Class 9. Riemannian metric.

A metric on a (real) v.b. associates a positive definite symmetric bilinear form on each fiber. i.e.

a section $M \to E^* \otimes E^*$ satisfies some conditions.

Basic properties

1) metrics alway exist.

2) g_1, g_2 are metrics, then $sg_1 + tg_2$ are also metris

$s, t \geq 0 \quad s^2 + t^2 \neq 0$

This implies the space of metrics is contractible because any metrics can be connected linearly.

by $tg_1 + (1-t)g_2$

3) The existence of metric reduces the str gp of the bundle transition functions from $GL(n, \mathbb{R})$ to $O(n)$ (Hermitian metric reduces $GL(n, \mathbb{C})$ to $U(n)$)

Def. A metric on TM is called a <u>Riemannian metric</u>.

also called a metric of M.

Recall that we write coordinates of

$$T\mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n \quad \text{as} \quad x_1, \cdots, x_n, \frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^n}$$

and $T^*\mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n$ as $x_1, \cdots, x_n, dx_1, \cdots, dx_n$

Locally, a vector field (a section $M \to TM$)

is written as $\sum_{k=1}^{n} V_k \frac{\partial}{\partial x^k}$ for $V_k : U \to \mathbb{R}$

(Here we omit the function $\varphi_U : \pi^{-1}(U) \to U \times \mathbb{R}^n$)

a 1-form (a section $M \to T^*M$)

is written as $\sum_{k=1}^{n} V_k \, dx^k$

Hence a metric (a section of $T^*M \otimes T^*M$)

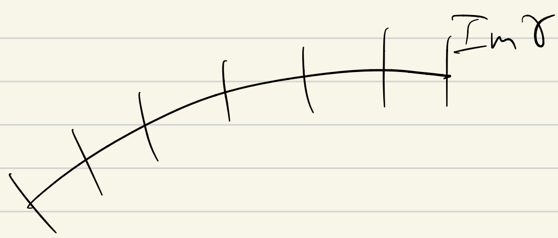
Can be written as $\sum_{i,j=1}^{n} g_{ij} \, dx^i \otimes dx^j$

with $\{g_{ij}(p)\}$ a sym, posi definite matric

$$g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = g_{ij}$$

Intuition of length of a path in $\mathbb{R}^n$

Let $\gamma : I = [0,1] \to \mathbb{R}^n$



Step 1. Break I into N segments

i.e. consider $t_i = \frac{i}{N}$ and $\gamma(t_i)$

Approximate $\operatorname{Im} \gamma\big|_{[t_i, t_{i+1}]}$ by $|\gamma(t_{i+1}) - \gamma(t_i)|$

$$\Delta t = \frac{1}{N}$$

Step 2. Use Taylor's thm to write

$$\frac{|\gamma(t_{i+1}) - \gamma(t_i)|}{\Delta t} = \left|\frac{d}{dt}\gamma\right|_{t=t_i} + O(\Delta t)$$

$$\sum_{i=0}^{N-1} |\gamma(t_{i+1}) - \gamma(t_i)| = \left(\sum \left|\frac{d}{dt}\gamma\right|_{t=t_i}\right)\Delta t + O(\Delta t)$$

Step 3 Take the limit $N \to \infty$, $\Delta t \to 0$

we get the length of γ

$$L_\gamma := \int_I \left|\frac{d}{dt}\gamma\right| dt \quad \text{(This is a definition)}$$

Def. Given a Riemannian metric g on M and a path $\gamma : I \to M$ we define the length of γ as $L_\gamma := \int_I \left(g\left(\frac{d}{dt}\gamma, \frac{d}{dt}\gamma\right)\right)^{\frac{1}{2}} dt$

Rem. $\frac{d}{dt}\gamma$ means the pushforward of (tangent map)

$\frac{\partial}{\partial t} \in T_t I$ to $\gamma_* \frac{\partial}{\partial t} \in T_{\gamma(t)} M$

Explicitly $U \subset I \quad V \subset M$

$\phi_U : U \longrightarrow \mathbb{R} \qquad \phi_V : V \longrightarrow \mathbb{R}^n$

$\phi_{U*} \; TI|_U \longrightarrow U \times \mathbb{R} \quad \phi_{U*} : TM|_V \longrightarrow V \times \mathbb{R}^n$

$\phi_V \circ \gamma \circ \phi_U^{-1} : \mathbb{R} \longrightarrow \mathbb{R}^n$

$$\gamma_* \frac{\partial}{\partial t} = \phi_{V*}^{-1} \circ (\phi_V \circ \gamma \circ \phi_U^{-1})_* \circ \phi_{U*} \frac{\partial}{\partial t}$$

Since the notation $\frac{\partial}{\partial t}$ already means $\phi_{U*}^{-1}\left(\frac{\partial}{\partial t}\right)$,
and $\phi_U = Id$,

we have

$$\frac{d}{dt}\gamma\Big|_p = \phi_{V*}^{-1} \circ (\phi_V \circ \gamma)_* \frac{\partial}{\partial t}\Big|_p$$

$$= \phi_{V*}^{-1}\left(p, \frac{d(x_1 \circ \phi_V \circ \gamma)}{dt}, \dots, \frac{d(x_n \circ \phi_V \circ \gamma)}{dt}\right)$$

$x_i : \mathbb{R}^n \longrightarrow \mathbb{R}$ coordinate map

Def The distance between $p, q \in M$ is defined by

$d(p,q) := \inf_\gamma L\gamma \quad \gamma$ is a path with $\gamma(0) = p$
$\gamma(1) = q$

Prop 1) $d(p,p)=0$ $d(p,q)\neq 0$ for $p\neq q$

2) $d(p,q)=d(q,p)$

3) $d(p,q)\leq d(p,r)+d(r,q)$ $r\in M$

Rem. 1). if $p\neq q$. we have a nbhd of p that is disjoint from q (Haursdorff). the path must intersects the boundary of the nbhd and in the nbhd the length is nonzero

Thm (Geodesic thm)

1) There is a smooth path γ from p to q with $l_\gamma = d(p,q)$, such path (or its image) is called <u>geodesic</u>

2) Any geodesic is an embedded, 1-dim submfd that can be reparameterized st. $g(\gamma_*\frac{\partial}{\partial t}, \gamma_*\frac{\partial}{\partial t})=1$

This is called a <u>Constant speed</u> geodesic

(Repara means we get a new path by composition $I \xrightarrow{\psi} I \xrightarrow{\gamma} M$)

3) A constant speed geodesic is characterized as follows

Let $U \subset M$, $\phi_U : U \to \mathbb{R}^n$ $g|_U = \phi_U^*(\sum g_{ij}\, dx^i \otimes dx^j)$

We write $\gamma_U : \gamma^{-1}(U) \cap I \longrightarrow \mathbb{R}^n$

$$t \longmapsto \phi_U \circ \gamma(t)$$
$$= (\gamma^1_U(t), \gamma^2_U(t), \dots, \gamma^n_U(t))$$

Then
$$\frac{d^2}{dt^2}\gamma^i_U + \sum_{j,k=1}^{n} \Gamma^i_{jk} \frac{d}{dt}\gamma^j_U \frac{d}{dt}\gamma^k_U = 0 \qquad (*)$$

for any i where

$$\Gamma^i_{jk} = \frac{1}{2}\sum_{l=1}^{n} g^{il}(\partial_j g_{lk} + \partial_k g_{jl} - \partial_l g_{jk})$$

$\{g^{ij}\}$ is the inverse matrix of $\{g_{ij}\}$

Γ^i_{jk} is called Christoffel symbols

4) Any curve satisfying $(*)$ is locally length minimizing.

This means $\exists\ C_0 > 1$ s.t. when p, q on the curves with $d(p,q) \le C_0^{-1}$, then $\exists$ a segment of curve connecting p and q with length $d(p,q)$