Class 6 tangent bundle and Cotangent bundle.

Recall the tangent bundle TM is defined by the bundle transtion function $g_{VU} = (h_{VU})_*$ where $h_{VU} = \phi_V^{-1} \circ \phi_U$ is the tranistion function of M

A section $s: M \to TM$ is called a vector field.

A derivation $\mathcal{L}$ is a map from $C^\infty(M, \mathbb{R})$ to itself s.t. 1) $\mathcal{L}(f+g) = \mathcal{L}(f) + \mathcal{L}(g)$ 2) $\mathcal{L}(r) = 0$ for constant r

3) $\mathcal{L}(fg) = \mathcal{L}(f)g + f\mathcal{L}(g)$

From a vector field s, we define a derivation $\mathcal{L}_s$ called Lie derivative of s

Locally $U \subset M$ $\phi_U : U \to \mathbb{R}^n$ $\phi_{U*} : TM|_U \to U \times \mathbb{R}^n$

$\phi_{U*} \circ s|_U : U \to U \times \mathbb{R}^n$ $\quad V_k : U \to \mathbb{R}$

$(Id, V_1, \dots, V_n)$

$$(\mathcal{L}_s f)|_U = \sum_k V_k \frac{\partial (f \circ \phi_U^{-1})}{\partial x^k}$$ indep of the choice of U.

So we usually write $s|_U = \sum V_k \frac{\partial}{\partial x^k}$

For another chart V. we write $\frac{\partial}{\partial y^k}$ as basis and $s|_V = \sum V_k' \frac{\partial}{\partial y^k}$. we have

$$\frac{\partial(f\circ\phi_V^{-1})}{\partial y^k} = \frac{\partial(f\circ\phi_U^{-1}\circ\phi_U\circ\phi_V^{-1})}{\partial y^k} = \sum_l \frac{\partial(f\circ\phi_U^{-1})}{\partial x^l}\frac{\partial(x_l\circ\phi_U\circ\phi_V^{-1})}{\partial y^k}$$

$\phi_U\circ\phi_V^{-1} : \mathbb{R}^n$ (with basis y^k) $\longrightarrow \mathbb{R}^n$ (with basis x^k)

So $\dfrac{\partial}{\partial y^k} = \displaystyle\sum_l \frac{\partial(x_l\circ\phi_U\circ\phi_V^{-1})}{\partial y^k}\frac{\partial}{\partial x^l}$ ($x_l : \mathbb{R}^n \to \mathbb{R}$ projection to l-th coordinate)

and $\displaystyle\sum_l V_l \frac{\partial}{\partial x_l} = \sum_k V_k \frac{\partial}{\partial x_k} = \sum_k \sum_l V_k' \frac{\partial(x_l\circ\phi_U\circ\phi_V^{-1})}{\partial y^k}\frac{\partial}{\partial x_l}$

$$\Longrightarrow V_l = \sum_k V_k' \frac{\partial(x_l\circ\phi_U\circ\phi_V^{-1})}{\partial y^k}$$

$$\Longrightarrow \begin{pmatrix} V_1 \\ \vdots \\ V_n \end{pmatrix} = (h_{UV})_* \begin{pmatrix} V_1' \\ \vdots \\ V_n' \end{pmatrix}$$

Conversely, given a derivation $\mathcal{L}$. we construct a vector field as follows

Let $V_k : U \to \mathbb{R}$ be defined by

$\mathcal{L}(x_k\circ\phi_U)$ where $x_k : \mathbb{R}^n \to \mathbb{R}$ is the projection

Let $\dfrac{\partial}{\partial x^k} : U \to TM|_U \cong U\times\mathbb{R}^n$ be the section

correspondy to basis of $\mathbb{R}^n$ set $S = \displaystyle\sum_k V_k \frac{\partial}{\partial x^k}$

Exercise : for any $f : M \to \mathbb{R}$

$$(\mathcal{L}f)|_U = (L_S f)|_U = \sum_k V_k \frac{\partial(f\circ\phi_U^{-1})}{\partial x^k}$$

A function $f: M \to \mathbb{R}$ define a vector field ∇f as follows.

for a chart $U \subset M$, $\phi_U : U \to \mathbb{R}^n$

Define $f_U = f \circ \phi_U^{-1} : \mathbb{R}^n \to \mathbb{R}$ Consider $(f_U)_* = \left(\frac{\partial f_U}{\partial x_1}, \dots \frac{\partial f_U}{\partial x_n}\right)$

Let $(\nabla f)|_U = \sum_k \frac{\partial f_U}{\partial x_k} \frac{\partial}{\partial x_k}$.

Def The <u>cotangent bundle</u> T^*M is defined by $g_{VU} = ((h_{VU})_*^{-1})^T$

A section $s : M \to T^*M$ is called a <u>1-form</u>

Similar to the definition of $\frac{\partial}{\partial x^k}$

Given $U \subset M$. we define $\{dx^k\}$ to be the sections corresponding to basis of $\mathbb{R}^n$.

$\phi_{U_*} : T^*M|_U \to U \times \mathbb{R}^n$

Then for $f: M \to \mathbb{R}$, define a 1-form df by

$$df = \sum \frac{\partial f_U}{\partial x^k} dx^k$$

Algebra of vector bundles (Chap 4)

Def Given a vector bundle $E \to M$ with bundle transition functions $\{g_{VU}\}$. Then define the <u>dual</u> bundle $E^* \to M$ by $\{(g_{VU}^{-1})^T\}$

Rem T^*M is the dual bundle of TM.

$E^*|_p \cong \mathrm{Hom}(E|_p, \mathbb{R})$ $(E^*)^* = E$

The direct sum of two bundles $E_1, E_2 \to M$ is given by $E_1 \oplus E_2 = \{(v_1, v_2) \in E_1 \times E_2 \mid \pi_1(v_1) = \pi_2(v_2)\}$

Rem. $(E_1 \oplus E_2)|_p = E_1|_p \oplus E_2|_p$

$$g_{VU} = g_{VU,1} \oplus g_{VU,2}$$

The bundle $\mathrm{Hom}(E_1, E_2)$ is defined by

$\mathrm{Hom}(E_1, E_2)|_p = \mathrm{Hom}(E_1|_p, E_2|_p)$

If $\dim E_1|_p = n$ $\dim E_2|_p = k$

then $\dim \mathrm{Hom}(E_1, E_2)|_p = nk$

$\mathrm{Hom}(E_1, E_2)|_U \cong U \times M(k, n)$

$M(k,n)$ $k \times n$ matrices

the bundle transition function sends $m_U \in M(k,n)$

to $m_V = \underset{k\times k}{g_{VU,2}}\, \underset{k\times n}{m_U}\, \underset{n\times n}{(g^{-1}_{VU,1})^T}$

$$\begin{bmatrix} & \end{bmatrix}\begin{bmatrix} & & \end{bmatrix}\begin{bmatrix} & & \\ & & \end{bmatrix}$$

The bundle $E_1 \otimes E_2$ can be defined by $\mathrm{Hom}(E_1^*, E_2)$