

Math 230a: Differential geometry.

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Main Reference: Differential geometry by Cliff Taubes Chap 1-16

Office Hours: Fan MW 4:20-5:20pm SC 505H

Keeley Tu 1:30-2:30 pm SC 333G

Style of this course:

0 Main difference with undergraduate DG: think object itself rather than its embedding

1. This is like a tour to a garden or a zoo,

which means you will meet many new things but can't understand them well at the first time.

Don't be afraid to discuss with classmates about basic definitions and examples, and go back to previous materials once and once again.

2. I study low-dimensional topology, so I will rearrange the materials in the book in a topologist way. That means I will focus more on the construction and motivation. The book contains many useful calculations. It's good to read and calculate by yourself.

3. Topics: Manifolds, Lie groups, vector bundles, metrics, geodesics, connections, curvatures, principal bundles, Yang-Mills equations, ...

Class 1. Introduction to manifolds (Cliff Chap 1)

Def 1. A topological manifold ^{of dimension n .} is a paracompact, Hausdorff topological space s.t. each point (pt) has a neighborhood (nbhd) that is homeomorphic to $\mathbb{R}^n$

Explanation for red lines:

- topological space: a set X with distinguished collection $\mathcal{O}$ of subsets of X called open sets s.t.

1) $\emptyset, X \in \mathcal{O}$

2) Any union of sets in $\mathcal{O}$ is still in $\mathcal{O}$

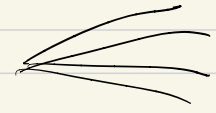
3) finite intersections of sets in $\mathcal{O}$ is still in $\mathcal{O}$

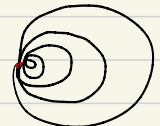
Ex: The standard topology on $\mathbb{R}^n$: $\mathcal{O}$ is the collection of sets given by unions and finite intersections of open balls

$$\{X \mid |x-p| < r \text{ for } p \in \mathbb{R}^n, r \in \mathbb{R}_+\}$$

- paracompact: every open cover has a locally finite refinement
- locally finite: each pt is in finitely many open sets.
- Hausdorff: any two pts have disjoint, open nbhds

These two conditions exclude some "wild" topo space:

1) Union of rays with irrational slopes 

2) Hawaiian ring  $\bigcup_{n \in \mathbb{N}} \{(x, y) \mid (x - \frac{1}{n})^2 + y^2 = \frac{1}{n^2}\}$
(not homeo to $\mathbb{R}^n$, still paracompact) (infinite union) (not Hausdorff)

3) Bug-eyed line $\mathbb{R} \cup \mathbb{R} / x \sim y \text{ if } x = y \neq 0$ 

4) The line of irrational slope in $\mathbb{R}^2 / (x, y) \sim (x+m, y+n) \text{ } m, n \in \mathbb{Z}$

- nbhd of pt: an open set containing the pt
- homeomorphism: A continuous, 1-1 map with continuous inv.
- Continuous: the preimage of an open set is an open set.

Usually we add the connected condition to a topo mfd

- Connected: it is not a disjoint union of two open sets

Fact: $\mathbb{R}^n$ is not homeo to $\mathbb{R}^m$ if $m \neq n$

So dim of a topo mfd is a definite number.

Ex of topo mfds:

1) $\mathbb{R}^n$, any open subset in $\mathbb{R}^n$

2) $S^n = \{x \in \mathbb{R}^n \mid |x| = 1\}$

3) $T^n = \mathbb{R}^n / (x_1, \dots, x_n) \sim (x_1 + m_1, \dots, x_n + m_n), m_1, \dots, m_n \in \mathbb{Z}$

4) product of manifolds. Indeed $T^n = \underbrace{S^1 \times S^1 \times \dots \times S^1}_{n \text{ copies}}$

Then we move to smooth manifolds.

Def 2. For a topo mfd X , the nbhd U of a pt with homeo

$\phi_U: U \rightarrow \mathbb{R}^n$ is called a (coordinate) chart.

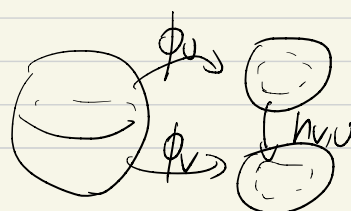
A collection of charts that covers X is

called a (coordinate) atlas.

If U, V are two charts of X . $U \cap V \neq \emptyset$, then

$h_{U,V} = \phi_V \circ \phi_U^{-1} : \underbrace{\phi_U(U \cap V)}_{\mathbb{R}^n} \rightarrow \underbrace{\phi_V(U \cap V)}_{\mathbb{R}^n}$ is also a homeo

It is called the (coordinate) transition function.



Given an atlas $\mathcal{U}$ and all trans functions $h_{V,U}$,

we can recover X by the quotient of the disjoint union

$$\coprod_{U \in \mathcal{U}} \mathbb{R}^n|_U / x \in \mathbb{R}^n|_U \sim h_{V,U}(x) \in \mathbb{R}^n|_V, \forall U, V \in \mathcal{U}$$

Idea: • Given the atlas and transition functions

we can always do computations locally

- We can understand the same manifold by different atlas

Def 3: An atlas is called a smooth structure if
all transition functions are smooth (infinitely differentiable)
or in C^∞

A topo mfd with a smooth str is called a smooth mfd

Two mfd's are diffeomorphic if $\exists$ a smooth map btw them
with a smooth inverse

Then we can do calculus on smooth manifolds

Fact. Smooth str is not unique for some topo mfd's.
does not exist for some topo mfd's.

For dimension n mfd,

If $n = 0, 1, 2, 3$, then we have existence and uniqueness
of the smooth str.

If $n \geq 4$, there exist counterexamples

Two basic examples :

$\mathbb{R}^n$: $\exists$ unique smooth str if $n \neq 4$.
uncountably many if $n = 4$.

S^n : $\exists$ unique smooth str if

$n = 1, 2, 3, 5, 6, 12, 56, 61$

non unique if $n \geq 7 \neq 56, 61$ odd

$n = 7 \quad \exists 28$ smooth str. (Milnor)

$n \geq 8 \quad < 140$ even

widely open : $n = 4$

3 : Poincaré conj Solved by Grigori Perelman (2002)

5, 6, 12 : Michel A. Kervaire and John W. Milnor (1963)

56 : Recent work of Daniel C. Isaksen (2019)

61 : Recent work of Guozhen Wang and Zhouli Xu (2017)

1, 2, 3, 4, 5, 6, 7, 8, ...

low-dimensional
topology

algebraic topology

even : complex geometry, symplectic geometry,
Kähler geometry, ...

odd : contact geometry, ...