Class 5 Tangent bundle

Recall the definition of vector bundle

Def 1: Let M be a smooth mfd of dim m.

A smooth mfd E is called a <u>(real) vector bundle</u> over M with fiber dimension n if we have the following

1) There is a smooth map $\pi: E \to M$

for each $p \in M$, $\exists\, U_p \subset M$ $\lambda_U: \pi^{-1}(U_p) \to \mathbb{R}^n$

s.t. for each $x \in U_p$. $\lambda_U: \pi^{-1}(x) \to \mathbb{R}^n$

is an diffeomorphism

2) There is a smooth map $\hat{o}: M \to E$ s.t. $\pi \circ \hat{o} = Id$

3) There is a smooth map $\mu: \mathbb{R} \times E \to E$ s.t.

a) $\pi(\mu(r,v)) = \pi(v)$

b) $\mu(r, \mu(r',v)) = \mu(rr', v)$

c) $\mu(1,v) = v$

d) $\mu(r,v) = v$ for $r \neq 1$ iff $v \in \operatorname{Im} \hat{o}$

We have an alternative way to define E

Def 2. Fix a locally finite open cover $\mathcal{U}$ of M

For $U, V \in \mathcal{U}$ s.t. $U \cap V \neq \phi$. choose a function

$g_{VU}: U \cap V \to GL(n, \mathbb{R})$ called <u>bundle transition function</u>

$$E = \bigcup_{U \in \mathcal{U}} U \times \mathbb{R}^n / (p,u) \in U \times \mathbb{R}^n \sim (p, g_{VU} \cdot u) \in V \times \mathbb{R}^n \quad p \in U \cap V$$

$\{g_{VU}\}_{U,V\in\mathcal{U}}$ need to satisfy the following condition

1) $g_{VU} = g_{UV}^{-1}$ $\quad U\cap V \neq \phi$

2) $g_{VU}\circ g_{UW}\circ g_{WU} = Id$ $\quad U\cap V\cap W \neq \phi$

called cocycle condition

The reason to introduce g_{VU} is the following

From Def I, for any $p\in M$, we have a nbhd $U\subset M$

s.t. $E|_U = \pi^{-1}(U) \xrightarrow{\cong} U\times\mathbb{R}^n$

$$v \longmapsto (\pi(v), \lambda_U(v))$$

If $v_1, v_2 \in E|_p$, then we can define

$$v_1+v_2 = (\lambda_U|_p)^{-1}(\underbrace{\lambda_U|_p(v_1)}_{\mathbb{R}^n} + \underbrace{\lambda_U|_p(v_2)}_{\mathbb{R}^n}) \in E|_p$$

The definition should be independent of the choice of U

if p is also in V, we set $e_i = \lambda_U|_p(v_i)$ $\quad i=1,2$

$$v_1+v_2 = (\lambda_U|_p)^{-1}(e_1+e_2) = (\lambda_V|_p)^{-1}(\lambda_V|_p\circ(\lambda_U|_p)^{-1}(e_1) + \lambda_V|_p\circ(\lambda_U|_p)^{-1}(e_2))$$

Let $\Psi_{VU} := \lambda_V\circ(\lambda_U)^{-1} : U\cap V\times\mathbb{R}^n \longrightarrow \mathbb{R}^n$

Then $\Psi_{VU}|_p(e_1+e_2) = \Psi_{VU}|_p(e_1) + \Psi_{VU}|_p(e_2)$

Since $e_1, e_2 \in\mathbb{R}^n$ can be arbitary, $\Psi_{VU}|_p$ is linear.

i.e. $\Psi_{VU}|_p \in GL(n,\mathbb{R})$

Hence we have a map $g_{VU}: U \cap V \to GL(n, \mathbb{R})$

$g_{VU}(p) = \psi_{VU}|_p$ this map is smooth because ψ_{VU} is.

$g_{VU} \circ g_{UV} = Id \quad g_{VU} \circ g_{UW} \circ g_{WV} = Id$

by construction. Conversely. Given $\{g_{VU}\}$. we can also recover the vector bundle in Def 1.

The second def is useful to construct the tangent bundle:

Def M is a smooth mfd of dim n. The <u>tangent bundle</u> of M is a vector bundle TM constructed as follows.

Fix a locally finite coordinate atlas $\mathcal{U}$ for M

For $U, V \in \mathcal{U}$ with $U \cap V \neq \phi$, we have diffeo

$$\phi_U: U \to \mathbb{R}^n \quad \phi_V: V \to \mathbb{R}^n$$

$h_{VU} = \phi_V^{-1} \circ \phi_U : \mathbb{R}^n \to \mathbb{R}^n$ is the transition function.

Let $g_{VU} = (h_{VU})_* =$ the Jacobian of h_{VU} $\in GL(n, \mathbb{R})$

$$h_{VU} \circ h_{UV} = Id \xRightarrow{\text{chain rule}} g_{VU} \circ g_{UV} = Id$$

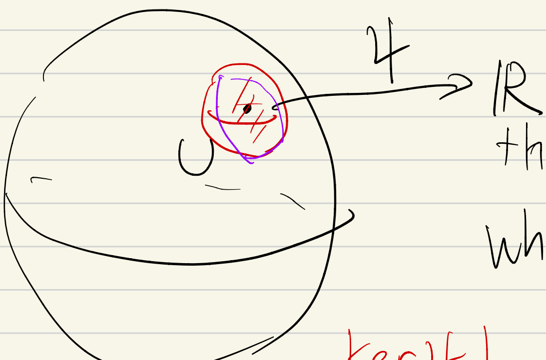
$$h_{VU} \circ h_{UW} \circ h_{WV} = Id \Rightarrow g_{VU} \circ g_{UW} \circ g_{WV} = Id$$

Rem The construction is indeed independent of $\mathcal{U}$ (see Cliff's book 3.4)

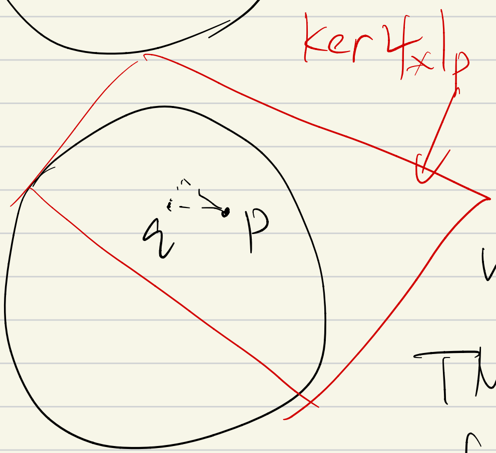
As a result. for any chart $U \subset M$. we have $TM|_U \xrightarrow{\cong} U \times \mathbb{R}^n$, we denote the diffeo by $(\phi_U)_*$

Prop If $M \subset \mathbb{R}^N$ is a submfd of dim n. by definition, for each $p \in M$. $\exists$ nbhd $U \subset \mathbb{R}^N$, $\psi: U \to \mathbb{R}^{N-n}$ with 0 as the regular value i.e. the Jacobian ψ_* is surjective at each pt in $\psi^{-1}(0)$, and $M \cap U = \psi^{-1}(0)$

$N=3$ $n=2$

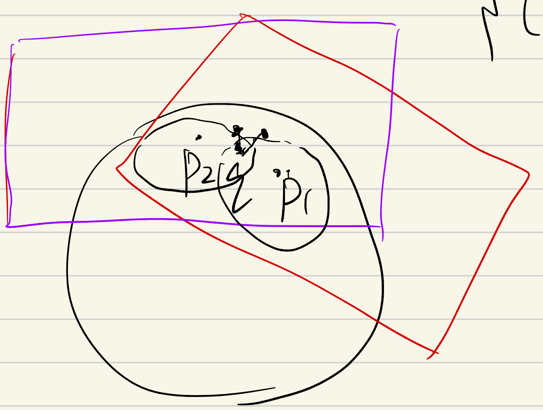


$\ker \psi_*|_p$ is identified with the subspace $\mathbb{R}^n$ tangent to M at p. which is independent of the choice of U



For a pt $q \in M$ near p. the projection of $q-p$ to $\ker \psi_*|_p$ is a diffeomorphism to a ball $\subset \ker \psi_*|_p$ which gives a chart for M

TM can be regarded as subset $\{(p,v) \in M \times \mathbb{R}^N \mid p \cdot v = 0\}$



Def A section $s: M \to TM$ is called a <u>vector field</u>

Let $C^\infty(M;\mathbb{R})$ be the space of smooth functions $M \to \mathbb{R}$ with addition and multiplication

A <u>derivation</u> is a map $\mathcal{L}$ from $C^\infty(M;\mathbb{R})$ to itself s.t. 1) $\mathcal{L}(f+g) = \mathcal{L}(f) + \mathcal{L}(g)$ 2) $\mathcal{L}(r) = 0$ for constant function r

3) $\mathcal{L}(fg) = (\mathcal{L}f)g + f(\mathcal{L}g)$ (Leibniz rule)

We can identify a derivation with a vector field

Given a vector field $s: M \to TM$. we construct a derivation $\mathcal{L}_s$ called <u>Lie derivative</u>

Let $U \subset M$ be a chart $\phi_U: U \to \mathbb{R}^n$

$$\phi_{U*}: TM|_U \xrightarrow{\cong} U \times \mathbb{R}^n$$

For $f: M \to \mathbb{R}$, define $f_U = f \circ \phi_U^{-1}: \mathbb{R}^n \to \mathbb{R}$

Suppose $\phi_{U*} \circ s|_U = (Id_U, v_1, v_2, \dots v_n)$

$v_k: U \to \mathbb{R}$

$$\text{Define } (L_s f)_U = \sum_{k=1}^{n} v_k \frac{\partial f_U}{\partial x^k} : U \to \mathbb{R}$$

Exer: check $L_s f: M \to \mathbb{R}$ is independent of the choice of U by the bundle transition function of TM

Because $(L_S f)_U = \sum_{k=1}^{n} V_k \frac{\partial f_U}{\partial x^k}$. We may write $S|_U = \sum_{k=1}^{n} V_k \frac{\partial}{\partial x^k}$ and write $\{\frac{\partial}{\partial x^k}\}$ as a basis of sections for $T\mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n$

This notation is important becauce in the future we usually do calculation locally and write a vector field as $\sum_{k=1}^{n} V_k \frac{\partial}{\partial x^k}$