

(Skip all materials about principal bundles
and come back to Chap 10 later)
Class 12 de Rham cohomology (Chap 12)

Basic idea of homology/cohomology:

Given a graded abelian group $C = \bigoplus_i C_i$ and an endomorphism

$$d: C \rightarrow C \text{ s.t. } d(C_i) \subset C_{i+1} \text{ (for cohomology)} \\ \subset C_{i-1} \text{ (for homology)}$$

$$\text{and } d^2 = 0$$

$$\text{Then } \text{Im} d = \{ c \in C \mid \exists a \in C, da = c \}$$

$$\subset \ker d = \{ c \in C \mid dc = 0 \}$$

$$H^i(C, d) = \ker(d: C_i \rightarrow C_{i+1}) / \text{Im}(d: C_{i-1} \rightarrow C_i)$$

(Notation $H_i(C, d)$ for homology)

$$H^*(C, d) = \bigoplus_i H^i(C, d) \quad H_*(C, d) = \bigoplus_i H_i(C, d) \\ \text{or } H^*(C, d)$$

Def. (C, d) is called a (co)chain complex. $H^*(C, d)$ is called the (co)homology group of (C, d)

elements in $\text{Im} d$ are called (co)boundaries

elements in $\ker d$ are called (co)cycles

For an element $c \in \ker d$, the corresponding element in H^* is denoted by $[c]$

Rem

① The chain complex may not be an invariant (of manifold or bundle),
but the homology and its rank are invariants (up to isomorphisms)

② When taking $\ker d / \text{Im} d$, we may lose some information in
 (C, d) , but that makes H^* simpler to study.

③ Sometimes, we don't need the grady, just start with (C, d)

with $d^2=0$ and take $H^*(C, d) = \ker d / \text{Im} d$.

If C is a vector space, d can be understood as a matrix

Then let's define the de Rham cohomology

Let M be a smooth manifold of dim n

For $k=0, 1, \dots, n$, let $\Omega^k = \Omega^k(M)$ be the space of

Sections $\omega: M \rightarrow \Lambda^k T^*M$

(At each point $p \in M$, we know ω_p is an antisymmetric multilinear map

$$\omega_p: \underbrace{T_p M \times T_p M \times \dots \times T_p M}_k \rightarrow \mathbb{R}$$

Antisym means $\omega_p(\dots, v_i, \dots, v_j, \dots) = -\omega_p(\dots, v_j, \dots, v_i, \dots)$

$$\dim(\Lambda^k T^*(M))|_p = \binom{n}{k}$$

Since sections are called k -form, Ω^k is the space of k -forms on M . In particular $\Omega^0 = C^\infty(M, \mathbb{R})$ is the space of smooth functions on M ($\Lambda^0 T^*(M) = \mathbb{R}$)

Previously, for a smooth function (a 0-form) $f: M \rightarrow \mathbb{R}$

we construct a 1-form df by local chart $df = \sum_{i=1}^n \frac{\partial f}{\partial x^i} dx^i$

d is a map: $\Omega^0 \rightarrow \Omega^1$

Goal: extend d to $\Omega^k \rightarrow \Omega^{k+1}$ s.t. $d^2=0$

This map is called the exterior derivative

($\Lambda^k V$ is called the exterior algebra for V)

Fact: Locally, any k -form can be written as

$$\sum_{i_1 < i_2 < \dots < i_k} f_{i_1 \dots i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k} \quad f_{i_1 \dots i_k}: U \rightarrow \mathbb{R}$$

($dx^{i_1} \wedge \dots \wedge dx^{i_k}$ are basis of sections on $\Lambda^k \mathbb{R}^n$ and we use $\phi_U: U \rightarrow \mathbb{R}$ to pull-back sections)

Define $d(\checkmark) = \sum_{i_1 < \dots < i_k} \sum_{m \notin \{i_1, \dots, i_k\}} \frac{\partial f_{i_1 \dots i_k}}{\partial x^m} dx^m \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k}$

$$= \sum_{i_1 < \dots < i_k < i_{k+1}} \left(\frac{\partial f_{i_2 i_3 \dots i_{k+1}}}{\partial x^{i_1}} - \frac{\partial f_{i_1 i_3 \dots i_{k+1}}}{\partial x^{i_2}} + \dots \right) dx^{i_1} \wedge \dots \wedge dx^{i_{k+1}}$$

Ex $w = y dx$ on $\mathbb{R}^2$
 $dw = dy \wedge dx$

↑
 sign comes from anti-sym.

This generalizes $df = \sum_m \frac{\partial f}{\partial x^m} dx^m$ (Need to check it is independent of charts)

$$d(df) = d\left(\sum_m \frac{\partial f}{\partial x^m} dx^m\right) = \sum_{m, l} \frac{\partial^2 f}{\partial x^m \partial x^l} dx^m \wedge dx^l$$

Since $\frac{\partial^2 f}{\partial x^m \partial x^l} = \frac{\partial^2 f}{\partial x^l \partial x^m}$ and $dx^m \wedge dx^l = -dx^l \wedge dx^m$

$d(df) = 0$ In general, we can check $d^2 = 0: \Omega^k \rightarrow \Omega^{k+2}$

Prop 1) $d(w_1 + w_2) = dw_1 + dw_2$

2) w_1, w_2 are k, l -forms

$$d(w_1 \wedge w_2) = dw_1 \wedge w_2 + (-1)^k w_1 \wedge dw_2$$

Def. w is closed if $dw = 0$ (cocycle)

w is exact if $w = d\alpha$ for some α . (coboundary)

The de Rham cohomology $H_{dR}^*(M) = \ker d / \text{Im} d$

It is a vector space of $\mathbb{R}$.

(If you know singular cohomology $H^*(M)$ for a topological space M , then we have $H_{dR}^*(M) \cong H^*(M; \mathbb{R})$

↑
in $\mathbb{R}$ coefficient

In general $H^*(M)$ can be defined over $\mathbb{Z}$.

$$\text{e.g. } H^i(\mathbb{R}P^3; \mathbb{Z}) = \begin{cases} \mathbb{Z} & i=0 \\ 0 & i=1 \\ \mathbb{Z}/2 & i=2 \\ \mathbb{Z} & i=3 \end{cases}$$

$$H_{dR}^i(\mathbb{R}P^3) = \begin{cases} \mathbb{R} & i=0 \\ 0 & i=1 \\ 0 & i=2 \\ \mathbb{R} & i=3 \end{cases}$$

If $\psi: M \rightarrow N$ is a smooth map. ψ induces a map

$$\psi^*: \Omega^k(N) \rightarrow \Omega^k(M) \text{ by}$$

$$\psi^* \omega(v_1, \dots, v_n) = \omega(\psi_* v_1, \dots, \psi_* v_n) \quad v_i \in T_p M \quad \psi_* \text{ tangent map}$$

which is well defined on $H_{dR}^k(N) \rightarrow H_{dR}^k(M)$ because

$$1) \text{ For } v \in T_p M \quad df(v) = v(f) := \mathcal{L}_v f$$

$$\text{because } df = \frac{\partial f}{\partial x^i} dx^i \quad v = v^i \frac{\partial}{\partial x^i} \quad \mathcal{L}_v f = v^i \frac{\partial f}{\partial x^i}$$

$$2) \psi^* df(v) = df(\psi_* v) = (\psi_* v)(f) = v(\psi^* f) = d(\psi^* f)(v) \\ \Rightarrow \psi^* df = d \psi^* f$$

$$3) d(\psi^*(dx^1 \wedge \dots \wedge dx^n)) = d(\psi^*(dx^1) \wedge \dots \wedge \psi^*(dx^n))$$

$$(0 = \psi^* d(dx^i) = d(\psi^* dx^i)) \rightarrow 0$$

$$d(w_1 + w_2) = dw_1 + dw_2 \quad d(w_1 \wedge w_2) = dw_1 \wedge w_2 + (-1)^k w_1 \wedge dw_2$$

$$4) \omega = \sum_{i_1 < i_2 < \dots < i_k} f_{i_1 \dots i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k}$$

$$d\omega = \sum_{i_1 < \dots < i_k} \sum_{m \notin \{i_1, \dots, i_k\}} \frac{\partial f_{i_1 \dots i_k}}{\partial x^m} dx^m \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k}$$

$$\psi^* \omega = \sum (\psi^* f) \psi^* (dx^{i_1} \wedge \dots \wedge dx^{i_k})$$

$$\psi^* d\omega = \sum \psi^* (df \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k})$$

$$= \sum \psi^* (df) \wedge \psi^* (dx^{i_1} \wedge \dots \wedge dx^{i_k})$$

$$= \sum d(\psi^* f) \wedge \psi^* (dx^{i_1} \wedge \dots \wedge dx^{i_k})$$

$$(d\psi^*(dx^{i_1} \wedge \dots \wedge dx^{i_k}) = 0)$$

$$= d(\sum (\psi^* f) \wedge \psi^* (dx^{i_1} \wedge \dots \wedge dx^{i_k}))$$

$$= d\psi^* \omega$$

$$\Rightarrow \psi^* d = d\psi^*$$

$$5) \text{ If } d\omega = 0, \text{ then } d(\psi^* \omega) = \psi^* d\omega = 0$$

$$\text{If } \omega_1 - \omega_2 = d\omega_0 \text{ then } \psi^*(\omega_1) - \psi^*(\omega_2) = \psi^* d\omega_0 = d(\psi^* \omega_0)$$

$$\Rightarrow \psi^* \text{ is well-defined on } \ker d / \text{Im } d$$

Prop If ψ is a diffeomorphism, ψ^* induces an iso between $H_{dR}^*(N)$

and $H_{dR}^*(M) \Rightarrow \text{If } H_{dR}^*(N) \neq H_{dR}^*(M), \text{ then } M \not\cong_{\text{diff}} N$

If ψ and ϕ are homotopic maps to N , i.e.

$$\exists \Phi : [0, 1] \times M \rightarrow N \text{ smooth, s.t. } \Phi(0, \cdot) = \phi$$

$$\text{and } \Phi(1, \cdot) = \psi, \text{ then } \psi^* = \phi^* : H_{dR}^*(N) \rightarrow H_{dR}^*(M)$$

Class 13 Exterior derivative and Lie derivative

Recall Ω^k is the space of k -form on M .

i.e. sections $M \rightarrow \Lambda^k T^*M$,

$$\Omega^0 = C^\infty(M; \mathbb{R})$$

Locally $w = \sum_{i_1 < i_2 < \dots < i_k} f_{i_1 \dots i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k}$

$$dw = \sum_{i_1 < \dots < i_k} \sum_{m \notin \{i_1, \dots, i_k\}} \frac{\partial f_{i_1 \dots i_k}}{\partial x^m} dx^m \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k}$$

$$= \sum_{i_1 < \dots < i_k} df \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k}$$

$d: \Omega^k \rightarrow \Omega^{k+1}$ is independent of charts

$$d^2 = 0. \quad H_{dR}^i(M) = \frac{(\ker d: \Omega^i \rightarrow \Omega^{i+1})}{(\text{Im } d: \Omega^{i-1} \rightarrow \Omega^i)}$$

Prop 1) If $\psi: M \rightarrow N$ is a diffeomorphism. ψ^* induces an iso between $H_{dR}^*(N)$ and $H_{dR}^*(M) \Rightarrow$ If $H_{dR}^*(N) \neq H_{dR}^*(M)$, then $M \not\cong_{\text{diff}} N$

2) If ψ and ϕ are homotopic map to N , i.e.

$$\exists \Phi: [0,1] \times M \rightarrow N \text{ smooth, s.t. } \Phi(0, \cdot) = \phi$$

$$\text{and } \Phi(1, \cdot) = \psi, \text{ then } \psi^* = \phi^*: H_{dR}^*(N) \rightarrow H_{dR}^*(M)$$

Pf: 1). Since $\psi^* d = d \psi^*$,

(This is called a (co)chain map)

ψ^* induces a well-defined map

$$\psi^*: H_{dR}^*(N) \rightarrow H_{dR}^*(M)$$

If ψ is a diffeo, $\exists \phi: N \rightarrow M$

s.t. $\psi \circ \phi = \text{Id}$ $\phi \circ \psi = \text{Id}$. Then ψ^*, ϕ^*

induce isomorphism between $H_{dR}^*(N)$ $H_{dR}^*(M)$

$$\begin{aligned} 2) \text{ Note that } T_{(t,p)}^*([0,1] \times M) &= T_t^*[0,1] \times T_p^*M \\ &= \mathbb{R} \times T_p^*M \end{aligned}$$

a 1-form on $T^*([0,1] \times M)$ can be written as

$$\alpha_t dt + \beta_t \quad \begin{aligned} \alpha_t &\in \Omega^0(M) = C^\infty(M; \mathbb{R}) \\ \beta_t &\in \Omega^1(M) \end{aligned}$$

$\omega \in \Omega^k(M)$ $\Phi^* \omega \in \Omega^k([0,1] \times M)$ can be written as

$$\Phi^* \omega = dt \wedge \alpha_t + \beta_t \quad \alpha_t \in \Omega^{k-1}(M) \quad \beta_t \in \Omega^k(M)$$

If $d\omega = 0$, then $0 = d(\Phi^* \omega)$

$$= -dt \wedge d^\perp \alpha_t + dt \wedge \frac{\partial}{\partial t} \beta_t + d^\perp \beta_t$$

where $d^\perp$ is the exterior derivative on $\{t\} \times M \subset [0,1] \times M$
 $d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^k \omega_1 \wedge d\omega_2$

This implies $\frac{\partial}{\partial t} \beta_t = d^\perp \alpha_t$, $d^\perp \beta_t = 0$

$d^\perp \beta_t = 0$ means $\Phi(t, \cdot)^* \omega$ is closed for any t

$$\psi^* \omega - \phi^* \omega = \int_0^1 \frac{\partial}{\partial t} \beta_t = \int_0^1 d^\perp \alpha_t = d \int_0^1 \alpha_t$$

So $\psi^* \omega - \phi^* \omega \in \text{Im } d$ $[\psi^* \omega] = [\phi^* \omega]$ $\square$

Cor. (Poincaré Len) Let $U \subset M$ be a contractible open set, i.e. $\exists$ smooth map $\Phi: [0,1] \times U \rightarrow M$

s.t. $\Phi(1, \cdot)$ is the inclusion

$\Phi(0, \cdot)$ maps U to a pt $p \in M$

Then $H_{dR}^*(U) = 0$

i.e. Any closed k -form ω ($d\omega = 0$) with $k \geq 1$

$\exists$ a $(k-1)$ -form α s.t. $d\alpha = \omega$

$$\text{Pf: } [\Phi(1, \cdot)^* \omega] = [\Phi(0, \cdot)^* \omega]$$

$$\parallel$$
$$\omega|_U$$

$$\parallel$$
$$\omega|_p$$

$$\Lambda^k T_p^* = 0 \text{ for } k \geq 1 \text{ so } \omega|_p = 0 \quad \square$$

Lie derivative on k -forms.

Suppose M is compact and V is a vector field

From vector field thm, we have a smooth map

$$G: (-\varepsilon, \varepsilon) \times M \rightarrow M \text{ s.t.}$$

$$1) G(0, p) = p$$

$$2) G_* \frac{\partial}{\partial t} = V$$

(We use the compactness to find $\varepsilon > 0$ for any pt $p \in M$)

For $\omega \in \Omega^k(M)$, consider $\phi^*\omega \in \Omega^k((- \varepsilon, \varepsilon) \times M)$

$$\phi^*\omega = dt \wedge \alpha_t + \beta_t \quad \alpha_t \in \Omega^{k-1}(M) \quad \beta_t \in \Omega^k(M)$$

Define $L_V \omega = \left. \frac{\partial}{\partial t} \beta_t \right|_{t=0}$

Fact (Cartan formula) $L_V \omega = (L_V d + d L_V) \omega$

where for a k form α , $L_V \alpha$ is a $k-1$ form defined by

$$L_V \alpha(V_1, \dots, V_{k-1}) = \alpha(V, V_1, \dots, V_{k-1})$$

$$\text{So } L_V \omega(V_1, \dots, V_k) = d\omega(V, V_1, \dots, V_k)$$

$\uparrow$
 k -form

$\uparrow$
 $(k+1)$ -form

$$+ d(\omega(V, V_1, \dots, V_k))$$

Note that $\alpha_0 = L_V \omega$

because $\phi^*\omega\left(\frac{\partial}{\partial t}, V_1, \dots, V_k\right)\Big|_{t=0}$

$$= \omega\left(\phi_* \frac{\partial}{\partial t}, \phi_* V_1, \dots, \phi_* V_k\right)\Big|_{t=0}$$

$$= \omega(V, \phi_* V_1, \dots, \phi_* V_k)$$

$$= dt \wedge \alpha_t\left(\frac{\partial}{\partial t}, V_1, \dots, V_k\right)\Big|_{t=0} + \cancel{\beta_t\left(\frac{\partial}{\partial t}, V_1, \dots, V_k\right)\Big|_{t=0}}$$

$$= \alpha_0(V_1, \dots, V_k)$$

Recall that we define L_V as a derivation

$$C^\infty(M; \mathbb{R}) \rightarrow C^\infty(M; \mathbb{R})$$

$$\overset{\parallel}{\Omega^0}$$

$$\overset{\parallel}{\Omega^0}$$

The definition here $L_V: \Omega^k \rightarrow \Omega^k$ can be regarded as an extension of derivation on $\Omega^* = \bigoplus_{k=0}^n \Omega^k$

It satisfies the Leibniz rule

$$L_V(\alpha \wedge \beta) = L_V \alpha \wedge \beta + \alpha \wedge L_V \beta$$

Ω^* is called the exterior algebra by wedge product $\wedge$

It is an example of superalgebra, which is a vector

space with $\mathbb{Z}/2 = \{0, 1\}$ grading and $\alpha \beta = (-1)^{\deg \alpha \deg \beta} \beta \alpha$ where $\deg \alpha, \deg \beta \in \{0, 1\}$ for homogeneous α, β .

Note that from $dx^i \wedge dx^j = -dx^j \wedge dx^i$,

We can show for α k -form, β l -form.

$$\alpha \wedge \beta = (-1)^{kl} \beta \wedge \alpha$$

$$\Omega^* = \underbrace{\bigoplus_{k \text{ even}} \Omega^k}_{\text{grading 0}} \oplus \underbrace{\bigoplus_{k \text{ odd}} \Omega^k}_{\text{grading 1}}$$

Note that the classes on

10/20 (Thu), 10/25 (Tue), and 10/27 (Thu)
will be online due to the travel.

The zoom link is _____

It will also be sent by email

The notes will be posted before the classes

Topics

10/20 Class 15 Curvature of a covariant derivative
(Chap 12.4-7)

10/25 Class 16 Metric compatible covariant derivative
(Chap 15.1-2)

10/27 Class 17 Levi-Civita connection
(Chap 15.3-4)

There are many computations. It is good to do it
by yourself

Class 14 Covariant derivative (Chap 11)

Let $\pi: E \rightarrow M$ be a vector bundle $s: M \rightarrow E$ a section
we want to find some derivative of s

Ex. If $E = M \times \mathbb{R}^n$, the section $s: M \rightarrow E$ is just

$$S(p) = (p, s_1(p), \dots, s_n(p)) \quad s_i: M \rightarrow \mathbb{R}$$

We can define $(s_i)_* : T_p M \rightarrow T_{s_i(p)} \mathbb{R} \cong \mathbb{R}$

$(s_i)_*$ can be regarded as a section of T^*M .

Hence we can construct $ds(p) := (p, (s_1)_*(p), \dots, (s_n)_*(p))$

as a section of $E \otimes T^*M = T^*M^{\otimes n}$

Then we extend the construction to general vector bundle

Def Let $C^\infty(M; E)$ be the space of sections $M \rightarrow E$.

A covariant derivative is a map

$$\nabla: C^\infty(M; E) \rightarrow C^\infty(M; E \otimes T^*M) \text{ obeying the}$$

$$\text{Leibniz rule } \nabla(fs) = s \otimes df + f \nabla s$$

for any $f \in C^\infty(M; \mathbb{R})$, $s \in C^\infty(M; E)$

Rem. The extra factor T^*M means, if we specify a

vector field V , we can define $\nabla_V: C^\infty(M; E) \rightarrow C^\infty(M; E)$

Construction of ∇

Take a locally finite open cover $\mathcal{U}$ of M and a partition

$$\text{of unity } \{\chi_\alpha: U_\alpha \rightarrow \mathbb{R}_{\geq 0}\} \quad \varphi_\alpha: E|_{U_\alpha} \rightarrow U_\alpha \times \mathbb{R}^n$$

$$\text{Set } \nabla S = \sum_{\alpha \in \mathcal{U}} \chi_{\alpha} \varphi_{\alpha}^{-1} (d\varphi_{\alpha}(S))$$

$$\text{where } \varphi_{\alpha}(S) : U_{\alpha} \rightarrow U_{\alpha} \times \mathbb{R}^n$$

$$\varphi_{\alpha}(S)(p) = (p, S_{\alpha 1}(p), \dots, S_{\alpha n}(p))$$

$$\text{and } d\varphi_{\alpha}(S) : U_{\alpha} \rightarrow U_{\alpha} \times (\mathbb{R}^n \otimes \mathbb{R}^m) \quad \begin{matrix} m = \dim M \\ n = \dim E|_p \end{matrix}$$

$$d\varphi_{\alpha}(S)(p) = (p, (S_{\alpha 1})_{*}(p), \dots, (S_{\alpha n})_{*}(p))$$

$$\varphi_{\alpha}^{-1} \text{ extends to } U_{\alpha} \times (\mathbb{R}^n \otimes \mathbb{R}^m) \rightarrow E \otimes T^{*}M|_U$$

$$\text{To check } \nabla fS = S \odot df + f \nabla S$$

it suffices to check over U_{α}

$$d\varphi_{\alpha}(fS)(p) = (p, (fS_{\alpha 1})_{*}(p), \dots)$$

$$S_{\alpha 1} : U_{\alpha} \rightarrow \mathbb{R} \quad (S_{\alpha 1})_{*}|_p : T_p U_{\alpha} \rightarrow T_{S_{\alpha 1}(p)} \mathbb{R}$$

$$f : U_{\alpha} \rightarrow \mathbb{R}$$

$$(fS_{\alpha 1})_{*} = df \otimes S_{\alpha 1} + f \otimes dS_{\alpha 1}$$

$$\text{as section } U_{\alpha} \rightarrow T^{*}U_{\alpha}$$

Another construction of ∇

Consider E as a subbundle of $M \times \mathbb{R}^N$

and write $S : M \rightarrow E$ as a section of $S : M \rightarrow M \times \mathbb{R}^N$

($E|_p$ is a linear subspace of $p \times \mathbb{R}^N$)

Let $\nabla S = \Pi_E ds$, where ds is a section of

$M \times \mathbb{R}^N \otimes T^*M$ and Π_E is the orthogonal projection to $E \otimes T^*M$ by the standard inner product in $\mathbb{R}^N$

$$\begin{aligned} \text{Since } \Pi_E S &= S, \quad \nabla f s = \Pi_E d(f s) \\ &= \Pi_E (f \otimes ds + df \otimes s) \\ &= S \otimes df + f \nabla S \end{aligned}$$

Rem. The covariant derivative is not unique.

Suppose ∇ and ∇' are two covariant derivatives

$$(\nabla - \nabla')(f s) = f(\nabla - \nabla')s$$

Claim $\nabla - \nabla'$ is a section of $\text{End}(E) \otimes T^*M$
 $:= \text{Hom}(E, E) \otimes T^*M$.

Lem. Let E, E' be two vector bundles over M

If α is a map from $C^\infty(M; E) \rightarrow C^\infty(M; E')$ that is linear over $C^\infty(M; \mathbb{R})$, i.e. $f\alpha = \alpha f$, then α is a section of $\text{Hom}(E, E')$

p.f. Locally, fix a basis $\{e_1, \dots, e_n\}$ for E and a basis $\{e'_1, \dots, e'_n\}$ for E'

Then we can write $\alpha e_j = \sum_i A_{ij} e'_i$

If s is a section of E $s = \sum_j s_j e_j$, $s_j: U \rightarrow \mathbb{R}$

$$\alpha S = \alpha \left(\sum_j s_j e_j \right) = \sum_j s_j \alpha(e_j) = \sum_{ij} s_j A_{ij} e_i$$

Then α is defined by $\{A_{ij}\}$ as a section of $\text{Hom}(E, E')$, change basis of E or E' will change

$\{A_{ij}\}$ by matrix multiplication. $\square$

we use α rather than $\frac{\alpha}{\alpha}$ ^{in latex} just for writing simplicity.

Conversely, given $\alpha \in C^\infty(M; \text{End}(E) \otimes T^*M)$,

we can construct $\nabla' = \nabla + \alpha$. This is still a covariant derivative

Rem The space of covariant derivatives is affine over $C^\infty(M; \text{End}(E) \otimes T^*M)$. This means these two spaces are isomorphic, but the isomorphism is canonical only when we pick an element ∇

Then we consider the transition of ∇ in different charts.

Locally, $\phi_U: U \rightarrow \mathbb{R}^n$, $\psi_U: E|_U \rightarrow U \times \mathbb{R}^n$, $S: M \rightarrow E$
define the covariant derivative ∇^0 by

$$\nabla^0 \phi_U S(p) = (p, dS_U(p))$$

$$S_U = \phi_U^{-1} \circ S: \mathbb{R}^n \rightarrow \mathbb{R}^n$$

For a general covariant derivative ∇ , we know

$$\varphi_U(\nabla S)(p) = (p, ds_U + \alpha_U S_U)$$

where α_U is a section of $(\text{End}(E) \otimes T^*M)|_U$

For another chart V with $U \cap V \neq \emptyset$

$$\text{we have } \varphi_V(\nabla S)(p) = (p, ds_V + \alpha_V S_V)$$

$S_V = g_{VU} S_U$ for bundle transition function

$$g_{UV}: V \cap U \rightarrow GL(n, \mathbb{R})$$

$$ds_V + \alpha_V S_V = d(g_{VU} S_U) + \alpha_V (g_{VU} S_U)$$

$$= g_{VU} ds_U + dg_{VU} S_U + \alpha_V g_{VU} S_U$$

$$= g_{VU} (ds_U + (g_{VU}^{-1} \alpha_V g_{VU} + g_{VU}^{-1} dg_{VU}) S_U)$$

Since ∇S is a section of $E \otimes T^*M$.

this should equal to $g_{VU} (ds_U + \alpha_U S_U)$

$$\text{Thus, } ds_U + (g_{VU}^{-1} \alpha_V g_{VU} + g_{VU}^{-1} dg_{VU}) S_U$$

$$= ds_U + \alpha_U S_U$$

$$\Rightarrow \alpha_U = g_{VU}^{-1} \alpha_V g_{VU} + g_{VU}^{-1} dg_{VU}$$

$$\Rightarrow \alpha_V = g_{UV}^{-1} \alpha_U g_{UV} + g_{UV}^{-1} dg_{UV}$$

$$= g_{VU} \alpha_U g_{VU}^{-1} + g_{VU} dg_{VU}^{-1}$$

Class 15 Curvature of a covariant derivative (Chap 12)

Def Let $C^\infty(M; E)$ be the space of sections $M \rightarrow E$.

A covariant derivative is a map

$$\nabla: C^\infty(M; E) \rightarrow C^\infty(M; E \otimes T^*M) \text{ obeying the}$$

$$\text{Leibniz rule } \nabla(fs) = s \otimes df + f \nabla s$$

$$\text{for any } f \in C^\infty(M; \mathbb{R}), s \in C^\infty(M; E)$$

Prop. (last class)

1) Covariant derivative exists

2) ∇, ∇' are two covariant derivatives

iff $\nabla - \nabla' = \alpha$ is a section of $\text{End}(E) \otimes T^*M$

Prop (last week) $\Omega^k = C^\infty(M; \wedge^k T^*M)$

There exists map $d: \Omega^k \rightarrow \Omega^{k+1}$ s.t. $d^2 = 0$

$$d(w_1 + w_2) = dw_1 + dw_2$$

$$d(w_1 \wedge w_2) = dw_1 \wedge w_2 + (-1)^{\deg w_1} w_1 \wedge dw_2$$

Today we combine ∇ and d together to define

exterior covariant derivative

$$d_\nabla: C^\infty(M; E \otimes \wedge^k T^*M) \rightarrow C^\infty(M; E \otimes \wedge^{k+1} T^*M)$$

$$k=0 \quad \wedge^0 T^*M = M \times \mathbb{R} \quad d_\nabla s = \nabla s$$

$k>0$ If the section of $E \otimes \wedge^k T^*M$ can be

written as $s \otimes w$, then define

$$d_{\nabla} s \otimes w = \nabla s \wedge w + s \otimes dw$$

$$\text{Then define } d_{\nabla} \left(\sum_i s_i \otimes w_i \right) = \sum_i d_{\nabla} (s_i \otimes w_i)$$

In general, $d_{\nabla}^2 \neq 0$

For $s \in C^{\infty}(M; E)$, we write $d_{\nabla}^2 s = F_{\nabla} s$,
where F_{∇} is called the curvature

$$\begin{aligned} F_{\nabla}(fs) &= d_{\nabla}^2(fs) = d_{\nabla}(f d_{\nabla} s + s \otimes df) \\ &= \underbrace{df \wedge d_{\nabla} s + f d_{\nabla}^2 s + d_{\nabla} s \wedge df + s \otimes d^2 f}_{=0} \\ &= f d_{\nabla}^2 s = f F_{\nabla} s \end{aligned}$$

Thus, F_{∇} is a section of $\text{End}(E) \otimes \wedge^2 T^*M$
by Lem in the last class.

For $s \otimes w \in C^{\infty}(M; E \otimes \wedge^k T^*M)$

$$\begin{aligned} d_{\nabla}^2(s \otimes w) &= d_{\nabla}(d_{\nabla} s \wedge w + s \otimes dw) \\ &= d_{\nabla}^2 s \wedge w - \underbrace{d_{\nabla} s \wedge dw}_{\substack{\text{section tensor 1-form} \\ (d(\alpha \wedge \beta) = d\alpha \wedge \beta - \alpha \wedge d\beta \text{ for 1-form } \alpha)}} + d_{\nabla} s \wedge dw + s \otimes d^2 w \\ &= F_{\nabla} s \wedge w \end{aligned}$$

Local description of F_D

$$\phi_U: U \rightarrow \mathbb{R}^m \quad \varphi_U: E|_U \rightarrow U \times \mathbb{R}^n \quad \dim M = m \quad \dim E|_p = n.$$

$$S: M \rightarrow \bar{E} \quad S_U = \phi_U^{-1} \circ S: \mathbb{R}^n \rightarrow \mathbb{R}$$

$$\nabla^0 \varphi_U S(p) = (p, dS_U(p))$$

$$\varphi_U(\nabla S)(p) = (p, dS_U + \alpha_U S_U)$$

where α_U is a section of $(\text{End}(E) \otimes T^*M)|_U$

$$d_{\nabla}^2 S = d_{\nabla}(\nabla S) \quad \varphi_U(d_{\nabla}^2 S)(p)$$

$$= (p, \cancel{d^2 S_U} + (d\alpha_U)S_U - \alpha_U \wedge dS_U + \alpha_U \wedge (dS_U + \alpha_U S_U))$$

also have α_U part because we use ∇ rather than ∇^0 .

$$d(w_1 \wedge w_2) = dw_1 \wedge w_2 + (-1)^{\deg w_1} dw_2 \wedge w_1$$

$$= (p, (d\alpha_U + \alpha_U \wedge \alpha_U) \cdot S_U)$$

$$\text{Thus, locally } (F_D)_U = d\alpha_U + \alpha_U \wedge \alpha_U$$

This is a matrix-valued 2-form.

i.e. a section of $(\text{End}(E) \otimes \Lambda^2 T^*M)|_U$

We look more carefully on the notation $\alpha_U \wedge \alpha_U$

because α_U is a matrix-valued 1-form.

Write $dx^1, \dots, dx^n$ for basis of sections on $T^*M|_U$

$$\alpha_U = \sum_k \alpha_{Uk} dx^k \quad \alpha_{Uk} \text{ are matrices (depend on } p)$$

$$\alpha_U \wedge \alpha_U = \sum_{ij} \alpha_{Ui} \alpha_{Uj} dx^i \wedge dx^j$$

matrix multiplication

$$\left(\begin{matrix} -dx^i \wedge dx^j \\ = dx^j \wedge dx^i \end{matrix} \right) \Downarrow \sum_{i < j} (\alpha_{Ui} \alpha_{Uj} - \alpha_{Uj} \alpha_{Ui}) dx^i \wedge dx^j$$

write $[\alpha_{U_i}, \alpha_{U_j}]$ for $\alpha_{U_i} \alpha_{U_j} - \alpha_{U_j} \alpha_{U_i}$,
called the commutator

$$\text{Also, } d\alpha_U = \sum_{i < j} (\partial_i \alpha_{U_j} - \partial_j \alpha_{U_i}) dx^i \wedge dx^j$$

$$(F_U)_U = \sum_{i < j} (\underbrace{\partial_i \alpha_{U_j} - \partial_j \alpha_{U_i} + [\alpha_{U_i}, \alpha_{U_j}]}_{\text{write as } (F_U)_{ij}}) dx^i \wedge dx^j$$

Check the result is independent of charts

$$(F_U)_V = d\alpha_V + \alpha_V \wedge \alpha_V \quad \text{From the last class}$$

$$\begin{aligned} \alpha_V &= g_{UV}^{-1} \alpha_U g_{UV} + g_{UV}^{-1} d g_{UV} \\ &= g_{VU} \alpha_U g_{VU}^{-1} + g_{VU} d g_{VU}^{-1} \end{aligned}$$

$$\Rightarrow (F_U)_U = g_{UV}^{-1} (F_U)_V g_{UV}$$

$$\left(\begin{aligned} \uparrow \\ d\alpha_V &= -g_{UV}^{-1} d g_{UV} \wedge g_{UV}^{-1} d g_{UV} - g_{UV}^{-1} d g_{UV} \wedge g_{UV}^{-1} \alpha_U g_{UV} \\ &\quad + g_{UV}^{-1} d \alpha_U g_{UV} - g_{UV}^{-1} \alpha_U g_{UV} \wedge g_{UV}^{-1} d g_{UV} \\ \alpha_V \wedge \alpha_V &= g_{UV}^{-1} d g_{UV} \wedge g_{UV}^{-1} d g_{UV} + g_{UV}^{-1} \alpha_U g_{UV} \wedge g_{UV}^{-1} d g_{UV} \\ &\quad + g_{UV}^{-1} d g_{UV} \wedge g_{UV}^{-1} \alpha_U g_{UV} + g_{UV}^{-1} \alpha_U \wedge \alpha_U g_{UV} \end{aligned} \right)$$

Meaning of F_U (for simplicity on a chart)

write $dx^1, \dots, dx^n$ for basis of sections of $T^*M|_U$.

∇S_U can be written as $\sum_k \nabla_k S_U dx^k$

$\nabla_k S_U$ is the covariant derivative of S_U along $\frac{\partial}{\partial x^k}$.

For the usual partial derivative, we have $\frac{\partial^2 f}{\partial x^i \partial x^j} = \frac{\partial^2 f}{\partial x^j \partial x^i}$

but for covariant derivative.

$$\nabla_i \nabla_j S_a - \nabla_j \nabla_i S_a = (F_{\nabla ij})_a S_a$$

$$\text{where } (F_{\nabla})_a = \sum_{i,j} (F_{\nabla ij})_a dx^i \wedge dx^j$$

$$= \frac{1}{2} \sum_{i,j} (F_{\nabla ij})_a dx^i \wedge dx^j$$

$$\text{Explicitly. } \nabla_i S_a = \partial_i S_a + \alpha_i S_a$$

$$\nabla_i \nabla_j S_a = \partial_i \partial_j S_a + \partial_i \alpha_j S_a + \alpha_j \partial_i S_a + \alpha_i \alpha_j S_a$$

$$(\nabla_i \nabla_j - \nabla_j \nabla_i) S_a = \partial_i \alpha_j - \partial_j \alpha_i + \alpha_i \alpha_j - \alpha_j \alpha_i$$