MATH 231A Final Exam

This final exam is due at 11:59 pm on Tuesday, Dec. 10, 2024. You are allowed to check lecture notes, references, and your submissions of previous assignments, but are not allowed to talk with classmates, the CA, or anyone else. You may use the theorems mentioned in the class if you state them correctly. If you want to use the results in your answers of previous submissions, please rewrite them and put them in your answer for the final exam.

Problem 1 (10 pts): For a pointed topological space X, describe the definitions of the augmented homology $\widetilde{H}_*(X)$ and the reduced homology $\overline{H}_*(X)$. Prove that there exists a canonical isomorphism between them.

Problem 2 (40 pts): Compute the (co)homology of the following spaces. You may use any techniques learned from the classes. Suppose m and k are positive integers.

1. $H_*(S^k)$ and $H^*(S^k)$ for k-dimensional sphere S^k.
2. $H_*(T^k)$ and $H^*(T^k)$ for k-dimensional torus T^k.
3. $H_*(\bigvee_{i=1}^m S^k)$ for the wedge product of spheres.
4. $H_*(\mathbb{RP}^2 \times \mathbb{RP}^3; \mathbb{Z}/2)$ for the projective spaces.

Problem 3 (20 pts): Suppose m, n are positive integers. Let Tor denote $\mathrm{Tor}_1^{\mathbb{Z}}$ and similarly for Ext. Compute the following groups.

1. $\mathrm{Tor}(\mathbb{Z}/n, \mathbb{Z}),\ \mathrm{Tor}(\mathbb{Z}, \mathbb{Z}/n),\ \mathrm{Tor}(\mathbb{Z}/n, \mathbb{Z}/m),\ \mathrm{Tor}(\mathbb{Z}/n, \mathbb{Q})$.
2. $\mathrm{Ext}(\mathbb{Z}/n, \mathbb{Z}),\ \mathrm{Ext}(\mathbb{Z}, \mathbb{Z}/n),\ \mathrm{Ext}(\mathbb{Z}/n, \mathbb{Z}/m),\ \mathrm{Ext}(\mathbb{Z}/n, \mathbb{Q})$.

Problem 4 (20 pts): Construct a CW complex X with homology as follows.

$$H_0(X) \cong \mathbb{Z},\ H_2(X) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/4,\ H_4(X) \cong \mathbb{Z},\ \text{and } H_n(X) = 0 \text{ for } n \neq 0, 2, 4.$$

Problem 5 (10 pts): For any finite CW complex X, show that $H^1(X)$ is a free abelian group.