

Class 16 Metric compatible covariant derivative (Chap 15)

In weeks 7-8, we study ① the exterior derivative

$$d: \Omega^k \rightarrow \Omega^{k+1}, \text{ where } \Omega^k = C^\infty(M; \Lambda^k T^*M) \\ = \{ \text{sections in } \Lambda^k T^*M \}.$$

② the covariant derivative $\nabla: C^\infty(M; E) \rightarrow C^\infty(M; E \otimes T^*M)$

③ the exterior covariant derivative

$$d\nabla: C^\infty(M; E \otimes \Lambda^k T^*M) \rightarrow C^\infty(M; E \otimes \Lambda^{k+1} T^*M)$$

and

④ the curvature F_∇ of $d\nabla$ as a section of $\text{End}(E) \otimes \Lambda^2 T^*M$

In weeks 5-6, we study metric g on E , which is

a section of $E^* \otimes E^*$ s.t. at each fiber, it is symmetric and positive definite.

Using partition of unity, we show the metric and the covariant derivative exist.

(the space of the metric is convex hence contractible and the space of covariant derivative is affine over $C^\infty(M; \text{End}(E) \otimes T^*M)$)

In this week, we study metric compatible covariant derivative, i.e. for any sections u, v of E ,

$$dg(u, v) = g(\nabla u, v) + g(u, \nabla v)$$

(some version of Leibniz rule)

Given $u, v \in C^\infty(M; E)$ $g(u, v)$ is a map $M \rightarrow \mathbb{R}$.

$dg(u, v)$ is a 1-form (we apply exterior derivative)

$\nabla u \in C^\infty(M; E \otimes T^*M)$, $\nabla u \otimes v \in C^\infty(M; E \otimes T^*M \otimes E)$

$g(\nabla u, v)$ means we apply g to two E components.

and hence also obtain a section in T^*M , i.e. a 1-form.

Following the notation in previous class, we write everything on chart U and omit subscript U for s_u, α_u

Let $e_1, \dots, e_n$ be a basis of orthonormal sections of E , where orthonormal means $g(e_i(p), e_j(p)) = \delta_{ij}$

It exists by Gram-Schmidt procedure.

Then $s = \sum_{k=1}^n s_k e_k$ $s_k: U \rightarrow \mathbb{R}$

- (In weeks 5-6, we write $s_1, \dots, s_n$ for a basis of sections and $v_1, \dots, v_k$ as functions $U \rightarrow \mathbb{R}$)
- Now the notation follows from those in weeks 7-8)

$$\nabla s = \sum_{i,j} (ds_i + \alpha_{ij}^j s_j) e_i$$

α_{ij}^j is component of α in previous class, α_{ij}^j is 1-form.

$$\text{especially } \nabla e_i = \sum \alpha_{ij}^j e_j \\ = \alpha_{ij}^j e_j$$

(write α_{ij}^j rather than α_{ij} because we want to omit $\sum$ later when an index appears twice, we take sum over this index)

$$\text{From } dg(u, v) = g(\nabla u, v) + g(u, \nabla v)$$

$$\text{we have } 0 = dg(e_i, e_j) = g(\nabla e_i, e_j) + g(e_i, \nabla e_j)$$

$$= g(\alpha_i^k e_k, e_j) + g(e_i, \alpha_j^l e_l)$$

$$= \alpha_i^j + \alpha_j^i$$

i.e. the matrix value 1-form $\alpha = \{\alpha_j^i\}_{i,j}$

$$\text{satisfies } \alpha^T + \alpha = 0$$

Given ∇ on E , we can define ∇ on E^* as follows.

Let s be a section of E and s^* be a section of E^* , we have a pairing $\langle s, s^* \rangle : M \rightarrow \mathbb{R}$

$d\langle s, s^* \rangle$ is a 1-form

Then we define ∇s^* by the following equation

$$d\langle s, s^* \rangle = \langle \nabla s, s^* \rangle + \langle s, \nabla s^* \rangle \quad (1)$$

(this equation holds for any s and fixed s^*)

We can also extend ∇ to $E^* \otimes E^*$ by

$$\nabla(s_1^* \otimes s_2^*) = \nabla s_1^* \otimes s_2^* + s_1^* \otimes \nabla s_2^* \quad (2)$$

Then we can define $\nabla g \in C^\infty(M; E^* \otimes E^* \otimes TM)$

Lemma ∇ is metric compatible iff $\nabla g = 0$

Pf: To define ∇g , we can first define ∇ on $E \otimes E$ by

$$\nabla(s_1 \otimes s_2) = \nabla s_1 \otimes s_2 + s_1 \otimes \nabla s_2 \quad (3)$$

define ∇ on $(E \otimes E)^* = E^* \otimes E^*$ by defining $\nabla(s_1^* \otimes s_2^*)$

$$d \langle s_1 \otimes s_2, s_1^* \otimes s_2^* \rangle = \langle \nabla(s_1 \otimes s_2), s_1^* \otimes s_2^* \rangle + \langle s_1 \otimes s_2, \nabla(s_1^* \otimes s_2^*) \rangle \quad (4)$$

for any $s_1 \otimes s_2$ and then extend linearly over $\mathbb{R}$

Note that when multiply with a map $f: M \rightarrow \mathbb{R}$, we define

$$\begin{aligned} \nabla f(s_1^* \otimes s_2^*) &= (s_1^* \otimes s_2^*) \otimes f + f \nabla(s_1^* \otimes s_2^*) \\ &= f s_1^* \otimes f s_2^* + f \nabla(s_1^* \otimes s_2^*) \end{aligned}$$

We need to check these two definitions are the same

i.e. (1-3) implies (4) and (1,3,4) $\Rightarrow$ (2)

Use the second definition, we have

$$dg(u, v) = d \langle u \otimes v, g \rangle$$

$$= \langle \nabla(u \otimes v), g \rangle + \langle u \otimes v, \nabla g \rangle$$

$$= \langle \nabla u \otimes v + u \otimes \nabla v, g \rangle + \langle u \otimes v, \nabla g \rangle$$

$$= g(\nabla u, v) + g(u, \nabla v) + \langle u \otimes v, \nabla g \rangle \quad \square$$

Then we consider the case $E = TM$. $E^* = T^*M$.

$$\nabla: C^\infty(M; TM) \rightarrow C^\infty(M; T^*M \otimes T^*M)$$

Define $\wedge$ to be the antisymmetrization

from $C^\infty(M; T^*M \otimes T^*M) \rightarrow C^\infty(M; \Lambda^2 T^*M)$

$$\text{by } \Lambda(w_1 \otimes w_2) = \frac{1}{2}(w_1 \otimes w_2 - w_2 \otimes w_1)$$

and extending linearly.

For a 1-form $w \in C^\infty(M; T^*M)$

we can define $dw \in C^\infty(M; \Lambda^2 T^*M)$

$$\text{and } \Lambda(dw) = \Lambda(\nabla w) \in C^\infty(M; \Lambda^2 T^*M)$$

(Rem. previously we define $d\nabla$ to be a map
 $C^\infty(M; E \otimes \Lambda^k T^*M) \rightarrow C^\infty(M; E \otimes \Lambda^{k+1} T^*M)$)

In the special case $E = T^*M$ we compose it with Λ
Cliff still uses the same notation, but we write as $\Lambda d\nabla$

Def The torsion tensor $T_\nabla : C^\infty(M; T^*M) \rightarrow C^\infty(M; \Lambda^2 T^*M)$

is defined by $T_\nabla w = \Lambda(dw) - dw$

∇ (on T^*M) is called torsion free if $T_\nabla = 0$, i.e.

$$\Lambda d\nabla w = dw$$

Thm. Given a Riemannian metric g on TM , there is
a unique metric compatible, torsion free covariant derivative

∇_L called the Levi-Civita connection.

Class 17 Levi-Civita connection

Given a Riemannian metric g on TM , a covariant derivative ∇ on TM is compatible with g if

$$\text{either 1) } dg(u, v) = g(\nabla u, v) + g(u, \nabla v)$$

for any vector fields u, v

or 2) $\nabla g = 0$ for the covariant derivative on $T^*M \otimes T^*M$

$$\text{induced by } \nabla(s_1^* \otimes s_2^*) = \nabla s_1^* \otimes s_2^* + s_1^* \otimes \nabla s_2^*$$

$$d\langle s, s^* \rangle = \langle \nabla s, s^* \rangle + \langle s, \nabla s^* \rangle$$

∇ is called torsion-free if the torsion tensor

$$T_\nabla = \text{Ad}_\nabla - d : C^\infty(M; T^*M) \rightarrow C^\infty(M; \Lambda^2 T^*M) \text{ vanishes,}$$

where d is the usual exterior derivative.

$$A(w_1 \otimes w_2) = \frac{1}{2}(w_1 \otimes w_2 - w_2 \otimes w_1)$$

Thm. Given a Riemannian metric g on TM , there is a unique metric compatible, torsion free covariant derivative

∇_{LC} called the Levi-Civita connection.

Here we don't distinguish ∇ on TM or T^*M because they can induce each other

Pf. 1 (in the next pages, using orthonormal basis)

In the last class, we compute in a locally orthonormal basis $e_1, \dots, e_n$. Suppose

$$\nabla e_i = \sum_j e_j \otimes \alpha_i^j = \sum_j \alpha_i^j e_j \quad \text{for short}$$

α_i^j 1-forms, ∇ is compatible with g iff $\alpha_i^j = -\alpha_j^i$ (1)

consider the dual basis $e^i = e_i^*$

$\langle e_i, e^j \rangle = \delta_{ij} : M \rightarrow \mathbb{R}$ suppose

$$\nabla e^i = e^j \otimes \beta_j^i = e^j \otimes \beta_{jk}^i e^k = \beta_{jk}^i e^j \otimes e^k$$

β_j^i 1-form, $\beta_{jk}^i : M \rightarrow \mathbb{R}$

$$0 = d\langle e_i, e^j \rangle = \langle \nabla e_i, e^j \rangle + \langle e_i, \nabla e^j \rangle$$

$$= \langle \alpha_i^k e_k, e^j \rangle + \langle e_i, e^k \otimes \beta_k^j \rangle$$

$$= \alpha_i^j + \beta_i^j$$

$\Rightarrow \nabla$ is compatible with g iff $\beta_i^j = -\beta_j^i$ (1')

$$Ad_\nabla e^i = A(\beta_{jk}^i e^j \otimes e^k)$$

$$= \beta_{jk}^i \left(\frac{1}{2} (e^j \otimes e^k - e^k \otimes e^j) \right)$$

$$= \frac{1}{2} (\beta_{jk}^i - \beta_{kj}^i) e^j \otimes e^k$$

Suppose $de^i = \sum_{j < k} \gamma_{jk}^i e^j \wedge e^k$

$$= \sum_{j < k} \gamma_{jk}^i \frac{1}{2} (e^j \otimes e^k - e^k \otimes e^j) = \sum_{j, k} \frac{1}{2} \gamma_{jk}^i e^j \otimes e^k$$

where we set $\gamma_{jk}^i = -\gamma_{kj}^i$ for $j > k$, $\gamma_{jj}^i = 0$

Then $T_{\nabla} e^i = \text{Ad}_{\nabla} e^i - de^i = \frac{1}{2} (\beta_{jk}^i - \beta_{kj}^i - \gamma_{jk}^i) e^j \otimes e^k$

∇ is torsion free iff $\beta_{jk}^i - \beta_{kj}^i - \gamma_{jk}^i = 0$ (2)

change indices, we have $\beta_{ki}^j - \beta_{ik}^j - \gamma_{ki}^j = 0$ (2')

$$\beta_{ij}^k - \beta_{ji}^k - \gamma_{ij}^k = 0 \quad (2'')$$

Also from (1'), we have $\beta_{ik}^j = -\beta_{jk}^i$

(2) + (2') - (2'')

$$\Rightarrow \beta_{jk}^i = \frac{1}{2} (\gamma_{jk}^i + \gamma_{ki}^j - \gamma_{ij}^k)$$

If there is another solution $(\beta')_{jk}^i$, suppose

$$\eta_{jk}^i = \beta_{jk}^i - (\beta')_{jk}^i, \text{ then } (2) \Rightarrow \eta_{jk}^i = \eta_{kj}^i$$

we have

$$(1') \Rightarrow \eta_{ik}^j = -\eta_{jk}^i$$

$$\eta_{ik}^j \stackrel{(2')}{=} -\eta_{jk}^i \stackrel{(2)}{=} -\eta_{kj}^i \stackrel{(1')}{=} \eta_{ij}^k \stackrel{(2)}{=} \eta_{ji}^k \stackrel{(1')}{=} -\eta_{ki}^j \stackrel{(2)}{=} -\eta_{ik}^j$$

$$\Rightarrow \eta_{ik}^j = 0, \text{ so the solution is unique.}$$

In the next two pages
Pf 2 (Using coordinate basis $\frac{\partial}{\partial x^i}$ and dx^i .)

Note that we don't have $g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) = g_{ij}$ in this case

But $d(dx^i) = 0$, so we don't have γ_{jk}^i terms.

$$\text{Suppose } \nabla \frac{\partial}{\partial x^i} = \Gamma_{ik}^j \frac{\partial}{\partial x^j} \otimes dx^k$$

(we use Γ_{ik}^j because we will show this is indeed the Christoffel symbol

$$\Gamma_{ik}^j = \frac{1}{2} g^{jl} (\partial_i g_{lk} + \partial_k g_{il} - \partial_l g_{ik})$$

From the pairing $\nabla \langle \frac{\partial}{\partial x^i}, dx^j \rangle$, we have

$$\nabla dx^i = -\Gamma_{jk}^i dx^j \otimes dx^k$$

$$T_\nabla = 0 \Rightarrow \Gamma_{jk}^i = \Gamma_{kj}^i$$

Then we need to show ∇ is compatible with g

$$\text{Suppose } g = g_{ij} dx^i \otimes dx^j \quad g_{ij} = g_{ji} : U \rightarrow \mathbb{R}$$

$$\{g^{ij}\} = (g_{ij})^{-1} \quad 0 = \nabla g = \nabla (g_{ij} dx^i \otimes dx^j)$$

$$= (\nabla g_{ij}) dx^i \otimes dx^j + g_{ij} (\nabla dx^i) \otimes dx^j + g_{ij} dx^i \otimes (\nabla dx^j)$$

For simplicity, we can consider $\nabla g = \langle \frac{\partial}{\partial x^l}, \nabla g \rangle$

$$\text{Note that } \nabla_L g_{ij} = \frac{\partial g_{ij}}{\partial x^l} = \partial_l g_{ij}$$

$$\nabla_L dx^i = -\Gamma_{lk}^i dx^k$$

Hence we have

$$0 = \partial_\ell g_{ij} dx^i \otimes dx^j - g_{ij} (\Gamma_{\ell p}^i dx^p \otimes dx^j + \Gamma_{\ell q}^j dx^i \otimes dx^q)$$

We can change indices and solve Γ_{jk}^i as in the first proof, and also prove the uniqueness similarly. Here for simplicity, I just show the Christoffel symbol satisfies the equation.

$$\text{Recall } \Gamma_{ik}^j = \frac{1}{2} g^{jl} (\partial_i g_{lk} + \partial_k g_{il} - \partial_l g_{ik})$$

$$\begin{aligned} g_{ij} \Gamma_{\ell p}^i &= \frac{1}{2} g_{ij} g^{ir} (\partial_\ell g_{rp} + \partial_p g_{ir} - \partial_r g_{ip}) \\ &= \frac{1}{2} \partial_\ell g_{jp} + \partial_p g_{ij} - \partial_j g_{ip} \end{aligned}$$

$$\text{because } g_{ij} g^{ir} = \delta_{jr}$$

$$\text{Then } g_{ij} (\Gamma_{\ell p}^i dx^p \otimes dx^j + \Gamma_{\ell q}^j dx^i \otimes dx^q)$$

$$= \frac{1}{2} (\partial_\ell g_{jp} + \partial_p g_{ij} - \partial_j g_{ip}) dx^p \otimes dx^j$$

$$+ \frac{1}{2} (\partial_\ell g_{iq} + \partial_q g_{il} - \partial_i g_{\ell q}) dx^i \otimes dx^q$$

$$\begin{aligned} \begin{matrix} p \rightarrow i \\ q \rightarrow j \end{matrix} &= \frac{1}{2} (\partial_\ell g_{ji} + \cancel{\partial_i g_{lj}} - \cancel{\partial_j g_{li}} + \partial_\ell g_{ij} + \cancel{\partial_j g_{li}} \\ &\quad - \cancel{\partial_i g_{lj}}) dx^i \otimes dx^j \\ &= \partial_\ell g_{ij} dx^i \otimes dx^j \end{aligned}$$

Class 18 Covariantly constant section and curvature

Last time, we show there is a unique covariant derivative on TM and T^*M that is compatible with a given metric g and torsion free, which is called Levi-Civita connection ∇_{LC} . We proved the existence and uniqueness using two different bases.

① orthonormal basis $e_1, \dots, e_n, e^i = e_i^*$
metric compatible condition is easier, but de^i may not be zero

② coordinate basis $\frac{\partial}{\partial x^i}, dx^i$
 $d(dx^i) = 0$, so the torsion free condition is easier, but $g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$ may not be δ_{ij}

In the second proof, we notice that

$$\nabla \frac{\partial}{\partial x^i} = \Gamma_{ik}^j \frac{\partial}{\partial x^j} \otimes dx^k, \quad \nabla dx^i = -\Gamma_{jk}^i dx^j \otimes dx^k$$

$$\text{where } \Gamma_{ik}^j = \frac{1}{2} g^{jl} (\partial_i g_{lk} + \partial_k g_{il} - \partial_l g_{ik})$$

is the Christoffel symbol. Recall it appears in the geodesic equation (weeks 5-6)

$$\frac{d^2 \gamma^i}{dt^2} + \Gamma_{jk}^i \frac{d\gamma^j}{dt} \frac{d\gamma^k}{dt} = 0 \quad \text{or} \quad \ddot{\gamma}^i + \Gamma_{jk}^i \dot{\gamma}^j \dot{\gamma}^k = 0$$

What is the relation between these two?

First, we give a definition for general v.b.

Def Let $\gamma: [0,1] \rightarrow M$ be a smooth path

A section of $E|_{\gamma} = \pi^{-1}(\gamma)$ is parallel if

$\nabla_{\dot{\gamma}} S := \langle \gamma^* \frac{\partial}{\partial t}, \nabla S \rangle$ vanishes, i.e.

the covariant derivative at tangent direction of γ vanishes. For a basis e_i of E

we write $\nabla S = (ds_i + \alpha^{ij} s_j) e_i$

$$\gamma^* \frac{\partial}{\partial t} = \frac{d\gamma^k}{dt} \frac{\partial}{\partial x^k} \quad \alpha^{ij} = \alpha^{ij}_k dx^k \quad \alpha^{ij}_k: U \rightarrow \mathbb{R}$$

$$\text{Then } \nabla_{\dot{\gamma}} S = \left(\frac{ds_i \circ \gamma}{dt} + \alpha^{ij}_k \frac{d\gamma^k}{dt} s_j \circ \gamma \right) e_i \quad (**)$$

This is a section of $E|_U$

$\nabla_{\dot{\gamma}} S = 0$ iff $\langle \nabla_{\dot{\gamma}} S, e^i \rangle = 0$ for any i

In particular if $E = TM$ $e_i = \frac{\partial}{\partial x^i}$ $S = \gamma_x \frac{\partial}{\partial t}$

$\nabla = \nabla_{LC}$ Then $s_i \circ \gamma = \frac{d\gamma^i}{dt}$

$$\nabla \frac{\partial}{\partial x^l} = \left(d \left(\frac{\partial}{\partial x^l} \right)_i + \alpha^{ij} \left(\frac{\partial}{\partial x^l} \right)_j \right) \frac{\partial}{\partial x^i} = \alpha^{il}_k \frac{\partial}{\partial x^k}$$

$$\left(\left(\frac{\partial}{\partial x^l} \right)_i = \delta_{il} \right) \Rightarrow \alpha^{il}_k = \Gamma^l_{ik}$$

$$\nabla_{\dot{\gamma}} \dot{\gamma} = 0 \Leftrightarrow \frac{d^2 \gamma^i}{dt^2} + \Gamma^i_{jk} \frac{d\gamma^j}{dt} \frac{d\gamma^k}{dt} = 0$$

i.e. For a geodesic γ , $\gamma_x \frac{\partial}{\partial t}$ is a parallel section with respect to ∇_{LC}

Recall in weeks 5-6, we use vector field thm to show that given any point $p = \gamma(0)$ $v_0 \in TM|_{\gamma(0)}$ we can always find a unique solution of γ satisfying the geodesic equation.

Similarly, for a general vector bundle, given a path γ and $v_0 \in E|_{\gamma(0)}$, we can find a unique parallel section s on $E|_\gamma$
(Solving $(\nabla \cdot \cdot) = 0$)

Let $v_1 = s(\gamma(1))$, it is called the parallel transportation of v_0 along γ

Moreover, if $v_0^1, v_0^2, \dots, v_0^n$ a basis of $E|_{\gamma(0)}$, we obtain a basis $v_1^1, \dots, v_1^n$ of $E|_{\gamma(1)}$ after parallel transportation.

i.e. ∇ and γ induces an isomorphism between $E|_{\gamma(0)}$ and $E|_{\gamma(1)}$

Question. does the isomorphism depend on γ for given $\gamma(0)=p, \gamma(1)=q$?

Answer : This is true when the curvature $F_D = 0$.
 We will come back to this topic after introducing principal bundle.

Then let's discuss curvature for metric compatible covariant derivative

Let $e_1, \dots, e_n$ be an orthonormal basis of E .

$$\nabla e_i = \alpha_i^j e_j \quad \alpha_i^j = \alpha_{ik}^j dx^k \quad \alpha = \{\alpha_{ik}^j\}$$

$$\alpha_i^j = -\alpha_j^i \text{ if } \nabla \text{ is compatible with } g$$

Recall $F_D = d\alpha + \alpha \wedge \alpha \in C^\infty(M; \text{End}(E) \otimes \wedge^2 T^*M)$

$$F_D e_i = e_j \otimes d\alpha_i^j + e_k \otimes \alpha_i^k \wedge \alpha_k^j$$

$$= e_j \otimes (d\alpha_i^j + \alpha_i^k \wedge \alpha_k^j)$$

$$(F_D)_i^j = d\alpha_i^j + \alpha_i^k \wedge \alpha_k^j \quad \alpha_j^i = -\alpha_i^j$$

$$\begin{aligned} (F_D)_j^i &= d\alpha_j^i + \alpha_j^k \wedge \alpha_k^i = -d\alpha_i^j + (-\alpha_k^i) \wedge (-\alpha_i^k) \\ &= -d\alpha_i^j - \alpha_i^k \wedge \alpha_k^j = -(F_D)_i^j \end{aligned}$$

For the dual basis e^i , we have $\nabla e^i = -\alpha_j^i e^j$

$$F_D e^i = e^j \otimes (-d\alpha_j^i + \alpha_k^i \wedge \alpha_j^k)$$

$$(F_D)_j^i = -d\alpha_j^i + \alpha_k^i \wedge \alpha_j^k = d\alpha_i^j + \alpha_i^k \wedge \alpha_k^j$$

This is compatible with previous notation.

For $\nabla = \nabla_{LC}$, write $(F_\nabla)^i_j = \frac{1}{2} R^i_{jkl} e^k \wedge e^l$
 $R^i_{jkl} = -R^i_{jlk}$ $R^i_{jkl} = -R^i_{kjl}$ $R^i_{jkl}: U \rightarrow \mathbb{R}$

Since ∇_{LC} is torsion free $d_\nabla w = dw$ for 1-form w
 where d_∇ denote $\wedge(\nabla w)$ and $\wedge$ is the antisymmetrization
 We can generalize $\wedge$ for k -tensor s.t.

$$\wedge(dx^1 \otimes \dots \otimes dx^n) = dx^1 \wedge \dots \wedge dx^n$$

Then we can check

$$\wedge d_\nabla(w_1 \wedge w_2) = \wedge d_\nabla w_1 \wedge w_2 + (-1)^{\deg w_1} w_1 \wedge \wedge d_\nabla w_2$$

Note that $d(w_1 \wedge w_2) = dw_1 \wedge w_2 + (-1)^{\deg w_1} w_1 \wedge dw_2$

So $\wedge d_\nabla w = dw$ for any form

We have $F_\nabla e^i = d_\nabla^2 e^i = e^j \otimes (F_\nabla)^i_j$

From $\wedge(F_\nabla e^i) = \wedge d_\nabla^2 e^i = (\wedge d_\nabla)^2 e^i = d^2 e^i = 0$

We have $\wedge(e^j \otimes (F_\nabla)^i_j) = e^j \wedge (F_\nabla)^i_j = 0$

This means $R^i_{jkl} e^j \wedge e^k \wedge e^l = 0$

So $R^i_{jkl} + R^i_{klj} + R^i_{ljk} = 0$ (1)

Change indices we have

$R^j_{kli} + R^j_{lik} + R^j_{ikl} = 0$ (2) $R^k_{ijl} + R^k_{jli} + R^k_{lji} = 0$ (3)

$R^l_{ijk} + R^l_{jki} + R^l_{kij} = 0$ (4) Since $R^i_{klj} = -R^k_{lij} = R^k_{ijl}$

(1) - (2) - (3) + (4) $\Rightarrow R^i_{jkl} = R^k_{lij}$

Class 19. Bianchi identity and Chern class.

Last time, we compute the curvature for ∇_L

Suppose $e^1, \dots, e^n$ are orthonormal basis of T^*M (locally)

$$\text{Write } F_{\nabla} e^i = d_{\nabla}^2 e^i = e^j \otimes (F_{\nabla})^i_j$$

$$(F_{\nabla})^i_j = \frac{1}{2} R^i_{jkl} e^k \wedge e^l \quad R^i_{jkl}: U \rightarrow \mathbb{R}$$

We prove the following identities

$$\text{Prop. 1) } R^i_{ikl} = -R^i_{jkl} \quad (\text{from metric compatible condition})$$

$$2) R^i_{jlk} = -R^i_{jkl} \quad (\text{from definition})$$

$$3) R^i_{jkl} + R^i_{klj} + R^i_{ljk} = 0 \quad (\text{from torsion free condition})$$

$$4) R^i_{jkl} = R^k_{lij} \quad (\text{from 1) - 3})$$

$$\text{Today we first prove 5) } \nabla_m R^i_{jkl} + \nabla_k R^i_{jlm} + \nabla_l R^i_{jmk} = 0$$

This is equivalent to $d_{\nabla} F_{\nabla} = 0$ (we omit the pf of equivalence)

$$\text{Recall } F_{\nabla} = d\alpha + \alpha \wedge \alpha$$

$$\text{This means } F_{\nabla} S = d_{\nabla}^2 S = (p, (d\alpha_U + \alpha_U \wedge \alpha_U) \cdot S_U)$$

$$d_{\nabla}(d_{\nabla}^2 S) = (p, d((d\alpha_U + \alpha_U \wedge \alpha_U) \cdot S_U) + \alpha_U \wedge ((d\alpha_U + \alpha_U \wedge \alpha_U) S_U))$$

$$\hookrightarrow d(d\alpha_U + \alpha_U \wedge \alpha_U) S_U + (d\alpha_U + \alpha_U \wedge \alpha_U) dS_U$$

$$+ \alpha_U \wedge (d\alpha_U + \alpha_U \wedge \alpha_U) S_U \quad \text{Note } d_{\nabla} S_U = (d + \alpha_U) S_U$$

$$\begin{aligned}
& \downarrow \\
& = \left(d(d\alpha_u + \alpha_u \wedge \alpha_u) + \alpha_u \wedge (d\alpha_u + \alpha_u \wedge \alpha_u) \right. \\
& \quad \left. - \alpha_u \wedge (d\alpha_u + \alpha_u \wedge \alpha_u) \right) \wedge S_u + (d\alpha_u + \alpha_u \wedge \alpha_u) \wedge d_{\nabla} S_u \\
& = \left(d^2 \alpha_u + d(\alpha_u \wedge \alpha_u) + \cancel{\alpha_u \wedge d\alpha_u} + \alpha_u \wedge \alpha_u \wedge \alpha_u \right. \\
& \quad \left. - \cancel{\alpha_u \wedge d\alpha_u} - \alpha_u \wedge \alpha_u \wedge \alpha_u \right) \wedge S_u \\
& \quad + (d\alpha_u + \alpha_u \wedge \alpha_u) \wedge d_{\nabla} S_u
\end{aligned}$$

Then we have $d_{\nabla}(F_{\nabla} S) = F_{\nabla} \wedge d_{\nabla} S_u$

Note $d_{\nabla}(F_{\nabla} S) = d_{\nabla} F_{\nabla} + (-1)^{\deg F_{\nabla}} F_{\nabla} \wedge d_{\nabla} S_u$

So $d_{\nabla} F_{\nabla} = 0$ (Note $d_{\nabla}(F_{\nabla} S) \neq d_{\nabla}(F_{\nabla} S)$)

From the definition of R_{ijkl}^i ,

we have $F_{\nabla} = \frac{1}{2} R_{ijkl}^i e_i \otimes e^j \otimes (e^k \wedge e^l)$

Since e_i are orthonormal basis, we write

$R_{ijkl} = R_{ijkl}^i$ (In $\frac{\partial}{\partial x^i}$ basis, $R_{ijkl} = g_{im} R_{ijkl}^i$)

Define the Riemannian tensor

$$Riem = \frac{1}{2} R_{ijkl} e^i \otimes e^j \otimes (e^k \wedge e^l)$$

$$= \frac{1}{4} R_{ijkl} (e^i \wedge e^j) \otimes (e^k \wedge e^l)$$

$$\nabla Riem = \frac{1}{4} \nabla_m R_{ijkl} (e^i \wedge e^j) \otimes (e^k \wedge e^l) \otimes e^m$$

Since $\nabla e^i = \alpha_j^i e^j$, we have

$$\nabla_m R_{ijkl} = \partial_m R_{ijkl} + \alpha^{in} R_{njkl} + \alpha^{jn} R_{inlk} + \alpha^{kn} R_{ijnl} + \alpha^{ln} R_{ijkn}$$

Define the Ricci tensor by $Ric = Ric_{ik} e^i \otimes e^k$

$$Ric_{ik} = \sum_j R_{ijjk}$$

Define the scalar curvature $R: M \rightarrow \mathbb{R}$

by trace of Ric :

$$R = \sum_i Ric_{ii} = \sum R_{ijij}$$

$(Ric_{ik} - \frac{1}{2} R \delta_{ik}) e^i \otimes e^k$ is called

the Einstein tensor. (However, the Einstein equation in physics doesn't use Riemannian metric i.e. not positive definite)

Chern classes.

For a complex vector bundle E with $E|_p \cong \mathbb{C}^n$ we can also define covariant derivative ∇ and the curvature F_∇

Define the k -th Chern class to be

$$C_k(E) = \frac{1}{(2\pi\sqrt{-1})^k} \text{tr}(\underbrace{F_\nabla \wedge \dots \wedge F_\nabla}_{k \text{ times}})$$

We define the trace. First, choose a basis

$e_1, \dots, e_n$ for E , and dual basis $e^1, \dots, e^n$

For any section S of $\overset{\text{End}(E)}{= E \otimes E^*}$, we can write $S = S_i^j e_i \otimes e^j$, define $\text{tr}(S) = S_i^i$

It is independent of basis.

If $\omega \in C^\infty(\text{End}(E) \otimes \wedge^k T^*M)$, we can check $d(\text{tr} \omega) = \text{tr}(d_\nabla \omega)$

If $\omega_i \in C^\infty(\text{End}(E) \otimes \wedge^{k_i} T^*M)$

define $\omega_1 \wedge \omega_2 \in C^\infty(\text{End}(E) \otimes \wedge^{k_1+k_2} T^*M)$ by $(\omega_1 \wedge \omega_2)e_1 = \omega_1 \wedge (\omega_2 e)$

Then $\text{tr}(w_1 \wedge w_2) \in C^\infty(\Lambda^{k_1+k_2} T^*M)$

Note $F_\nabla \in C^\infty(\text{End}(E) \otimes \Lambda^2 T^*M)$

$$\text{tr}(\underbrace{F_\nabla \wedge \dots \wedge F_\nabla}_k) \in C^\infty(\Lambda^{2k} T^*M)$$

$$d(\text{tr}(\underbrace{F_\nabla \wedge \dots \wedge F_\nabla}_k)) = k \text{tr}(d_\nabla F_\nabla \wedge \dots \wedge F_\nabla) = 0 \quad (\text{Bianchi identity})$$

Thm. the de Rham cohomology class

$$[\text{tr}(\underbrace{F_\nabla \wedge \dots \wedge F_\nabla}_k)] \text{ does not depend on}$$

the choice of ∇

pf. Recall any two covariant derivatives ∇, ∇' are differed by $\alpha \in C^\infty(\text{End}(E) \otimes T^*M)$

Consider $\nabla^t = \nabla + t\alpha$

$$F_{\nabla^t} = F_\nabla + t d_\nabla \alpha + t^2 \alpha \wedge \alpha$$

$$d_{\nabla^t} t\alpha = d_\nabla t\alpha + (t\alpha \wedge \alpha + \alpha \wedge t\alpha)$$

$$\frac{\partial}{\partial t} F_{\nabla^t} = d_{\nabla^t} t\alpha$$

$$\frac{\partial}{\partial t} \text{tr}(\bar{F}_\nabla^t \wedge \dots \wedge F_\nabla^t)$$

$$= k \text{tr} (d\nabla^t \alpha \wedge \bar{F}_\nabla^t \wedge \dots \wedge F_\nabla^t)$$

$$= k d \text{tr}(\alpha \wedge \bar{F}_\nabla^t \wedge \dots \wedge F_\nabla^t) \quad \left(\begin{array}{l} \text{Since} \\ d\nabla^t F_\nabla^t = 0 \end{array} \right)$$

$$\text{Hence } C_k(\bar{F}_{\nabla'}) - C_k(\bar{F}_\nabla)$$

$$= d \int_0^1 \frac{k}{(2\pi i)^k} \text{tr}(\alpha \wedge \bar{F}_\nabla^t \wedge \dots \wedge F_\nabla^t)$$