Math 230a : Differential geometry

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Course Page : http://scholar.harvard.edu/fanye/classes/math230a-differential-geometry

Main Reference : Differential geometry by Clifford H. Taubes Chap 1-16

Office Hours : Fan Tuesday 1:30 - 2:30 pm Science Center 505H

Clair Sunday 10:30 - 11:30 am by zoom

More information: see syllabus on the page.

Style of this course :

1. This is like a tour to a garden or a zoo, which means you will meet many new things but can't understand them well at the first time. Don't be afraid to discuss with classmates about basic definitions and examples, and go back to previous materials once and once again.

2. I study low-dimensional topology, so I will rearrange the materials in the book in a topologist way. That means I will focus more on the construction and motivation. The book contains many useful calculations It's good to read and calculate by yourself.

3. Topics : Manifolds, Lie groups, vector bundles, metrics, geodesics, connections, curvatures, principal bundles, Yang-Mills equations, ...

Class 1. Introduction to manifolds (Clift Chap 1)

Def 1. A topological manifold of dimension n is a paracompact, Hausdorff topological space s.t. each point (pt) has a neighborhood (nbhd) that is homeomorphic to $\mathbb{R}^n$

Explanation for red lines:

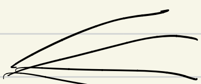
- topological space: a set X with distinguished collection $\mathcal{O}$ of subsets of X called open sets s.t
  1) $\phi, X \in \mathcal{O}$
  2) Any union of sets in $\mathcal{O}$ is still in $\mathcal{O}$
  3) finite intersections of sets in $\mathcal{O}$ is still in $\mathcal{O}$

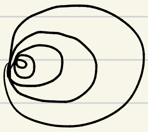
Ex: The standard topology on $\mathbb{R}^n$: $\mathcal{O}$ is the collection of sets given by unions and finite intersections of open balls

$$\{x \mid |x-p| < r \text{ for } p \in \mathbb{R}^n, r \in \mathbb{R}_+\}$$

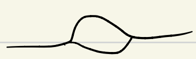
- paracompact: every open cover has a locally finite subcover
- locally finite: each pt is in finitely many open sets.
- Hausdorff: any two pts have disjoint, open nbhds

These two conditions exclude some "wild" topo space:

1) Union of rays with irrational slopes 

2) Hawaiian ring  $\bigcup_{n \in \mathbb{N}} \{(x,y) \mid (x-\frac{1}{n})^2 + y^2 = \frac{1}{n^2}$

(infinite union)

3) Bug-eyed line $\mathbb{R} \cup \mathbb{R} / x \sim y$ if $x = y \neq 0$ 

4) The line of irrational slope in $\mathbb{R}^2 / (x,y) \sim (x+m, y+n)$ $m, n \in \mathbb{Z}$

- nbhd of pt: an open set containing the pt
- homeomorphism: A continuous, 1-1 map with continuous inv
- Continuous: the preimage of an open set is an open set.

Usually we add the connected condition to a topo mfd

- Connected: it is not a disjoint union of two open sets

Fact: $\mathbb{R}^n$ is not homeo to $\mathbb{R}^m$ if $m \neq n$

So dim of a topo mfd is a definite number.

Ex of topo mfds:

1) $\mathbb{R}^n$, any open subset in $\mathbb{R}^n$

2) $S^n = \{ x \in \mathbb{R}^n \mid |x| = 1 \}$

3) $T^n = \mathbb{R}^n / (x_1, \cdots, x_n) \sim (x_1 + m_1, \cdots, x_n + m_n)$, $m_1, \cdots, m_n \in \mathbb{Z}$.

4) product of manifolds. Indeed $T^n = \underbrace{S^1 \times S^1 \times \cdots \times S^1}_{n \text{ copie}}$

Then we move to smooth manifolds.

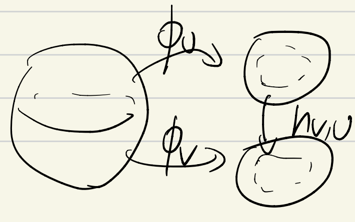
Def 2. For a topo mfd X, the nbhd of a pt is called a (coordinate) chart. A collection of charts that covers X is called a (coordinate) atlas.

If U, V are two charts of X. $U \cap V \neq 0$, then

$\exists\ \phi_U : U \to \mathbb{R}^n$, $\phi_V : V \to \mathbb{R}^n$ homeos

$$h_{V,U} = \phi_V \circ \phi_U^{-1} : \underset{\mathbb{R}^n}{\phi_U(U \cap V)} \longrightarrow \underset{\mathbb{R}^n}{\phi_V(U \cap V)}$$ is also a homeo

$\phi_U \subset \mathbb{R}^n$, $\phi_V(U \cap V) \subset \mathbb{R}^n$



It is called the (coordinate) transition function.

Given an atlas $\mathcal{U}$ and all trans functions $h_{V,U}$.

we can recover X by the quotient of the disjoint union

$$\coprod_{U \in \mathcal{U}} \mathbb{R}^n|_U \Big/ x \in \mathbb{R}^n|_U \sim h_{VU}(x) \in \mathbb{R}^n|_V, \ \forall U, V \in \mathcal{U}$$

Idea: • Given the atlas and transition functions we can always do computations locally

• We can understand the same manifold by different atlas

Def 3: An atlas is called a <u>smooth structure</u> if all transition functions are smooth (infinitely differentiable) or in C^∞

A topo mfd with a smooth str is called a <u>smooth mfd</u>

Two mfds are <u>diffeomorphic</u> if $\exists$ a smooth map btw them with a smooth inverse

Then we can do calculus on smooth manifolds

Fact. Smooth str is not unique for some topo mfds.

does not exist for some topo mfds.

For dimension n mfd,

If $n = 0, 1, 2, 3$, then we have existence and uniqueness of the smooth str.

If $n \geq 4$. there exist counterexamples

Two basic examples :

$\mathbb{R}^n$: $\exists$ unique smooth str if $n \neq 4$.

uncountably many if $n = 4$.

S^n : $\exists$ unique smooth str if

$n = 1, 2, 3, 5, 6, 12, 56, 61$

non unique if $n \geq 7$ $\neq 56, 61$ odd

$n = 7$ $\exists$ 28 smooth str. (Milnor)

$n \geq 8$ < 140 even

widely open : $n = 4$

3 : Poincaré conj Solved by Grigori Perelman (2002)

5, 6, 12 : Michel A. Kervaire and John W. Milnor (1963)

56 : Recent work of Daniel C. Isaksen (2019)

61 : Recent work of Guozhen Wang and Zhouli Xu (2017)

$\underbrace{1, 2, 3, 4,}_{\text{low-dimensional topology}} \underbrace{5, 6, 7, 8, \cdots\cdots}_{\text{algebraic topology}}$

even : Complex geometry. symplectic geometry.

Kähler geometry, ...

odd : contact geometry, ...