

Class 20 Principal bundle (Chap 10)

Before introducing principal bundle, let's review materials

in previous lectures:

Part I: manifolds

topological manifold: M para, Haus topo space

s.t. each pt has nbhd homeo to $\mathbb{R}^n$

pt $\in U \subset M$ $\phi_U: U \xrightarrow{\cong} \mathbb{R}^n$ (Global view)

Construct by charts and transition functions (Local view)

Atlas $\mathcal{U}$, for $U, V \in \mathcal{U}$

$$h_{vu} = \phi_V \circ \phi_U^{-1}: \underbrace{\phi_U(U \cap V)}_{\cong \mathbb{R}^n} \xrightarrow{\cong} \underbrace{\phi_V(U \cap V)}_{\cong \mathbb{R}^n}$$

(They satisfy the cocycle condition $h_{vu} \circ h_{uv} = \text{Id}$)

$$h_{vu} \circ h_{uw} \circ h_{wv} = \text{Id}$$

$$M = \coprod_{U \in \mathcal{U}} \mathbb{R}^n_U / x \in \mathbb{R}^n_U \sim h_{vu}(x) \in \mathbb{R}^n_V$$

Smooth mfd: h_{vu} are smooth

(The smoothness only makes sense for maps btw $\mathbb{R}^n$)

Inverse / Implicit function thm: $\psi: \mathbb{R}^m \rightarrow \mathbb{R}^n$

If for any pt in $\psi^{-1}(a)$, the Jacobian $\left\{ \frac{\partial \psi^i}{\partial x^j} \right\}$

is surjective, then $\psi^{-1}(a)$ is a smooth mfd

Lie group: smooth manifold with group structure.

s.t. $(a, b) \mapsto ab$. $a \mapsto a^{-1}$ are smooth

$$\text{Ex: } GL(n, \mathbb{R}), SL(n, \mathbb{R}), \underbrace{O(n), SO(n)}_{\text{compact}}, \\ GL(n, \mathbb{C}), SL(n, \mathbb{C}), \underbrace{U(n), SU(n)}_{\text{compact}}$$

Part II. vector bundle

$$E \text{ smooth mfd} : \pi : E \rightarrow M$$

$$\textcircled{1} \forall p \in M \exists U \subset M \text{ containing } p$$

$$\text{s.t. } \varphi_U : E|_U = \pi^{-1}(U) \longrightarrow U \times \mathbb{R}^n \text{ diffeo}$$

$$\textcircled{2} \text{ zero section } \hat{0} : M \rightarrow E \quad \pi \circ \hat{0} = \text{Id}$$

$$\textcircled{3} \exists \text{ smooth map } \mu : \mathbb{R} \times E \rightarrow E \\ \text{or } \mathbb{C} \times E \rightarrow E$$

satisfying some conditions.

(Global view)

(locally finite)

Construct by charts and bundle transition functions

$$g_{UV} : U \cap V \longrightarrow GL(n, \mathbb{R}) \text{ with cocycle cond}$$

$$E = \coprod_{U \in \mathcal{U}} U \times \mathbb{R}^n / (p, u) \in U \times \mathbb{R}^n \sim (p, g_{UV} u) \in V \times \mathbb{R}^n$$

(Local view)

v.b. are generalization of product manifold $M \times \mathbb{R}^n$

it is also a mfd. but with a fiber str

tangent bundle: $g_{vu} = (h_{vu})_* \quad TM$

cotangent bundle: $g_{vu} = ((h_{vu})_*^{-1})^T \quad T^*M$
 $\dim M = \dim \text{fiber}$

Sections: $S: M \rightarrow E$

$C^\infty(M; E)$ or $C^\infty(E)$ space of sections

operation: $E^*, E \oplus E_2, E_1 \otimes E_2, \text{Hom}(E_1, E_2)$
 $\text{Sym}^k E \quad \wedge^k E$

$$\text{End}(E) = \text{Hom}(E, E) = E^* \otimes E$$

Def Principal bundle P over M with fiber G
 $\uparrow$ mfd. $\uparrow$ Lie gp

has the following structures

1) a smooth map $m: G \times P \rightarrow P$ with

① $m(e, p) = p$ e is identity in G

② $m(h, m(g, p)) = m(hg, p)$

usually write $m(g, p)$ by pg^{-1}

2) a smooth map $\pi: P \rightarrow M$ surjective with

$$\pi(pg^{-1}) = \pi(p) \quad \pi \text{ is called projection}$$

3) Any pt in M has nbhd U with

$$\varphi_U: P|_U = \pi^{-1}(U) \longrightarrow U \times G \text{ diffeo}$$

$$\text{s.t. if } \varphi_U(p) = (\pi(p), h(p))$$

$$\text{then } \varphi_U(p g^{-1}) = (\pi(p), h(p) g^{-1})$$

(Global view)

Def 2 Given a locally finite open cover $\mathcal{U}$ of M
and $g_{UV}: U \cap V \rightarrow G$ satisfying

$$g_{VU} = g_{UV}^{-1} \quad g_{VU} g_{UV} g_{WV} = \text{Id}$$

$$P = \bigsqcup_{U \in \mathcal{U}} U \times G / (p, g) \in U \times G \sim (p', g_{UV} g) \in V \times G$$

$$\text{The } G \text{ action } m(h, (p, g)) = (p, gh)$$

Gen generalization of product principal bundle $M \times G$

Def A bundle homomorphism $f: P \rightarrow P'$ over M

$$\text{satisfies } \pi'(f(p)) = \pi(p)$$

$$f(p g^{-1}) = f(p) g^{-1}$$

P and P' are iso if f has an inverse

Def: A fiber bundle E is a smooth mfd with

① $\pi: E \rightarrow B$. B called base manifold

② $\forall p \in B$. $\exists U \subset B$ containing p

st. $\exists \varphi_U: \pi^{-1}(U) \rightarrow U \times F$ diffeo

F mfd. called fiber (Global view)

Rem Can also be defined by bundle transition function

$$g_{UV}: U \cap V \rightarrow \text{Diff}(F)$$

$\swarrow$ diffeo group

(Local version)

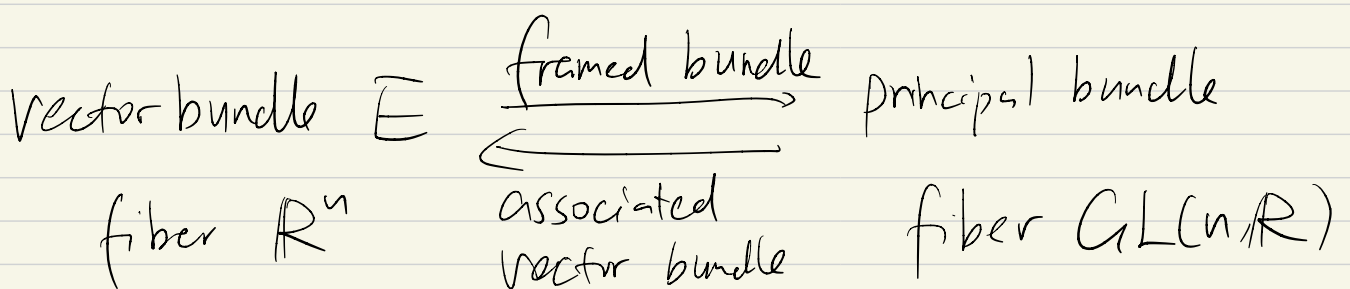
A vector bundle is a fiber bundle with fiber $\mathbb{R}^n, \mathbb{C}^n$

+ scalar multiplication

A principal bundle is a fiber bundle with fiber G

+ group multiplication.

Note that $GL(n, \mathbb{R})$ contains linear transformation of $\mathbb{R}^n$



Framed bundle of E : recall the transition function of E is $g_{vu}: U \cap V \rightarrow GL(n, \mathbb{R})$

$$E = \coprod_{U \in \mathcal{U}} U \times \mathbb{R}^n / (p, u) \in U \times \mathbb{R}^n \sim (p, g_{vu}u) \in V \times \mathbb{R}^n$$

$$P = \coprod_{U \in \mathcal{U}} U \times G / (p, g) \in U \times G \sim (p', g_{vu}g) \in V \times G$$

$$\text{Let } G = GL(n, \mathbb{R}) \quad P = P_{GL(E)}$$

Associated vector bundle: a representation of G

is a homomorphism $\rho: G \rightarrow GL(n, \mathbb{R})$

$$\text{i.e. } \rho(e) = \text{Id} \quad \rho(gh) = \rho(g)\rho(h) \quad \rho(g) = \rho(g)^{-1}$$

Given $g_{vu}: U \cap V \rightarrow G$ for P

Construct vector bundle $\bar{E}$ by

$$g_{vu}: U \cap V \rightarrow G \xrightarrow{\rho} GL(n, \mathbb{R})$$

$$\bar{E} = P \times_P \mathbb{R}^n$$

Class 21 Review

Last time, we review Part I manifold and

Part II vector bundle, and construct principal bundle

	mfd M	v.b. E	p.b. P
Chart (Global)	$\phi_u: U \rightarrow \mathbb{R}^m$	$\psi_u: E _U \rightarrow U \times \mathbb{R}^n$	$\psi_u: P _U \rightarrow U \times G$
transition (Local)	$h_{uv}: \phi_u(U \cap V) \subset \mathbb{R}^m$ $\downarrow$ $\phi_v(U \cap V) \subset \mathbb{R}^m$	$g_{uv}: U \cap V \rightarrow GL(n, \mathbb{R})$	$g_{uv}: U \cap V \rightarrow G$
action	/	scalar action $\mathbb{R} \times E \rightarrow E$ $\mathbb{C} \times E \rightarrow E$	group action $G \times E \rightarrow E$

fiber bundle: $F \rightarrow E: \psi_u: E|_U \rightarrow U \times F$

$\downarrow$

B no action

vector bundle $\xleftrightarrow{\text{framed bundle}}$ principal bundle

$\xleftarrow{\text{associate v.b.}}$

need a rep: $\rho: G \rightarrow GL(n, \mathbb{R}^n)$

Today, we review more constructions in

Part III metrics, IV derivatives

V metric compatible derivatives.

Part III metrics

A metric g on E is a section $M \rightarrow E^* \otimes E^*$ s.t.

1) $g|_p(v, v) > 0$ for $v \neq 0 \in E|_p$ (positive definite)

2) $g|_p(v, w) = g|_p(w, v)$ $v, w \in E|_p$ (Symmetric)

Fact. 1) metric always exists

(construct locally and use partition of unity to glue globally)

2) g_1, g_2 are metrics $\Rightarrow s g_1, t g_1 + (1-t) g_2$ are metrics
 $s, t > 0$

$\Rightarrow$ the space of metrics is convex, contractible

3) The existence of metric can help us choose

an orthonormal basis $e_1, \dots, e_n$ locally (only on U)
which is different from the coordinate basis

$\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}$ of $TM|_U$ $dx^1, \dots, dx^n$ of $T^*M|_U$

Hence we can choose transition function

$$g_{UV} : U \cap V \rightarrow O(n)$$

If further more E is orientable, i.e. $\bigwedge^{\dim E} E \cong M \times \mathbb{R}$

then we have $\det g_{UV} > 0$. $g_{UV} : U \cap V \rightarrow SO(n)$

Then we can construct principal bundle

$PO(n)$ or $PSO(n)$ just by g_{vu} .

Conversely, if we construct an associate bundle from some principal bundle with fiber $SO(n)$, then the vector bundle is orientable

In Part III, we also study Riemannian metric (on TM) and introduce the length L_γ of a smooth path

$$\gamma: I=[0,1] \rightarrow M \quad L_\gamma = \int_I g\left(\gamma_{x \frac{\partial}{\partial t}}, \gamma_{x \frac{\partial}{\partial t}}\right)^{\frac{1}{2}} dt$$

$\gamma_*: TI \rightarrow TM$ is the tangent map (push forward)

$$\text{For } p, q \in M \quad d(p, q) = \inf_{\substack{\gamma(0)=p \\ \gamma(1)=q}} L_\gamma$$

Geodesic thm says if M is compact, then

the path minimizing L_γ always exists, called geodesic.

It is embedded (no $\angle$ or α)

and we can set $g\left(\gamma_{x \frac{\partial}{\partial t}}, \gamma_{x \frac{\partial}{\partial t}}\right) = 1$

Locally, it is obtained by solving the geodesic

$$\text{equation} \quad \ddot{\gamma}^i + \Gamma_{jk}^i \dot{\gamma}^j \dot{\gamma}^k = 0 \quad \dot{\gamma} = \frac{d\gamma}{dt} \quad \ddot{\gamma} = \frac{d^2\gamma}{dt^2}$$

$$\Gamma_{jk}^i = \frac{1}{2} \sum_{l=1}^n g^{il} (\partial_j g_{lk} + \partial_k g_{jl} - \partial_l g_{jk})$$

called Christoffel symbol

This equation can be regarded as Euler-Lagrange equation from the functional

$$S: \{ \gamma: I \rightarrow \mathbb{R}^n \mid \gamma(0)=p, \gamma(1)=q \} \rightarrow \mathbb{R}$$

$$\gamma \longmapsto L\gamma$$

i.e. For any $\eta: I \rightarrow \mathbb{R}^n$ $\eta(0)=\eta(1)=0$

$$\gamma_s = \gamma + s\eta \quad \left. \frac{dS(\gamma_s)}{ds} \right|_{s=0} = 0$$

Later, when we introduce LC conn, this equation can be written as $\nabla_{\dot{\gamma}}^{\text{LC}} \dot{\gamma} = 0$

i.e. $\dot{\gamma}$ itself is parallel section along γ about ∇^{LC}

Part IV derivatives.

$\Omega^k(M) = C^\infty(M; \wedge^k T^*M)$ the space of k -forms

We construct exterior derivative $d: \Omega^k \rightarrow \Omega^{k+1}$

$$\begin{aligned} \text{by (locally)} \quad d(f dx^1 \wedge \dots \wedge dx^k) &= df \wedge dx^1 \wedge \dots \wedge dx^k \\ &= \frac{\partial f}{\partial x^m} dx^m \wedge dx^1 \wedge \dots \wedge dx^k \end{aligned}$$

$$\text{Since } \frac{\partial^2 f}{\partial x^i \partial x^j} = \frac{\partial^2 f}{\partial x^j \partial x^i}, \text{ we have } d^2 = 0$$

$$H_{dR}^k(M) = \ker(d: \Omega^k \rightarrow \Omega^{k+1}) / \operatorname{Im}(d: \Omega^{k-1} \rightarrow \Omega^k)$$

de Rham cohomology, it is an $\mathbb{R}$ -vector space
only depends on the diffeo type of M

(Indeed also doesn't depend on smooth str
because $H_{dR}^k(M) = H_{\text{sing}}^k(M; \mathbb{R})$)

$f: M \rightarrow N$ induces

① push forward $f_*: T_p M \rightarrow T_{f(p)} N$

② pull-back of bundle $\pi: E \rightarrow N$ $f^* E$

also pull-back of principal bundle $f^* P$ $f^* P_{GL(E)} = P_{GL(f^* E)}$

③ pull-back of forms $f^*: \Omega^k(N) \rightarrow \Omega^k(M)$

$$f^* w(v_1, \dots, v_k) = w(f_* v_1, \dots, v_k)$$

④ pull-back of de Rham cohomology class

$$\text{because } df^* w = f^* dw \Rightarrow f^*: H_{dR}^k(N) \rightarrow H_{dR}^k(M)$$

homotopic maps f, g induce the same map.

$$(\Phi: I \times M \rightarrow N, \Phi(0, -) = f, \Phi(1, -) = g)$$

Lie derivative: $L_v: \Omega^k \rightarrow \Omega^k$ for v vector field

$$\text{Cartan formula } L_v w = (v \lrcorner d + d \lrcorner v) w$$

$$v \lrcorner \alpha(v_1, \dots, v_{k-1}) = \alpha(v, v_1, \dots, v_{k-1})$$

Covariant derivative : $\nabla : C^\infty(E) \rightarrow C^\infty(E \otimes T^*M)$

satisfying $\nabla(fs) = s \otimes df + f \nabla s$

If $v \in TM$, we can define $\nabla_v : C^\infty(E) \rightarrow C^\infty(E)$
similar to the partial derivative along v in $\mathbb{R}^n$

Fact. ① Cov. der exists (construct locally and
use partition of unity, locally. $S = (S_1, \dots, S_n)$

$$S_i : U \rightarrow \mathbb{R} \quad (S_i)_* : T_p U \rightarrow T_{S_i(p)} \mathbb{R} \cong \mathbb{R}$$

$((S_1)_*, \dots, (S_n)_*)$ is a section of $(E \otimes T^*M)|_U$

written as ds . In particular, for $M \times \mathbb{R}$

the exterior derivative $d : \Omega^0 \rightarrow \Omega^1$ is a cov der.

② Any ∇, ∇' , we have $\nabla' - \nabla = \alpha$

$\alpha \in C^\infty(\text{End}(E) \otimes T^*M)$ matrix-valued 1-form.

③ Combining d , we have

$$d_\nabla : C^\infty(E \otimes \Lambda^k T^*M) \rightarrow C^\infty(E \otimes \Lambda^{k+1} T^*M)$$

$$d_\nabla^2 S = F_\nabla S \quad \text{the curvature}$$

$$\text{Locally } F_\nabla = d\alpha + \alpha \wedge \alpha$$

(on TM . $\nabla_i = \nabla_{\frac{\partial}{\partial x^i}}$ $\nabla_i \nabla_j - \nabla_j \nabla_i = F_\nabla$)

④ Given g on TM, $\exists$ unique ∇ on TM satisfies

$$\int \nabla g = 0 \quad (\text{metric compatible})$$

$$\int T_{\nabla} = d_{\nabla} - d = 0 \quad (\text{torsion free}), \text{ called}$$

we also study curvature of LC conn

⑤ Bianchi identity $d_{\nabla} F_{\nabla} = 0$

$$C_k(E) = \left[\frac{1}{(2\pi\sqrt{-1})^k} \text{tr} \left(\underbrace{F_{\nabla} \wedge \dots \wedge F_{\nabla}}_k \right) \right] \in H_{2k}^{*}(M)$$

indep of choice of ∇ .

Class 22 Connection on principal bundle (Chaps 11.4)

Suppose $\pi: P \rightarrow M$ is a principal bundle with fiber G .

First, we study the tangent space of G at e , denoted by

$\mathfrak{g} = T_e G$, called Lie algebra of G (lie(G) in Cliff's book)

Since G is a Lie group, for any fixed $g \in G$, define

$L_g: G \rightarrow G$, $R_g: G \rightarrow G$ they are diffeomorphism.
 $h \mapsto gh$ $h \mapsto hg$

$(L_g)_*: T_e G \rightarrow T_g G$ is a vector space isomorphism.

$TG \cong G \times T_e G$ by the following map.

For $(g, v) \in G \times T_e G$, define $f(g, v) = (L_g)_* v \in T_g G$

Def A vector field $v: G \rightarrow TG$ is left-invariant if

$$v(gh) = (L_g)_* v(h)$$

Note that left-invariant v.f. is 1-1 cor to vector in $T_e G$

Let $\mathfrak{g} = \text{Lie}(G)$ be the set of all left-inv v.f.

Recall that there is a 1-1 correspondence btw

vector field v and derivative $L_v: C^\infty(M; \mathbb{R}) \rightarrow C^\infty(M; \mathbb{R})$

Given v, w v.f. we can define $[v, w]$ by

$$L_{[v, w]}(f) = L_v L_w(f) - L_w L_v(f)$$

Then ① $[ax+by, z] = a[x, z] + b[y, z]$

$$[x, ay+bz] = a[x, y] + b[x, z]$$

$$a, b \in \mathbb{R} \quad x, y, z \text{ v.f.}$$

② $[x, x] = 0 \Rightarrow [x, y] = -[y, x]$

③ $[v, [w, u]] + [u, [v, w]] + [w, [u, v]] = 0$
(Jacobi identity)

$[-, -]$ is called Lie bracket (of vector fields)

In the case of $\mathfrak{g}$, we can check for left inv v, w ,

$[v, w]$ is still left inv, so we have a map

$$[-, -] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g},$$

$\mathfrak{g}$ is called Lie algebra associated to G ,

as a vector space, it is just $T_e G$

Ex.	G	$\mathfrak{g}$
$GL(n, \mathbb{R})$ ($\det \neq 0$) <u>$GL(n, \mathbb{C})$</u>		$gl(n, \mathbb{R}) = M(n, \mathbb{R})$ $gl(n, \mathbb{C}) = M(n, \mathbb{C})$ $[x, y] = xy - yx$
$SL(n, \mathbb{R})$ ($\det = 1$) <u>$SL(n, \mathbb{C})$</u>		$sl(n, \mathbb{R}) \subset M(n, \mathbb{R})$ $\text{tr } x = 0$ (traceless)
$O(n)$ ($AA^T = Id$)		$x + x^T = 0$ (skew-symmetric)
$U(n)$ ($AA^* = Id$)		$x + x^* = 0$
$SO(n)$ ($AA^T = Id$ $\det A = 1$)		$so(n)$ $\text{tr } x = 0$ $x + x^T = 0$ $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$
$SU(2)$ ($AA^* = Id$ $\det A = 1$)		$SU(2) = SU(3)$ $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$
$SU(n)$ ($AA^* = Id$ $\det A = 1$)		$su(n)$ $\text{tr } x = 0$ $x + x^* = 0$ Pauli matrices $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$\pi: P \rightarrow M$ is a smooth (surjective map), we can

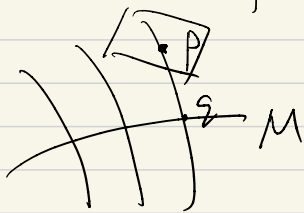
consider the tangent map $T\pi_x: T_p P \rightarrow T_{\pi(p)} M$

$\ker \pi_x$ is a sub-(vector)-bundle of TP

s.t. sections in $\ker \pi_x$ are sent to zero sections of TM

So $(\ker \pi_x)$ over $P|_q = \pi^{-1}(q) \cong G$ for $q \in M$

is isomorphic to $TP|_q \cong T_e G = \mathfrak{g}$



also, we consider $\pi^* TM$ as a bundle over P .

We have a sequence of vector bundles

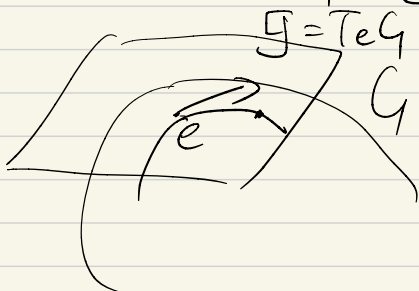
$$\ker \pi_x \rightarrow TP \rightarrow \pi^* TM$$

s.t. at each fiber, it is exact $(\ker(TP \rightarrow \pi^* TM)) = \text{Im}(\ker \pi_x \rightarrow TP)$

Indeed $\ker \pi_x \cong P \times \mathfrak{g}$

For $(p, x) \in P \times \mathfrak{g}$, consider the map $t \mapsto p \exp(tx)$

where $\exp: \mathfrak{g} \rightarrow G$ is the exponential map



define $f(p, x) \in \ker \pi_x$

to be the tangent vector at $t=0$

Any $g \in G$ define $\psi_g: P \xrightarrow{\cong} P$ by $p \mapsto pg^{-1}$

It induces $(\psi_g)_x: T_p P \rightarrow T_{pg^{-1}p} P$

Then we have $f(pg^{-1}, (Lg)_*(Rg^{-1})_*x) = (\psi_g)_*(f(p, x))$

$$g \times g^{-1}$$

Def a connection A on P is one of the following:

- 1) A section $P \rightarrow T^*P \otimes \ker \pi_*$
(or $\text{Hom}(TP, \mathfrak{g})$ where $\mathfrak{g}$ is the product v.b. $P \times \mathfrak{g}$)

satisfies ① $\langle A|_p, f(p, x) \rangle = x \in \mathfrak{g}$

$$\textcircled{2} \langle \psi_g^*(A|_{pg^{-1}}), v \rangle = g \langle A|_p, v \rangle g^{-1} \quad \forall g \in G, v \in T|_p$$

- 2) Horizontal subspaces $H \subset TP$ tangent to fibers of π i.e.

$$TP = H \oplus \ker \pi_*$$

(A choice of splitting in $\ker \pi_* \rightarrow TP \rightarrow \pi^*TM$)

H is isomorphic to π^*TM , but not canonically.

Also. H need to be preserved under $(\psi_g)_*$

1) $\Rightarrow$ 2). Let H_A be kernel of $A : TP \rightarrow \ker \pi_*$

2) $\Rightarrow$ 1) Use H to define A by projection

If A and A' are two connections, then $A - A'$ is

a map from π^*TM to $\ker \pi_* = P \times \mathfrak{g}$

which is compatible with the action of g

Class 23 connection and covariant derivative

Last time, we introduced Lie algebra of a Lie gp as left-invariant vector field. the space is identified with $T_e G = \mathfrak{g}$

$$L_g : G \xrightarrow{\cong} G \quad R_g : G \xrightarrow{\cong} G \\ h \mapsto gh \quad h \mapsto hg$$

$$\text{Left inv} : (L_g)_* v = v \quad (\text{usually write } v \text{ for v.f. but } x \text{ for element in } \mathfrak{g})$$

Define $\text{ad} : G \rightarrow GL(\mathfrak{g})$

$$G \times \mathfrak{g} \rightarrow \mathfrak{g} \quad g \cdot x = (L_g)_* (R_{g^{-1}})_* x = g \times g^{-1}$$

This is called adjoint representation of G

Recall a rep of G is a homomorphism

$$\rho : G \rightarrow GL(n, \mathbb{R}) \quad \text{now } \mathfrak{g} \text{ itself is a vector space (with a Lie bracket)}$$

(consider tangent map, we have $\text{ad} : \mathfrak{g} \rightarrow gl(\mathfrak{g})$,

$$\text{where } gl(\mathfrak{g}) = \text{End}(\mathfrak{g}), \quad \text{ad}(x)(y) = [x, y] \\ = M(\text{d} \ln g, \mathbb{R}) = xy - yx$$

This is called the adjoint representation of Lie algebra $\mathfrak{g}$)

Recall for G -principal bundle P and a rep $\rho: G \rightarrow GL(n, \mathbb{R})$, we can define an associated vector bundle $P \times_{\rho} \mathbb{R}^n$ by transition functions

$$g_{UV}: U \cap V \rightarrow G \xrightarrow{\rho} GL(n, \mathbb{R})$$

We can use adjoint rep to obtain $P \times_{\text{ad}} \mathfrak{g}$

It is a vector bundle over M

We want to argue for two connections A, A'

the difference $A' - A$ is a section of $P \times_{\text{ad}} \mathfrak{g}$

Note $\mathfrak{g}$ can be regarded as a subspace of matrices, this is related to the matrix-valued 1-form in

$$\nabla' - \nabla = \alpha \in C^{\infty}(M; \text{End}(E) \otimes \wedge^1 T^*M)$$

Recall the definition of connections:

we have an exact sequence of v.b. over P

$$\ker \pi_* \rightarrow TP \rightarrow \pi^* TM$$

$$\ker \pi_* \cong P \times \mathfrak{g} \text{ (product } \mathfrak{g} \text{ bundle)}$$

a connection is either ① a bundle map $TP \rightarrow \ker \pi_*$

s.t. $\ker \pi_* \hookrightarrow TP \rightarrow \ker \pi_*$ is identity.

or ② a subbundle H s.t. $H \oplus \ker \pi_* = TP$

Both satisfy some G -equivariant conditions.

H is called horizontal space, because

it is tangent to the fiber G

From ①. A is a $\mathfrak{g}$ -valued 1-form on P

$$\psi_g : P \xrightarrow{\cong} P \quad G\text{-equivariant means} \\ p \mapsto pg^{-1}$$

$$\langle \psi_g^*(A|_{pg^{-1}}), v \rangle = g \langle A|_p, v \rangle g^{-1} \quad \forall g \in G, v \in T P|_p$$

$$\textcircled{2} \quad G\text{-equivariant of } H \text{ means } (\psi_g)_* H_p = H|_{pg^{-1}}$$

$$H_A = \ker A \quad H \cong \pi^* TM$$

The difference $A' - A : TP \rightarrow \ker \pi_*$

s.t. $\ker \pi_* \hookrightarrow TP \xrightarrow{A'-A} \ker \pi_*$ is zero

So it induces a map $\pi^* TM = TP / \ker \pi_* \rightarrow \ker \pi_*$

that is G -equivariant, so this gives a section

$$\text{In } P \times_{\text{ad } G} = \{(p, x) \in P \times \mathfrak{g}\} / (p, x) \sim (pg^{-1}, g \cdot x \cdot g^{-1})$$

$\uparrow$
from G -equivariant condition

We show a general argument about associated bundle

Let $V = \mathbb{R}^n$ or $\mathbb{C}^n$ $\text{End}(V) = GL(n, \mathbb{R})$ or $GL(n, \mathbb{C})$

Let $\rho: G \rightarrow \text{End}(V)$ be a representation

$$\begin{aligned} E &= P \times_{\rho} V = \{(p, v) \in P \times V\} / (p, g) \sim (pg^{-1}, \rho(g)v) \\ &= P \times V / \text{action of } G \end{aligned}$$

Then a section $s: M \rightarrow E$ is the same as

$$G\text{-equiv map } \mathcal{S}: P \rightarrow V$$

Given a connection A (or H_A) on P , we can define cov der ∇ on E as follows.

First, take $ds: TP \rightarrow V$, regarded as

a section $\underline{V} \otimes T^*P$, where $\underline{V}$ denotes the bundle

$$P \times V. \text{ We have } \langle (\mathcal{U}_g)^*(ds|_{pg^{-1}}), w \rangle = \rho(g) \langle ds|_p, w \rangle$$

for $w \in TP|_p$
(or TM)

For $V|_p \in \pi^* TM|_p$, we pick the unique horizontal

lift $V_A|_p \in H_p \subset T_p P$ s.t. $\pi_* V_A|_p = V|_p$

$$\text{we have } (\mathcal{U}_g)_* V_A|_p = V_A|_{pg^{-1}}$$

Let $\nabla\$_$ be the G -equiv map from π^*TM to V

$$\text{by } \langle \nabla\$, v \rangle = \langle d\$, v_A \rangle$$

we have

$$\langle \nabla((\pi^*f)\$), v \rangle = \langle d((\pi^*f)\$), v_A \rangle$$

$$= \$ \langle \pi^*df, v_A \rangle + \pi^*f \langle d\$, v_A \rangle$$

$$= \$ \langle df, v \rangle + \pi^*f \langle \nabla\$, v \rangle$$

Hence, $\nabla\$_$ induces a covariant derivative $\nabla\$_$

Locally, we have $\varphi_U: P|_U \rightarrow U \times G$

a connection A on P can be written explicit

$$A = \varphi_U^*(\bar{g}^1 dg + \bar{g}^1 \alpha_U g)$$

($\mathfrak{g}$ -valued 1-form on $U \times G$, not just U)

α_U is a $\mathfrak{g}$ -valued 1-form on U

$\bar{g}^1 dg$ comes from T^*G part $\bar{g}^1$ identify $T_g G$ with $T_e G$

$\bar{g}^1 \alpha_U g$ comes from G -equivariant.

Class 24 Horizontal lift

$\pi: P \rightarrow M$ principal bundle with fiber G

we have $\ker \pi_* \hookrightarrow TP \twoheadrightarrow \pi^* TM$ exact

a connection A is a G -equivariant map

$$TP \rightarrow \ker \pi_* \text{ s.t. } \ker \pi_* \hookrightarrow TP \xrightarrow{A} \ker \pi_*$$

is identity. take horizontal space H_A as $\ker A$

we have $\pi_*: H_A \subset TP \rightarrow TM$

is isomorphism.

Given $V \in TM$, we get $V \in \pi^* TM$,

pick $V_A \in H_A$ s.t. $\pi_* V_A = V$

V_A is called horizontal lift of V .

on associated bundle $E = P \times_{\rho} V$, we can define

∇S by lift $S: M \rightarrow E$ to

G -equiv $\mathbb{S}: P \rightarrow V$ and $d\mathbb{S}: TP \rightarrow V$

$$\langle \nabla S, V \rangle = \langle d\mathbb{S}, V_A \rangle$$

Given a path $\gamma: [0,1] \rightarrow M$, we want to

lift γ to $\gamma_A: [0,1] \rightarrow P$ horizontally.

Prop. Given $t_0 \in [0,1]$, $p \in P|_{\gamma(s)}$, $\exists$ unique

$\gamma_A: [0,1] \rightarrow P$ satisfying

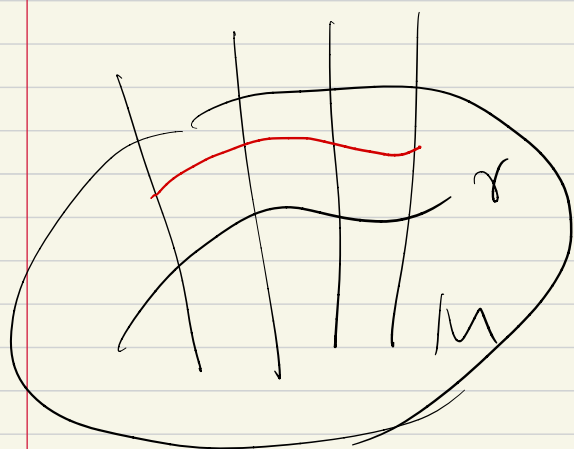
$$1) \gamma_A(t_0) = p$$

$$2) \pi(\gamma_A) = \gamma$$

$$3) \dot{\gamma}_A \subset H_A$$

Moreover, this lift is G -eqn. i.e.

the path $\gamma'_A: t \mapsto \gamma_A(t)g^{-1}$ is the horizontal lift of γ with $\gamma'_A(t_0) = pg^{-1}$



Recall in the case of E, ∇

a section S is parallel along γ if $\nabla_{\dot{\gamma}} S = 0$

If $E = P \times_p V$ is the associated bundle

a section $s: M \rightarrow E$ corresponds to G -eqn. $S: P \rightarrow V$

$$S(\gamma(t_0)) \in E|_{\gamma(t_0)} \quad \langle \nabla S, \dot{\gamma} \rangle = 0$$

$$\Rightarrow \langle ds, \dot{\gamma}_A \rangle = 0$$

$\Rightarrow$ partial derivative along $\dot{\gamma}_A$ vanishes

To prove the existence of horizontal lift.

We need local model of the connection

Locally, we have $\varphi_U: P|_U \rightarrow U \times G$

a connection A on P can be written explicit

$$A = \varphi_U^* (\bar{g}^T dg + \bar{g}^T \alpha_U g)$$

($\mathfrak{g}$ -valued 1-form on $U \times G$, not just U)

α_U is a $\mathfrak{g}$ -valued 1-form on U

This α_U is the same one as in $\nabla s_U = ds_U + \alpha_U s_U$

In another chart $V \times G$, we have

$$(x, g_V) = (x, g_{VU}(x) g_U)$$

$$\bar{g}_V^T dg_V + \bar{g}_V^T \alpha_V g_V$$

$$= \bar{g}_U^T dg_U + \bar{g}_U^T (\bar{g}_{VU}^T \alpha_V g_{VU} + \bar{g}_{VU}^T dg_{VU}) g_U$$

$$\Rightarrow \alpha_U = \bar{g}_{VU}^T \alpha_V g_{VU} + \bar{g}_{VU}^T dg_{VU}$$

In chart $U \times G$, we have

$$\gamma_A(t) = (\gamma(t), g(t))$$

$$\dot{\gamma}_A(t) = (\dot{\gamma}(t), g^{-1}(t) \dot{g}(t))$$

identify $T_g G$ with $T_e G$ by g^{-1}

$\dot{\gamma}_A \in \mathcal{H}_A \Rightarrow$ the action

$$\dot{g}^{-1}(t) dg(t) + \dot{g}^{-1}(t) \alpha_u(\gamma(t)) g(t) \text{ vanishes}$$

$$\Rightarrow g^{-1}(t) \dot{g}(t) + \dot{g}^{-1}(t) \langle \alpha_u(\gamma(t)), \dot{\gamma}(t) \rangle g(t) = 0$$

$$\Rightarrow \dot{g}(t) + \langle \alpha_u(\gamma(t)), \dot{\gamma}(t) \rangle g(t) = 0$$

This is an ODE, given $g(t_0)$, $\exists$ unique solution

Classification of principal G -bundle:

Thm: If $\psi, \phi : M \rightarrow N$ are homotopic,

i.e. $\exists \Phi : [0,1] \times M \rightarrow N$ $\Phi(0,-) = \psi$ $\Phi(1,-) = \phi$

For $\pi : P \rightarrow N$, $\psi^* P$ is isomorphic to $\phi^* P$

Pf: Fix a connection A on $\Phi^* P \rightarrow [0,1] \times M$

Given $p \in \psi^* P$, let $\gamma_{A,p}(t)$ be the

horizontal lift of $\gamma(t) = \bar{\Phi}(t, \pi(p))$

(This path has the same image on M)

Then $\gamma_{A,p}(1) \in \phi^*P$

The map $p \mapsto \gamma_{A,p}(1)$ is an isomorphism from ψ^*P to ϕ^*P , it is G -equiv

because $\gamma_{A,pg^{-1}}(t) = \gamma_{A,p}(t) g^{-1}$

Cor. If M is homotopic to $\mathbb{R}^n$ (or contractible), then any principal bundle over M is product G bundle

Fact For any G , $\exists$ a universal classifying space BG (unique up to homotopy), s.t.

$\{\text{principal } G\text{-bundle over } M\} / \sim$

$\Leftrightarrow \{\text{maps from } X \text{ to } BG\} / \text{homotopy} = [X, BG]$

Ex: $G = S^1 = U(1)$ $BG = \mathbb{CP}^\infty$ ($\mathbb{CP}^n \hookrightarrow \mathbb{CP}^{n+1}$)

$G = O(n)$ $BG = \text{Grass}(n, \mathbb{R}^\infty)$ ($\mathbb{R}^n$ in $\mathbb{R}^\infty$)

Another understanding of characteristic classes
(for p.b. or v.b. by framed bundle)

$$H^*(\mathbb{CP}^\infty) = \mathbb{Z}[x] \quad \deg x = 2$$

first Chern class of P is the pull-back
of the generator $x \in H^2(\mathbb{CP}^\infty)$ by

$$M \rightarrow BU(1) = \mathbb{CP}^\infty$$

Class 25 Yang-Mills equation

$\pi: P \rightarrow M$ principal bundle

$$0 \rightarrow \ker \pi_x \xrightarrow{A} TP \rightarrow \pi^* TM \rightarrow 0 \quad \text{exact}$$

$\begin{array}{ccc} & \nwarrow A & \\ \pi^* & & \uparrow \\ 0 & \dashrightarrow & H_A \end{array}$

A connection $A: TP \rightarrow \ker \pi_x$ G-equ

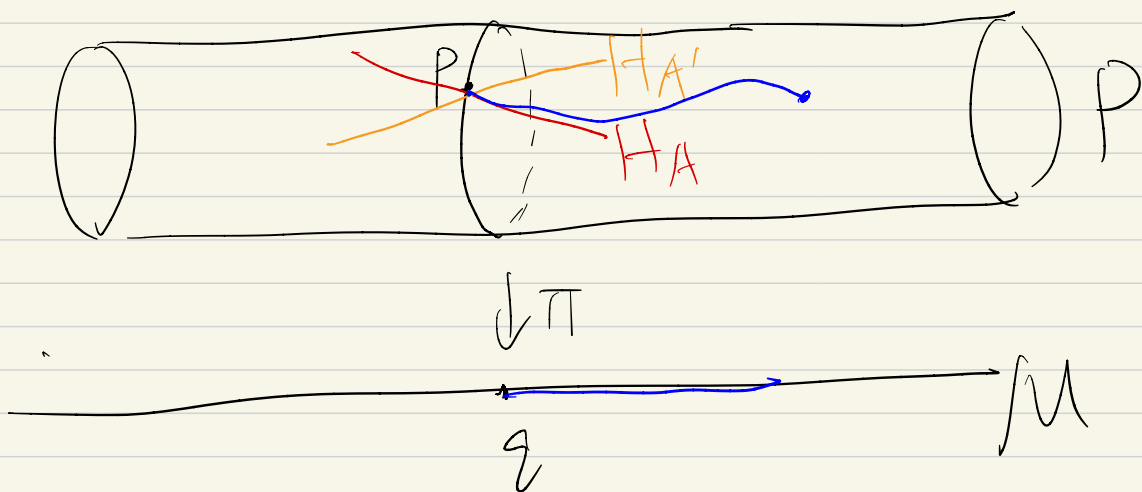
$\ker A = H_A$ horizontal space

$$H_A \cong \pi^* TM \quad \ker \pi_x \oplus H_A = TP$$

$$V \in TM \quad V \in \pi^* TM$$

$V_A \in H_A$ s.t. $\pi_x V_A = V$ horizontal lift of V

$$G = S^1$$



$$\nabla_A : C^\infty(\text{ad } P) \rightarrow C^\infty(\text{ad } P \otimes T^*M)$$

$$\text{ad } P = P \times_{\text{ad } G} G = P \times G / (p, x) \sim (pg^{-1}, g^x g^{-1})$$

$$\langle \nabla_A S, V \rangle = \langle dS, V_A \rangle$$

For a path $\gamma : [0, 1] \rightarrow M$ and $p \in P|_{\gamma(0)}$,

We can find unique horizontal lift $\gamma_{A,p} : [0, 1] \rightarrow P$

$$\text{s.t. } \pi \circ \gamma_{A,p} = \gamma \quad \gamma_{A,p}(0) = p \quad \dot{\gamma}_{A,p} \in H_A$$

$$\text{Locally (on } P|_U \cong U \times G), \quad A = g^{-1} dg + g^{-1} \alpha g$$

$$\text{with } \alpha u = g v u^{-1} \alpha v g v u + g v u^{-1} dg v u$$

$$\alpha u \in C^\infty(\mathfrak{g} \otimes T^*U)$$

matrix valued 1-form

We can consider the exterior covariant derivative

$$d_A : C^\infty(\text{ad } P \otimes \Lambda^k T^*M) \rightarrow C^\infty(\text{ad } P \otimes \Lambda^{k+1} T^*M)$$

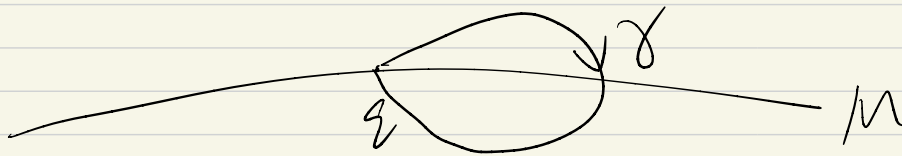
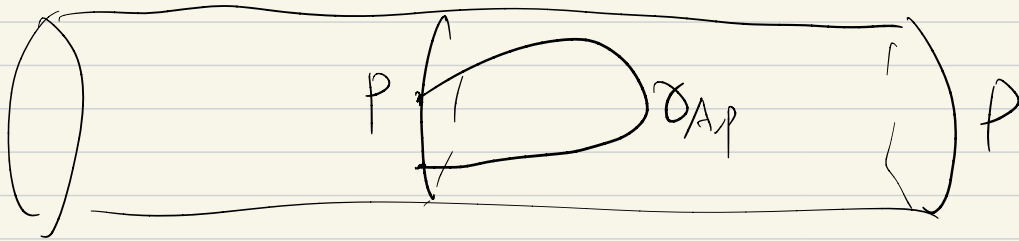
$$\text{and } F_A S = d_A^2 S \text{ for } S \in C^\infty(\text{ad } P)$$

$$\text{Locally, } F_A = d\alpha u + \alpha u \wedge \alpha u \in C^\infty(\text{ad } P \otimes \Lambda^2 T^*M)$$

matrix valued 2-form.

If $F_A = 0$ A is called a flat connection

In such case, $\gamma_{A,p}(1)$ only depends on homotopy class of γ



Define $\text{hol}_{A,p} : \pi_1(M, q) \longrightarrow G$ holonomy map
 $[\gamma] \longmapsto \gamma_{A,p}(1)$

where $\pi_1(M, q) = \underbrace{\{ \gamma : [0, 1] \rightarrow M \mid \gamma(0) = \gamma(1) = q \}}_{\text{homotopy}}$

we have $\text{hol}_{A,p} g^{-1}([\gamma]) = g \text{hol}_{A,p}([\gamma]) g^{-1}$

$$\text{hol}_{A,p}([\gamma_1] \cdot [\gamma_2]) = \text{hol}_{A,p}([\gamma_1]) \cdot \text{hol}_{A,p}([\gamma_2])$$

$$[\gamma_1] \cdot [\gamma_2] = [\gamma]$$

$$\text{s.t. } \gamma(t) = \begin{cases} \gamma_1(2t) & t \in [0, \frac{1}{2}] \\ \gamma_2(2t-1) & t \in [\frac{1}{2}, 1] \end{cases}$$

So flat connection determines a representation of

$\pi_1(M)$ up to conjugation

Conversely, the universal cover $\tilde{M}$ ($\pi_1(\tilde{M}) = \{e\}$)

is a principal $\pi_1(M)$ bundle over M

a representation $\rho: \pi_1(M) \rightarrow G$

induces a principal G -bundle

$$M \times_{\rho} G = \tilde{M} \times G / (g, g) \sim (\gamma \cdot g, \rho(\gamma) g \rho(\gamma)^{-1})$$

Let $(\tilde{\gamma}, e)$ be the horizontal lift of $\gamma = \pi_1(\tilde{\gamma})$

This defines a flat connection.

$$\{\text{flat conn}\} / \text{iso} \iff \{\pi_1 \text{ rep}\} / \text{conj}$$

Given g on TM , we can define the volume form $dvol$ by picking orthonormal basis

$$e^1, \dots, e^n \text{ of } T^*M \text{ and } dvol = e^1 \wedge \dots \wedge e^n.$$

There exists an operator $*$: $\Omega^k \rightarrow \Omega^{n-k}$, $n = \dim M$

$$\text{s.t. } v \wedge *v = |v|^2 dvol,$$

where $|\cdot|$ is induced by g .

$$\text{Explicitly, } *(e^1 \wedge \dots \wedge e^k) = e^{i_{k+1}} \wedge \dots \wedge e^n$$

$$*(e^{i_1} \wedge \dots \wedge e^{i_k}) = \pm e^{i_{k+1}} \wedge \dots \wedge e^{i_n}$$

$$\text{where } \{i_1, \dots, i_k, \dots, i_n\} = \{1, \dots, n\}$$

we have $\star^2 = (-1)^{k(n-k)} : \Omega^k \rightarrow \Omega^k$

In particular, let $n=4$ $k=2$. we have $\star^2 = 1$

$$\Omega^2 = \Omega^+ \oplus \Omega^- \quad \star = \pm 1 \text{ on } \Omega^\pm$$

$\Omega^\pm$ is generated by $e_1 \wedge e_2 \pm e_3 \wedge e_4$

$$e_1 \wedge e_3 \pm e_4 \wedge e_2$$

$$e_1 \wedge e_4 \pm e_2 \wedge e_3$$

F_A is a matrix valued 2-form.

we also have $\star F_A$.

$$\text{Let } \bar{F}_A^\pm = \frac{1}{2} (F_A \pm \star F_A) \quad \star \bar{F}_A^\pm = \pm \bar{F}_A^\pm$$

A is anti-self-dual if $\bar{F}_A^+ = 0$

$$\Leftrightarrow \bar{F}_A = -\star F_A$$

This is the Yang-Mills equation

recall the space of ∇ is affine over

$$C^\infty(\text{End}(E) \otimes T^*M)$$

$\bar{F}_A^+ = 0$ is an equation on the space of conn

the space of conn is also affine over

$$C^\infty(\mathfrak{g} \otimes T^*M) \quad \mathfrak{g}\text{-valued 1-form}$$

(Note we should start with $\text{confd } M$ with $\mathfrak{g}$ on TM)

In particular $G = U(1)$ $\mathfrak{g} = \{c \in \mathbb{C} \mid c + c^* = 0\}$
 $= i\mathbb{R}$

A is affine over $C^\infty(i\mathbb{R} \otimes T^*U)$

$F_A^+ = 0$ is indeed the Maxwell's equations

for electromagnetism in physics

$G = SU(2)$ $\mathfrak{g} = \{x \in M(2, \mathbb{C}) \mid \text{tr} x = 0, x + x^* = 0\}$
nonabelian
 $= \text{span} \left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$

Pauli matrices

$F_A^+ = 0$ is the original one studied in physics

Class 26 Topological invs from Yang-Mills equation

Last time, we focus on $\dim M = 4$ with g on TM and introduce the Yang-Mills equation $F_A^\dagger = 0$.

Let's review the set-up (replace M by X)

Suppose X is closed (compact + no boundary)
and oriented (orientable + an orientation)

$\pi: P \rightarrow X$ is a principal $G = SU(2)$ bundle.

Given g on TX , define $dvol$ by orthonormal basis

$$dvol = e^1 \wedge \dots \wedge e^n$$

Define $\star: \Omega^k \rightarrow \Omega^{4-k}$ s.t. $v \wedge \star v = |v|^2 dvol$

$\star^2 = 1$ on Ω^2 split it into $\Omega^+ \oplus \Omega^-$

Given $w \in \Omega^2$, we have $w \pm \star w \in \Omega^\pm$

For a conn A on P , we can define ∇_A, d_A, F_A

$F_A^\dagger = \frac{1}{2}(F_A + \star F_A)$ is the ASD part of F_A .

Fact. Principal $SU(2)$ bundle on 4d manifold X

has $c_1(P) = c_1(P \times_{\text{ad}} \mathfrak{g}) = 0$, and is classified by

$$c_2(P) = \frac{1}{(2\pi i)^2} [\text{tr}(F_A \wedge F_A)] \in H^4(X; \mathbb{Z}) \cong \mathbb{Z}$$

The orientation on X means picking a generator in $\mathbb{Z}$

Then we can write $c_2(P)$ as an integer $k \in \mathbb{Z}$

Let A_k be the space of all conns on P

with $c_2(P) = k$. it is affine over $C^\infty(\text{ad}P \otimes T^*X)$

Define $\text{Ad}P = P \times_{\text{Ad}G} = P \times G / (p, h) \sim (p\psi^{-1}, ghg^{-1}) \forall g$

It is a bundle of Lie group ($\text{ad}P = P \times_{\text{ad}G}$)

Let $G_k = C^\infty(\text{Ad}P)$ called gauge group

There is an action $G_k \times A_k \rightarrow A_k$

$$(g, A) \mapsto g^*A = A + g^{-1}dg$$

This induces $F_{g^*A} = g \cdot F_A g^{-1}$

so A is ASD $\Rightarrow g^*A$ is also ASD.

Let $M_k = \{\text{ASD conn in } A_k\} / G_k$

be the moduli space. the set of ASD conn

and G_k are both infinite dimensional,

but $\dim M_k$ is finite

Note that M_k also depends on g because x is.
 For any g , M_k is not always manifold

To prove M_k is a smooth manifold for generic g .

(generic: roughly means dense set in the space of metrics)

we have to do completion of A_k and G_k

under some norms, s.t. they become Hilbert manifold

(Locally Hilbert space = infinite dim space
 Complete with inner product)

Then we can apply infinite dim version of
 implicit function thm and Sard thm.

For generic g , M_k is a smooth manifold with
 (possibly containing singular part)

$$\dim M_k = 8k - 3(b_2^+ - b_1^+ + 1)$$

$$\uparrow$$

$$SU(2) \cong \mathbb{R}^3$$

$$\dim H_{dR}^1(X)$$

$$H_{dR}^2(X) = H_{dR}^+(X) \oplus H_{dR}^-(X)$$

$$b_2^+ = \dim H_{dR}^+(X)$$

Ex. $X = S^4$ round metric $k=1$ $M_k = \mathbb{B}^5$

$$X = \overline{\mathbb{CP}}^2 \text{ (}\mathbb{CP}^2 \text{ with opposite orientation)}$$

$g = \text{Fubini-Study metric induced from } \mathbb{C}^n$

$$k=1 \quad M_k = \text{Cone of } \overline{\mathbb{CP}}^2$$

$$[0,1) \times \overline{\mathbb{CP}}^2 / \{0\} \times \overline{\mathbb{CP}}^2$$



The cone pt is the reducible solution
(cones from $\mathcal{S}' = U(1) \subset SU(2)$ connection)

when $\dim M_K < 8$, after adding boundary,
 M_K is compact (> 8 , some bubble phenomenon)
of $\text{codim } 8$

In particular, if $\dim M_K = 0$, this is just
finitely many points. we can introduce orientations
on M_K . and in the case of $\dim M_K = 0$,
we count points with signs.

Fact: under some assumptions (like $b_2^+(X) > 0$),
the number of pts in M_K with $\dim M_K = 0$
is independent of g . Hence this is an invariant
that only depends on the diffeomorphism type of X

Sometimes, we also consider $SO(3)$ principal bundle
and count solutions of $\overline{F}_A^+ = 0$ (note $SO(3) = SU(2)$)

$\exists$ smooth manifold X_1, X_2 s.t. $X_1 \cong_{\text{homeo}} X_2$,
but the number of solutions are different. $X_1 \not\cong_{\text{diff}} X_2$

These are called exotic pair

Ex For K3 surface, the signed counting of solution is 1, but $\exists X_k \cong_{\text{homeo}} K3$, with the counting $(2k+1)$ for any $k \in \mathbb{Z}$.

Donaldson diagonal thm: Suppose X is a closed, oriented, connected, simply-connected smooth 4-manifold,
 $(\pi_1(X) = \{e\})$

Consider the cup product (wedge product on $\mathbb{Z}$ coeff)

$$H^2(X; \mathbb{Z}) \times H^2(X; \mathbb{Z}) \longrightarrow H^4(X; \mathbb{Z})$$

as a bilinear form, Q_X .

If Q_X is negative, then $Q_X = \begin{pmatrix} -1 & & \\ & -1 & \\ & & \ddots \\ & & & -1 \end{pmatrix}$
up to change basis

i.e. If Q_X is not diagonalizable.

there is no smooth structure on X .

The proof uses the moduli space M_1

$$\dim M_1 = 8 - 3 = 5. \quad \partial M_1 = X$$

and M_1 has cones on $\overline{\mathbb{CP}}^2$