Class 8 Metrics on vector bundles. (Chap 7)

Basic idea: For a real/complex vector bundle $E \to M$.

The fiber at each $pt \in M$ is $\mathbb{R}^n$ or $\mathbb{C}^n$

with vector space structures.

A metric on E is a positive definite symmetric bilinear form for each fiber

Def 1 A <u>metric</u> g on a real vector bundle $E \to M$ is a section $M \longrightarrow E^* \otimes E^*$. s.t. for each $p \in M$

1) $g|_p(v,v) > 0$ $\quad v \neq 0 \in E|_p$ (positive definite)

2) $g|_p(v,w) = g|_p(w,v)$ (symmetric)

$v, w \in E$.

Rem: By 2), we can choose a basis of $E|_p \cong \mathbb{R}^n$ s.t. the matrix representing $g|_p$ is diagonal.

i.e. $g|_p(v,w) = w^T \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} v = \sum_{i=1}^{n} \lambda_i v_i w_i$

By 1): $\lambda_i > 0$

Def 2 An <u>Hermitian metric</u> g on a complex vector bundle $E \to M$ is a section $M \to \overline{E}^* \otimes E^*$ s.t. for $p \in M$

1) $g(v,v) \in (0, \infty)$ $\quad v \neq 0 \in E|_p$

2) $g(v, cw) = c\, g(v,w)$

3) $g(cv, w) = \bar{c}\, g(v,w)$

4) $\overline{g(v,w)} = g(w,v)$

Rem Similarly, at each pt, we can choose a basis s.t.

$g|_p$ is diagonal $g|_p(v,w) = \sum_{i=1}^{n} \lambda_i \overline{v_i} w_i$ $\lambda_i \in (0,\infty)$

Prop: Any real / cpx v.b. admits an (Hermitian) metric

We use partition of unity to construct it.

pf. For a chart $U \subset M$. $\lambda_U : \pi^{-1}(U) \to \mathbb{R}^n (\mathbb{C}^n)$

Given a map $g_U : U \to U \times (\mathbb{R}^* \otimes \mathbb{R}^*)$ satisfying

the positive definite, symmetric condition on each fiber.

we obtain a metric on U by

$$g_U|_p(\lambda_U v, \lambda_U w) \quad v, w \in E|_p$$

We can choose a locally finite atlas $\mathcal{U}$ of M

and construct the metric on each $U \in \mathcal{U}$

To put them together, we notice that if g_1, g_2 are

two metrics, then $s g_1 + t g_2$ is still a metric

for $s,t \geq 0$. s,t are not both 0.

(Just check Condition 1) & 2))

Lem. Suppose $\mathcal{U}$ is a locally finite atlas of M. Then there exists a set of smooth functions $\{\chi_\alpha : M \to [0,\mathbb{R}]\}_{U_\alpha \in \mathcal{U}}$ called a <u>partition of unity</u>, satisfying the following conditions

1) χ_α has support in U_α (i.e. $\{p \in M \mid \chi_\alpha(p) \neq 0\} \subset U_\alpha$)

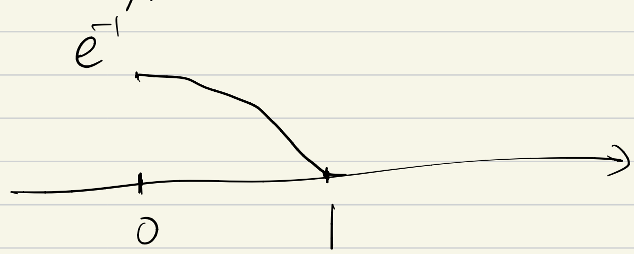
2) $\sum_\alpha \chi_\alpha(p) = 1$ for any $p \in M$.

Note that the locally finite condition + 1) implies at each p only finitely many χ_α is nonzero.

Pf: (See Cliff's book Appendix 1.2)

First we construct a function $\chi : [0,\infty) \to \mathbb{R}$

$$\chi(t) = \begin{cases} e^{\frac{1}{t-1}} & t < 1 \\ 0 & t \geq 1 \end{cases}$$



This function is smooth.

$$\lim_{t \to 1^-} \frac{d(e^{\frac{1}{t-1}})}{dt} = \lim_{t \to 1^-} e^{\frac{1}{t-1}} \cdot \left(-\frac{1}{(t-1)^2}\right) = 0$$

Similar for higher derivatives

Then for $\phi_{U_\alpha} : U_\alpha \to \mathbb{R}^n$. we construct $G_\alpha : M \to \mathbb{R}$

$$G_\alpha(p) = \begin{cases} \chi\left(\frac{|\phi_{U_\alpha}(p)|}{r_{U_\alpha}}\right) & p \in U_\alpha \\ 0 & p \notin U_\alpha \end{cases}$$

We can choose r_{U_α} s.t. $\{\varphi_{U_\alpha}^{-1}(B(r_{U_\alpha}))\}_\alpha$ covers M.

Then $\sum_\alpha \sigma_\alpha$ is nowhere zero

We define $\chi_\alpha = \left(\sum_\alpha \sigma_\alpha\right)^{-1} \sigma_\alpha$ $\square$

Using χ_α, we can define $g = \sum_\alpha \chi_\alpha g_{U_\alpha}$ as a metric on M $\square$

Recall a v.b. can be constructed by bundle transition functions $\{g_{VU} : U \cap V \to GL(n, \mathbb{R})\}$ with some conditions

If we have a metric, then we can choose $\operatorname{Im} g_{VU} \subset O(n) \subset GL(n, \mathbb{R})$ as follows

First, locally we have $\varphi_U : \pi^{-1}(U) \to U \times \mathbb{R}^n$ and a basis of sections $\{s_i : U \to \pi^{-1}(U)\}$

Since we have a metric on each fiber, we can use the Gram-Schmid procedure to obtain another basis of sections $\{s_i'\}$ which are orthogonal with respect to the metric at each fiber. Then we define a new diffeo $\varphi_U' : \pi^{-1}(U) \to U \times \mathbb{R}^n$ by setting

$$(\varphi_U')^{-1}(p, (v_1, \cdots, v_n)) = \sum_{i=1}^{n} v_i s_i'(p)$$

Then the bundle transition function $g_{VU} = \varphi'_V \circ (\varphi'_U)^{-1}$

sends orthogonal basis to orthogonal basis

so $g_{VU} : U \cap V \to O(n)$

If furthermore a real v.b. E is orientable, i.e.

$\Lambda^n E$ is isomorphic to $M \times \mathbb{R}$, then we have

$\det g_{VU}(p) > 0$. so $g_{VU} : U \cap V \to SO(n)$

(The groups $GL(n, \mathbb{R})$, $O(n)$, $SO(n)$ for g_{VU}

are called <u>structure groups</u> of the bundles)

The existence of metric $\Longleftrightarrow$ str gp can be reduced to $O(n)$

- - — metric + orientation $\Longleftrightarrow$ str. gp - - - - $SO(n)$

- - Hermitian metric for complex v.b. $\Longleftrightarrow$ $U(n) \subset GL(n, \mathbb{C})$

(almost complex str) $\subsetneq GL(2n, \mathbb{R})$

Hermitian metric + orientation $\Longleftrightarrow$ $SU(n)$

- - - symplectic str. $\Longleftrightarrow$ Some str gps.

More facts: 1) a metric induces an isomorphism $E \xrightarrow{\cong} E^*$

2) an Hermitian metric induces a $\mathbb{C}$ anti-linear isomorphism

$$E \xrightarrow{\cong} E^*$$

3) an metric on E induces a metric on E^*, $E^{\otimes k}$

$\Lambda^k E$, $\mathrm{Sym}^k E$. metrics on E_1, E_2 induce

a metric on $E_1 \oplus E_2$ $\mathrm{Hom}(E_1, E_2)$. $E_1 \otimes E_2$

4) For a smooth map $f: M \to N$ and a pull-back

$f^*E \longrightarrow M$ of $E \to N$, the metric on E

induces a metric on f^*E.

Def. A metric on TM is called a <u>Riemannian metric</u>.

also called a metric of M.

Rem. Metrics on the submfds can be induced by the

metric of the ambient space.

Any M can be embedded into $\mathbb{R}^N$; the metric

of $\mathbb{R}^N$ also induces a metric on M