Class 4 Vector bundle (Chap 3)

Manifold + group = Lie group

Manifold + vector space = vector bundle

Def 1: Let M be a smooth mfd of dim m.

A smooth mfd E is called a <u>(real) vector bundle</u> over M with fiber dimension n if we have the following

1) There is a smooth map $\pi : E \to M$

for each $p \in M$, $\exists\, U_p \subset M$ $\lambda_U : \pi^{-1}(U_p) \to \mathbb{R}^n$

s.t. for each $x \in U_p$. $\lambda_U : \pi^{-1}(x) \to \mathbb{R}^n$

is an diffeomorphism

2) There is a smooth map $\hat{o} : M \to E$ s.t. $\pi \circ \hat{o} = Id$

3) There is a smooth map $\mu : \mathbb{R} \times E \to E$ s.t.

a) $\pi(\mu(r, v)) = \pi(v)$

b) $\mu(r, \mu(r', v)) = \mu(rr', v)$

c) $\mu(1, v) = v$

d) $\mu(r, v) = v$ for $r \neq 1$ iff $v \in \operatorname{Im} \hat{o}$

Ex. The product bundle (also called the trivial bundle)

$E = M \times \mathbb{R}^n$

Rem 1) π is called the (bundle) <u>projection map</u>

For $W \subset M$, write $E|_W = \pi^{-1}(W)$

For $p \in M$, $E|_p$ is called a <u>fiber</u> over p

λ_U defines a diffeo $\varphi_U : E|_U \to U \times \mathbb{R}^n$ by

$(\pi(v), \lambda_U(v))$ φ_U is called <u>local trivialization</u> of E

2) $\hat{0}$ or its image is called the zero section

$\lambda_U \circ \hat{0} = 0 \in \mathbb{R}^n$

3) μ corresponds to the scalar multiplication in the vector space

we usually write $\mu(r, v)$ as rv

Def 2: Let E, E' be two vector bundles over M.

A <u>section</u> of E is a smooth map $s : M \to E$ s.t. $\pi \circ s = Id$

A <u>homomorphism</u> $\phi : E \to E'$ is a smooth map s.t.

1) $\pi'(\phi(v)) = \pi'(v)$

2) $\phi(rv) = r\phi(v)$ (Hence $\phi(\hat{0}(p)) = \hat{0}'(p)$)

A <u>bundle isomorphism</u> is a homomorphism with an inverse $g : E' \to E$

Question: Is any bundle isomorphic to the trivial bundle?

Counterexample: Möbius bundle over S^1

$$\left\{ (\theta, v) \in S^1 \times \mathbb{R}^2 : \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} v = v \right\}$$

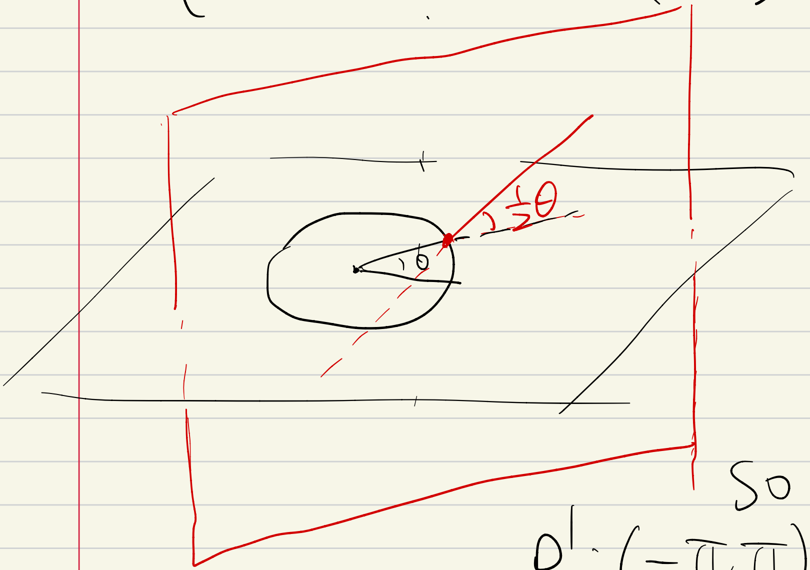
We have a smooth map

$\rho : (0, 2\pi) \times \mathbb{R} \longrightarrow E$

$(\theta, t) \longmapsto (\theta, t\cos(\tfrac{1}{2}\theta), t\sin(\tfrac{1}{2}\theta))$

$$\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} \begin{pmatrix} \cos\frac{1}{2}\theta \\ \sin\frac{1}{2}\theta \end{pmatrix} = \begin{pmatrix} \cos\theta\cos\frac{1}{2}\theta + \sin\theta\sin\frac{1}{2}\theta \\ \sin\theta\cos\frac{1}{2}\theta - \cos\theta\sin\frac{1}{2}\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos\frac{1}{2}\theta \\ \sin\frac{1}{2}\theta \end{pmatrix}$$



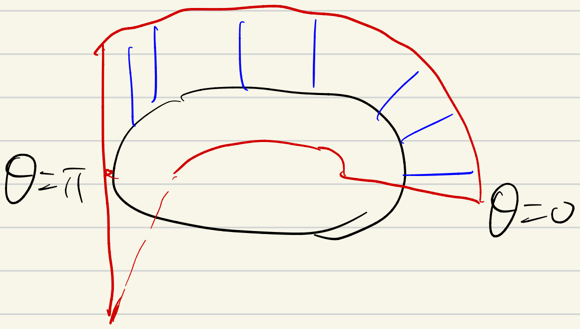
ρ is not continuous at

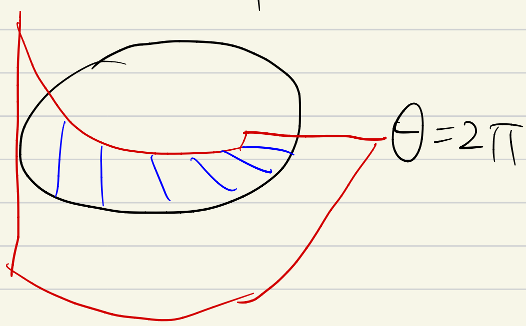
$\theta = 0 = 2\pi$

So we need the second chart

$\rho' : (-\pi, \pi) \times \mathbb{R} \longrightarrow E$

using the same map





We can find a nonzero section of $S^1 \times \mathbb{R} = E'$

(just use $s : S^1 \longrightarrow E'$ $s(\theta) = (\theta, 1)$

But we cannot find a nonzero section of Möbius bundle

(Intuitively, when we move $s(0)$ smoothly to $s(2\pi)$, we will have $s(0) \neq s(2\pi)$)

i.e. $\exists$ a bundle isomorphism $\phi: M \times \mathbb{R}^n \longrightarrow E$

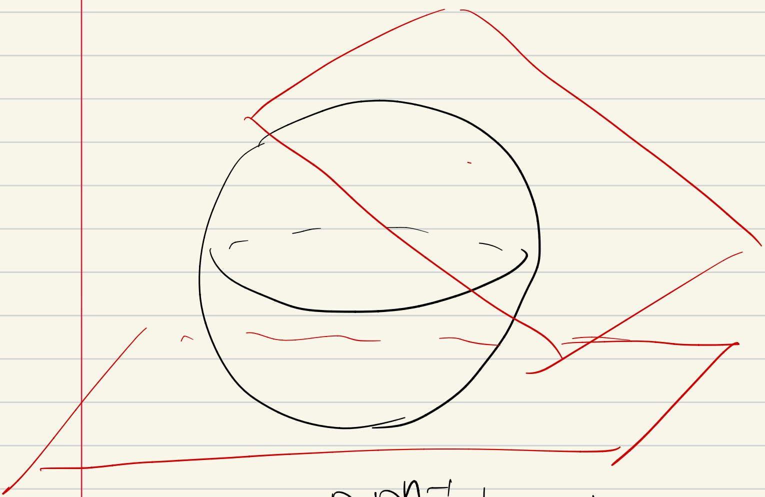
$$(x, \sum a_i e_i) \longmapsto \sum a_i s_i(x)$$

More examples of vector bundles.

- Let $S^2 = \{x \in \mathbb{R}^3 \mid |x| = 1\}$ $E = \{(x,v) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid \substack{|x|=1 \\ x \cdot v = 0}\}$

$\pi: E \to S^2$ $\quad \pi(x,v) = x$ $\quad r(x,v) = (x, rv)$ $r \in \mathbb{R}$

$\hat{0}: S^2 \longrightarrow E$ $\quad \hat{0}(x) = (x, 0)$



- Let $\mathbb{RP}^{n-1}$ be the space of lines through $0 \in \mathbb{R}^n$ (a pt is a line) It can also be regarded as $S^{n-1}/\{\pm 1\}$ where -1 acts on x by $-x$.

$\mathbb{RP}^2$ $\quad E = \{(\pm x, v) \in \mathbb{RP}^{n-1} \times \mathbb{R}^n \mid x // v\}$