Towards isomorphisms among Floer homologies

joint w/ Baldwin-Li-Sivek

(KM = Kronheimer-Mrowka
OS = Ozsváth-Szabó)

draw grid diagram first

(also table of SHI)

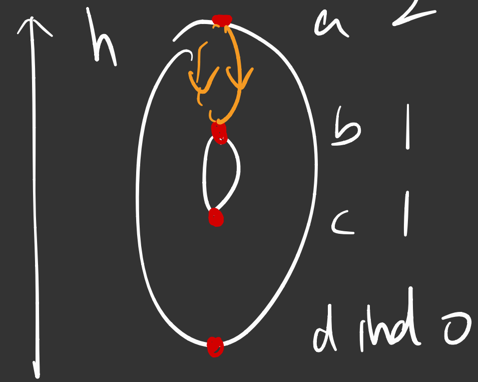
- (Elienberg-Steenrod '45) axioms of homology theories
- singular   topological space
- cellular   CW
- Morse   smooth mfd
- Without dim axiom, we get
$$\left(H_n(pt) = \begin{cases} 0 & n>0 \\ \mathbb{Z} & n=0 \end{cases} \right)$$
<u>generalized</u> homology theory.
(K-theory, bordism)

<u>Singular</u> : (easy to define, hard to compute)
(Any topo space) infinitely many generators and differentials

<u>cellular</u> : (hard to define, easier to compute)
(CW cpx) finite gens, diff by deg of maps

<u>Morse</u> : use Morse function,
(smooth mfd) finite gens, explicit diff

Morse theory



gen : critical pts

diff : # flow lines btw critical pts

$$C_\cdot = \mathbb{Z}\langle a, b, c, d\rangle$$

$$\partial a = (+1 + (-1))\, b = 0$$

$$H_*(T^2) = H_*(C) = \mathbb{Z}\langle \overset{2}{a}, \overset{1}{b}, \overset{1}{c}, \overset{0}{d}\rangle$$

(Floer ('88-89) generalized
Morse theory to ∞-dim space
now called Floer homology)

(Need compactness results
to be finite)

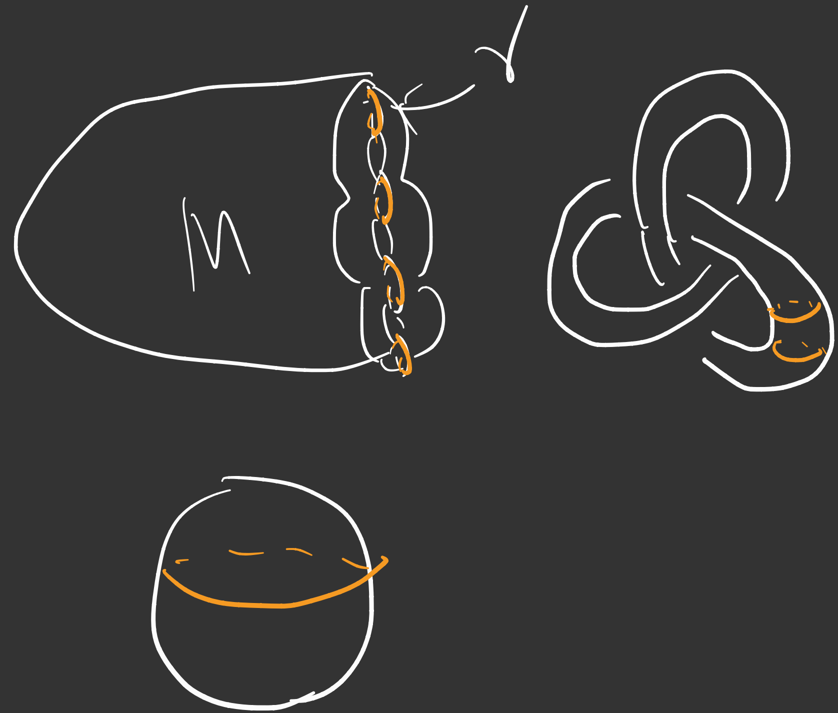
Floer theory
∞-dim space $\mathcal{B}$, functional
$\mathcal{F}: \mathcal{B} \longrightarrow \mathbb{R}$ or $\mathbb{R}/\mathbb{Z}$
Consider critical pts (grad $\mathcal{F} = 0$)
diff : counting soln of some
PDE (ASD, SW J-holo, etc)

- symp world : Ham, Lag, etc
- low-dim : instanton monopole.

Heegaard Floer
(Embedded contact homology)

	closed Y	knot K	suture (M,γ)
instanton	F, KM'08 (I^w) $I^\#$	F, KM'08 KHI	KM'08 SHI
monopole	KM'97-07 $\widetilde{HM}$ $\check{HM}$ $\widehat{HM}$ $\overline{HM}$	KM'08 KHM	KM'08 SHM
Heegaard Floer	OS'01 $\widehat{HF}$ HF^+ HF^- HF^∞	OS'02 Ras'03 $\widehat{HFK}$	Juh'06 (Ni'06) SFH

Ex of sutured manifold



Rem (best to say ;;)

- instanton/monopole gauge theory
  better functoriality (3+1 TQFT)
- HF more computable.
- instanton relates to SU(2) rep
  and Khovanov homology
- monopole better compactness,
  (Floer homotopy theory)
- HF extends to 2d
  (bordered theory)

Conj (KM'08) Over $\mathbb{C}$,
For (M,γ), $SHI = SHM = SFH$
For Y, K, similar (special cases)

- instanton only over $\mathbb{C}$
- HF over $\mathbb{Z}$
  but many properties over $\mathbb{Z}/2$
- don't expect iso over $\mathbb{Z}$ or $\mathbb{Z}/2$

(since instanton $SO(3)$ theory
HM/HF S^1 theory)
$H^*(SO(3), \mathbb{C})$ similar $H^*(S^1, \mathbb{C})$
but over $\mathbb{Z}/2$ diff.

potential counterexample: Poincaré sphere

Thm (Lekili '13, Baldwin–Sivek '21)

SHM = SFH

Rem. Based on thms for closed 3-mfds

HF = HM (Kutluhan–Lee–Taubes '10)

or HM = ECH = HF

(Taubes '08) (Colin–Ghiggini–Honda '12

latest version '23)

with Yuan Yao for appendix

Thm A (Baldwin–Li–Y. '20) '23

let $SFC(\mathcal{H})$ be the cpx with

$H_*(SFC) = SFH(M,\gamma)$

Then $\dim_{\mathbb{C}} SFC \geq \dim_{\mathbb{C}} SHI$

Here $\mathcal{H}$ is any admissible

Heegaard diagram for (M,γ)

Sivek Zemke

Thm B (BLSYZ in progress)

(If $H_2(M)=0$ in Thm A,)

$\exists$ a filtered diff d_I on SFC

s.t. $H_*(SFC, d_I) \cong SHI$

Thm C (BLSYZ in progress)

If K is a knot in S^3,

$d_I = d_1 + d_3 + \cdots$ by filtration

then $d_1 = d_{SFC}$

In particular $\dim_{\mathbb{C}} SFH \geq \dim_{\mathbb{C}} SHI$

Rem

- Grid diagram $\rightsquigarrow$ diffs are combinatorial
- Expect $H(d_I) = H(d_1)$

  (S.S. collapses at E_2)

  $\rightsquigarrow SFH = SHI$
- For (M,γ) in Thm B, d_1

  is 1-1 to diff in SFC

  (Prob signs $\mathbb{Z}/2$ vs. $\mathbb{C}$)
- Only uses formal properties

  $\rightsquigarrow$ axiomatic Floer theory