = v$

and $\dfrac{d^2\gamma^i}{dt^2} + \sum_{j,k} \Gamma^i_{jk} \dfrac{d\gamma^j}{dt}\dfrac{d\gamma^k}{dt} = 0$

(Here we write $\gamma = \gamma_U = \phi_U \circ \gamma_{(p,v)} (-\varepsilon, \varepsilon) \longrightarrow \mathbb{R}^n$)

The reason to choose such S is because the following properties

Prop.

1) $g(\gamma_*\frac{\partial}{\partial t}, \gamma_*\frac{\partial}{\partial t})$ is constant

pf. $\partial_t g(\gamma_*\frac{\partial}{\partial t}, \gamma_*\frac{\partial}{\partial t}) = \partial_t g_{ij}\dot\gamma^i\dot\gamma^j$

(when an index appear twice, it means we take Σ over the index)

(for simplicity, we write $\dot\gamma^i = \dfrac{d\gamma_U^i}{dt}$ $\ddot\gamma^i = \dfrac{d^2\gamma_U^i}{dt^2}$)

$\gamma_U = \phi_U \circ \gamma$.

$= g_{ij}\ddot\gamma^i\dot\gamma^j + g_{ij}\dot\gamma^i\ddot\gamma^j + \dot\gamma^k\partial_k g_{ij}\dot\gamma^i\dot\gamma^j = 0$

Compute $\partial_t\, g_{ij}\dot\gamma^i\dot\gamma^j$

$$= \boxed{g_{ij}\ddot\gamma^i\dot\gamma^j + g_{ij}\dot\gamma^i\ddot\gamma^j} + \dot\gamma^k\partial_k g_{ij}\dot\gamma^i\dot\gamma^j = 0$$

① $\ddot\gamma^i + \Gamma^i_{jk}\dot\gamma^j\dot\gamma^k = 0$ ② $g_{ij} = g_{ji}$ | ④ $\{g^{il}\} = \{g_{il}\}^{-1}$

③ $\Gamma^i_{jk} = \frac{1}{2}g^{il}(\partial_j g_{lk} + \partial_k g_{jl} - \partial_l g_{jk})$ | $g_{il}g^{il} = \delta_{il} = \begin{cases}1 & i=l\\ 0 & i\neq l\end{cases}$

② $-2g_{ij}\ddot\gamma^i\dot\gamma^j \overset{①}{=} 2g_{ij}\Gamma^i_{mk}\dot\gamma^m\dot\gamma^k\dot\gamma^j$

change index

$m \mapsto j$

$j \mapsto m$

$$2g_{im}\Gamma^i_{jk}\dot\gamma^j\dot\gamma^k\dot\gamma^m$$

$$\overset{③}{=} g_{im}g^{il}(\partial_j g_{lk} + \partial_k g_{jl} - \partial_l g_{jk})\dot\gamma^j\dot\gamma^k\dot\gamma^m$$

$$= \delta_{ml}(\partial_j g_{lk} + \partial_k g_{jl} - \partial_l g_{jk})\dot\gamma^j\dot\gamma^k\dot\gamma^m$$

$$= (\partial_j g_{mk} + \partial_k g_{jm} - \partial_m g_{jk})\dot\gamma^j\dot\gamma^k\dot\gamma^m$$

$$= \partial_k g_{ij}\dot\gamma^i\dot\gamma^j\dot\gamma^k$$

2) $\gamma_*\frac{\partial}{\partial t}\Big|_{t=0} = v \Rightarrow g(\gamma_*\frac{\partial}{\partial t}, \gamma_*\frac{\partial}{\partial t}) = |v|^2$

for all t (constant speed)

3) Fix $\mu > 0$, let $\eta(t) = \gamma_{(p,v)}(\mu t)$. Then

$\eta(0) = p$ $\eta_*\frac{\partial}{\partial t} = \mu v$ η also satisfy the geodesic equation. η is defined for $t \in (\mu^{-1}\varepsilon, \mu^{-1}\varepsilon)$

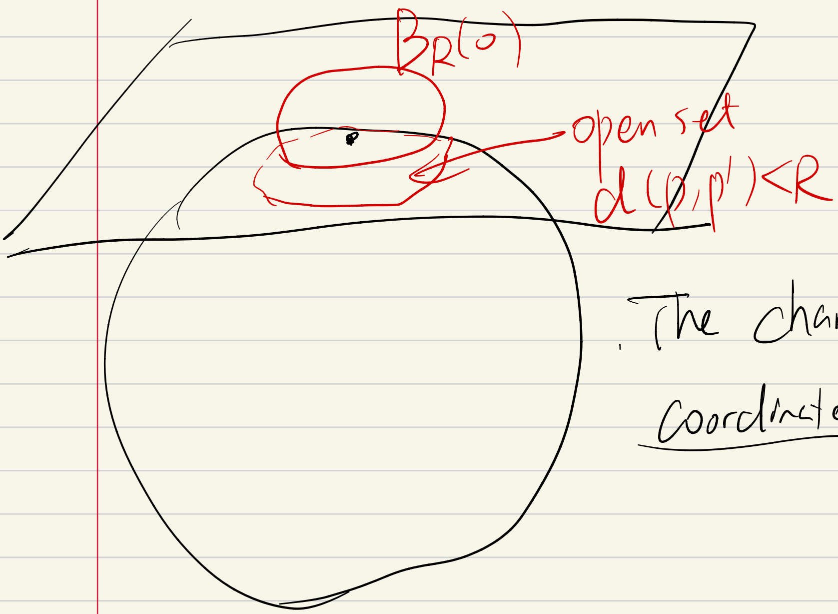
Then $\exists R$ s.t. the map $v \longmapsto \gamma_{(p,\frac{2}{\varepsilon}v)}(\frac{\varepsilon}{2})$

define a diffeo between $B_R(0) \subset TM|_p \cong \mathbb{R}^n$

and the open set of pts p' in M with

$d(p,p') < R$. This map is called the

<u>exponential map</u> at p $\quad exp_p : TM|_p \to M$



Note that this is an inverse map of a coordinate chart

The charts are called <u>Gaussian coordinates</u> or <u>normal coordinates</u>

Claim: The pullback metric $\hat{g} = exp^* g$

has the form $dr \otimes dr + r^2 g_r \leftarrow$ metric on S^{n-1}

(If M is compact, we can use finitely many Gaussian charts to cover M. hence we have a minimal value of $\varepsilon > 0$. This will imply Lem 1)

Pf of Claim: Cliff's book Section 9.3