Class 10 Geodesic (Chap 8)

Recall a path in M is a smooth map $\gamma : I = [0,1] \to M$. the length of γ is

$$L_\gamma := \int_I \left|\frac{d\gamma}{dt}\right| dt = \int_I g\left(\gamma_* \frac{\partial}{\partial t}, \gamma_* \frac{\partial}{\partial t}\right)^{\frac{1}{2}} dt$$

in $\mathbb{R}^n$ ... in M

where $\frac{\partial}{\partial t}$ is the second coordinate of $TI \cong I \times \mathbb{R}$ and $\gamma_* : TI \to TM$ is the tangent map (push forward) The distance $d(p,q) = \inf_{\substack{\gamma(0)=p \\ \gamma(1)=q}} L_\gamma$

Thm (Geodesic thm) M compact manifold

1) $\exists\, \gamma$ achieves $d(p,q)$. called geodesic

2) A geodesic is embedded, and can be reparameterized so that $g(\gamma_* \frac{\partial}{\partial t}, \gamma_* \frac{\partial}{\partial t}) = 1$

Ex.

3) Let $U \subset M$, $\phi_U : U \to \mathbb{R}^n$ $g|_U = \phi_U^*(\sum g_{ij}\, dx^i \otimes dx^j)$

We write $\gamma_U : \gamma^{-1}(U) \cap I \to \mathbb{R}^n$

$$t \longmapsto \phi_U \circ \gamma(t) = (\gamma_U^1(t), \gamma_U^2(t), \dots, \gamma_U^n(t))$$

Then
(*)
$$\frac{d^2}{dt^2}\gamma_U^i + \sum_{j,k=1}^n \Gamma^i_{jk} \frac{d}{dt}\gamma_U^j \frac{d}{dt}\gamma_U^k = 0$$

geodesic equation

for any i where

$$\Gamma^i_{jk} = \frac{1}{2}\sum_{l=1}^n g^{il}(\partial_j g_{lk} + \partial_k g_{jl} - \partial_l g_{jk})$$

$\{g^{ij}\}$ is the inverse matrix of $\{g_{ij}\}$

Γ^i_{jk} is called Christoffel symbols

4) Any path satisfying $(*)$ is locally length minimizing.

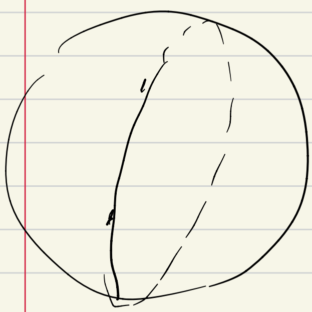
Ex

1) Geodesic in $\mathbb{R}^n$ with respect to the standard metric:

$$\partial_i g = 0. \quad (*) \Rightarrow \frac{d^2\gamma_i}{dt^2} = 0 \quad \Rightarrow \gamma_i(t) = at + b$$

i.e. geodesics in $\mathbb{R}^n$ are straight lines

2) Geodesic in S^2 with the induced metric from $\mathbb{R}^3$



the short segment of the great circle.

(see Cliff's book §.4.2 for Computations)

If p, q are two pole pts. then geodesics are not unique

Pf of Thm.

Step 1. $\mathbb{R}^n$ with the standard metric g

Consider the balls centered at the origin and there boundary spheres. Then $g = dr \otimes dr + r^2 g_S \rightarrow \mathbb{R}^n$

g_S is the induced metric on S^{n-1} with radius 1.

Let $\gamma : I \rightarrow \mathbb{R}^n$ be a path in a ball with radius R.

$\gamma(0) = 0 \in \mathbb{R}^n$ $\gamma(1)$ is on the boundary

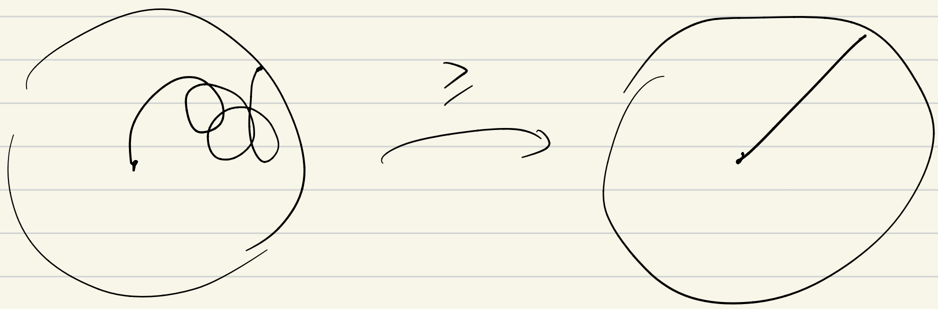
Write $\gamma(t)$ as $(r(t), \Theta(t))$ $r: I \to \mathbb{R}_{>0}$ $\Theta: I \to S^{n-1}$

$$g(\gamma_* \frac{\partial}{\partial t}, \gamma_* \frac{\partial}{\partial t}) = \left|\frac{\partial r}{\partial t}\right|^2 + r^2 g_S(\Theta_* \frac{\partial}{\partial t}, \Theta \frac{\partial}{\partial t})$$

$$\geq \left|\frac{\partial r}{\partial t}\right|^2$$ the equation holds iff Θ is constant

$$l_\gamma \geq \int_I \left|\frac{\partial r}{\partial t}\right| dt \geq \int_I \frac{\partial r}{dt} dt = R$$

the equation holds when γ is a straight line.



Moreover, we can reparameterize γ s.t. $g(\gamma_* \frac{\partial}{\partial t}, \gamma_* \frac{\partial}{\partial t}) = 1$

Step 2.

Lem (Special coordinates) If M is compact, then $\exists R > 0$ st. for each $p \in M$, $\exists$ chart $\phi_U: U \to \mathbb{R}^n$ s.t. $\phi_U(p) = 0$ and on a small ball B_R of $\mathbb{R}^n$, the metric g on $\phi_U^{-1}(B_R)$ is $\phi_U^*(dr \otimes dr + r^2 g_S)$ g_S metric on on S^{n-1}

Assuming the Lemma, we prove that for a fixed $p \in M$ and any $q \in M$ there exists a length minimizing curve from p to q.

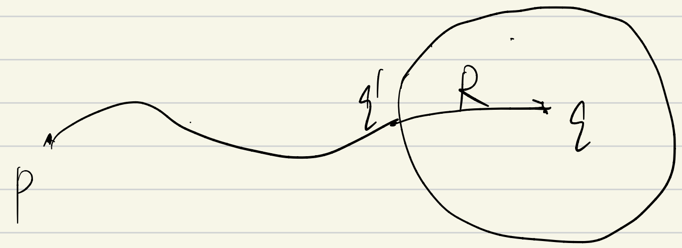
a) $d(p,q) \leq \frac{R}{2}$, such curve exists because q is in B_R for p

b) Suppose we already show the existence for q with $d(p,q) \leq \frac{NR}{2}$, we prove the existence for $\frac{NR}{2} < d(p,q) \leq \frac{(N+1)R}{2}$

There is a path γ_0 from p, q s.t. $l_{\gamma_0} < d(p,q) + \frac{R}{2}$

Suppose $q' = \mathrm{Im}\,\gamma_0 \cap \partial B_R(q)$

Then $d(p,q') \le l_{\gamma_0} - R < d(p,q) - \frac{R}{2} \le \frac{NR}{2}$



Let $q'' \in \partial B_R(q)$ s.t. $d(p, q'')$ is minimal.

we know $d(p, q'') \le d(p, q') < \frac{NR}{2}$

Then $\exists$ geodesics from p to q'', q'' to q.

the union of them is the geodesic from p to q

because it can't be shorten inside or outside the $B_R(q)$

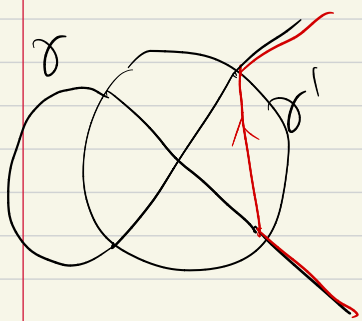
We can reperaterize it so that it has constant speed

Step 3. Nbhd of each pt on the geodesic is a straight line

some B_R, so γ is an immersion (locally injective)

If $\mathrm{Im}\,\gamma$ has self-intersection, then locally it is

two straight lines in B_R, if we replace γ by γ',



then $l_{\gamma'} < l_\gamma$. which contradicts the minimal

length condition Hence γ is an embedded curve

(locally injective and no self-intersection) we explain more on Lem 1.

Thm2 (vector field thm) Let $S: X \to TX$ be a vector field. For any $q \in X$, $\exists\, \varepsilon > 0$ and a unique path $\sigma_q : (-\varepsilon, \varepsilon) \to X$ s.t.

1) $\sigma_q(0) = q$ 2) $\sigma_{q*}\frac{\partial}{\partial t} = S(\sigma_q(t))$

Moreover, if $U \subset X$ is an open set with compact closure, then $(q,t) \longrightarrow \sigma_q(t)$ is a smooth map from $U \times [0,\varepsilon)$ to X s.t. for each t, $q \mapsto \sigma_q(q)$ is a diffeo from U to an open set of X

Pf. Cliff's book Appendix 8.1 $\square$

We take $X = TM$ locally $q = (p, v) \in X$

Define $S(p,v) = (v_i, -\Gamma^i_{jk} v_j v_k)$

Then we obtain a unique path

$$\sigma_q : (-\varepsilon, \varepsilon) \longrightarrow TM \quad \text{s.t. } \sigma_q(0) = q = (p,v)$$

$$\sigma_{q*}\frac{\partial}{\partial t} = S(\sigma_q(t))$$

This implies a unique path $\gamma_{(p,v)} : (-\varepsilon, \varepsilon) \to M$

s.t. $\gamma(0) = p$ $\dot{\gamma}_* \frac{\partial}{\partial t}\Big|_{t=0} = v$

and $$\frac{d^2\gamma^i}{dt^2} + \sum_{j,k} \Gamma^i_{jk} \frac{d\gamma^j}{dt}\frac{d\gamma^k}{dt} = 0$$

(Here we write $\gamma = \gamma_U = \phi_U \circ \gamma_{(p,v)} (-\varepsilon, \varepsilon) \longrightarrow \mathbb{R}^n$)