Class 2: Smooth manifolds (Chap 1)

Review: (some people use second countable, i.e. countable topo basis
this is equivalent to paracompact with countably many connected components)

Def 1. A topo mfd of dim n is a paracompact Hausdorff topo space s.t. each pt has a nbhd that is homeo to $\mathbb{R}^n$. such nbhd is called a chart

The collection of charts is called an atlas.

For two charts U, V. $\exists$ transition function

$$h_{V,U} : \underset{\mathbb{R}^n}{\underset{\cap}{\phi_U(U \cap V)}} \longrightarrow \underset{\mathbb{R}^n}{\underset{\cap}{\phi_V(U \cap V)}}$$

If $h_{V,U}$ is smooth (infinitely differentiable), then the atlas is called a smooth str.

A topo mfd with an smooth str is called a smooth mfd.

Rem. $\exists$ topo mfd that is not smoothable

$\exists$ topo mfd that has non unique smooth str

Def 2. Suppose M and N are smooth mfds.

Let $\mathcal{U}$ and $\mathcal{V}$ be locally finite atlases corresponding to smooth str of M and N (we use the paracompact condition)

A map $h: M \to N$ is called smooth if each map in

$$\{\phi_V \circ h \circ \phi_U^{-1} : h(U) \cap V \neq \emptyset\}_{(U,\phi_U) \in \mathcal{U},\ (V,\phi_V) \in \mathcal{V}}$$

is smooth

Ex: Consider the manifold $\mathbb{R}'$ with only one chart $\phi_1 : \mathbb{R}' \xrightarrow{x^3} \mathbb{R}$

and also the manifold $\mathbb{R}''$ with only one chart $\phi_2 : \mathbb{R}'' \xrightarrow{x} \mathbb{R}$

Define $f : \mathbb{R}' \xrightarrow{x^{\frac{1}{3}}} \mathbb{R}''$, then $\phi_2 \circ f \circ \phi_1^{-1} = x : \mathbb{R} \to \mathbb{R}$, f, g are smooth inverses

$g : \mathbb{R}'' \xrightarrow{x^3} \mathbb{R}'$. $\phi_1^{-1} \circ g \circ \phi_2 = x : \mathbb{R} \to \mathbb{R}$ $\mathbb{R}'$ is diffeo to $\mathbb{R}''$

The reason to define the smooth str is to do calculus on mfds

Thm 1 (Inverse function thm)

Let $U \subset \mathbb{R}^n$ be an open set and let $\psi : U \to \mathbb{R}^n$ be a smooth map. Choose a pt $p \in U$. Let ψ_* be the matrix $\left\{ \frac{\partial \psi^j}{\partial x^i} \right\}_{i,j \in [1,n]}$. It is called the Jacobian

If ψ_* is invertible, then $\exists$ nbhd $V \subset \mathbb{R}^n$ of $\psi(p)$ and a smooth map $G : V \to U$ s.t. $G(\psi(p)) = p$ and

- $G \circ \psi = \mathrm{Id}$ on nbhd $U' \subset U$ of p
- $\psi \circ G = \mathrm{Id}$ on V

i.e. ψ has a local inverse around p.

Idea: The smoothness assumption reduces a non-linear question to a linear one (first derivative). We will use Taylor's thm with remainder to prove it

Pf: $\psi(x) = \psi(p) + (\psi_*|_p)(x-p) + R(x-p)$ Remainder

$|R(u)| \le c|u|^2$ $|R(u) - R(v)| \le c|u-v|(|u|+|v|)$

Set $u = x-p = \psi_*^{-1}(\psi(x) - \psi(p) - R(u))$

Set $\psi(x) = q$. We need to find a pt

$u_q = \psi_*^{-1}(q - \psi(p) - R(u_q))$ if q is near $\psi(p)$

We introduce contraction mapping thm to find u_q

Lem 1. Let $r>0$ $\delta\in(0,1)$. $B=\{x\in\mathbb{R}^n \mid |x|<r\}$

If $f: B\to B$ satisfies

1) $|f(x)|<\delta r \quad \forall x\in B$

2) $|f(x)-f(y)|\le\delta|x-y| \quad \forall x,y\in B$

then $\exists$ unique $p\in B$ s.t. $f(p)=p$

Pf: Take any $x_0\in B$. Set $x_{n+1}=f(x_n)$.

Since $|x_n-x_m|<\delta^{n-m}r$ for $n>m$, we know

$\{x_n\}_{n\ge0}$ is a Cauchy sequence, which converges in B

Let p be the limit, then $f(p)=p$ $(f(x_n)=x_{n+1})$

If q also satisfies $f(q)=q$. then

$$|p-q|=|f(p)-f(q)|\le\delta|p-q| \Rightarrow p=q. \quad \square$$

We come back to the pf of inverse function thm.

Take $f(u)=\mathcal{F}_*^{-1}(q-\mathcal{F}(p)-R(u))$

Then f maps a small ball to itself because $|R(u)|\le c|u|^2$

$$\left(|u|\le\frac{1}{2c},\ |R(u)|\le\frac{1}{4c},\ \delta=\frac{1}{2}\right)$$

$$|f(u)-f(v)|\le|R(u)-R(v)|\le c|u-v|(|u|+|v|)<\delta|u-v|$$

when $|u|, |v|$ are small

We apply Lem 1 to obtain the unique element u_q s.t. $f(u_q)=u_q$

We can also use Lem 1 to analyze the function $q\mapsto u_q$.

to show it is smooth; see A 1.1.1 of Cliff's book.

Def 3: Let $U \subset \mathbb{R}^n$ be an open set. $\psi: U \to \mathbb{R}^m$ a smooth map

$a \in \mathbb{R}$ is a <u>regular value</u> of ψ if the Jacobian ψ_* is surjective

at any pt in $\psi^{-1}(a)$

Thm 2: (Sard thm) Let U, ψ be defined in Def 3. Then the set of

regular values of ψ have full measure.

Ex: $\psi: \mathbb{R}^n \to \mathbb{R}$ $\quad (x_1, \cdots, x_n) \to \sum_{i=1}^{n} x_i^2$

$$\psi_* = \left(\frac{\partial \psi}{\partial x_1}, \cdots, \frac{\partial \psi}{\partial x_n}\right) = (2x_1, \cdots, 2x_n)$$

$\forall r > 0$ is a regular value

Thm 3 (Implicit function thm) Let $U \subset \mathbb{R}^n$; $\psi: U \to \mathbb{R}^{n-m}$ $\quad n-m \geq 0$

and a is a regular value. Then $\psi^{-1}(a) \subset U$ is a smooth mfd

of dim m. For $p \in \psi^{-1}(a)$, $\exists$ a ball $B \in \mathbb{R}^n$ with center p

and a diffeo $\phi: B \to \mathbb{R}^n$

s.t. $\phi(B \cap \psi^{-1}(a)) = \{x_{m+1} = \cdots = x_n = 0\} \cap B_\varepsilon(0)$ for some $\varepsilon > 0$

Pf: Let $v_1, \cdots, v_{n-m}$ span $\ker \psi_*|_p$ and let π be

the projection: $\mathbb{R}^n \to \ker \psi_*|_p \cong \mathbb{R}^m$

Define $\phi: B \to \mathbb{R}^n$ $\quad \phi(x) = (\psi(x), \pi(x))$

Then $\phi_*|_p$ is invertible and we apply

the inverse function thm to find some local inverse G

Implicit function thm is a good way to construct

smooth mfds.

Def 4: A <u>submfd</u> M in $\mathbb{R}^n$ is a subset s.t. $\forall p \in M$ $\exists U_p \subset \mathbb{R}^n$ and $\psi_p : U_p \to \mathbb{R}^{n-m}$ with 0 as a regular value and $M \cap U_p = \psi_p^{-1}(0)$

Suppose N is a smooth mfd of dim n. A subset $M \subset N$ is a <u>submfd</u> if for any $p \in M$, $\exists U_p \subset N$

$\phi_p : U_p \to \mathbb{R}^n$ s.t. $\phi_p(M \cap U_p)$ is a submfd of $\mathbb{R}^n$.

Cor: Let $f : N \to M$ be a smooth map. Let $p \in N$. If for any $q \in f^{-1}(p)$, any chart $U_q \subset N$ with $\phi_q : U_q \to \mathbb{R}^n$, any chart $U_p \subset M$ with $\phi_p : U_p \to \mathbb{R}^m$, the Jacobian of

$\phi_p \circ f \circ \phi_q^{-1} : \mathbb{R}^n \to \mathbb{R}^m$ at $\phi_q(q)$ is surjective, then $f^{-1}(p)$ is a smooth submfd of N.

Fact: For any smooth mfd M, $\exists \phi : M \to \mathbb{R}^N$ $N \gg 0$ s.t. $\phi(M)$ is a submfd of $\mathbb{R}^N$,

and ϕ is a diffeomorphism.

Such ϕ is called an embedding.

The proof use a partition of unity. (see Appendix 1.2)