MATH 230A Final Exam

This final exam is due at 11:59 pm on Sunday, December 10, 2023. You are allowed to check lecture notes, textbooks, and your submissions of previous assignments, but are not allowed to talk with classmates, the CA, or anyone else. You may use the theorems mentioned in the class if you state them correctly. If you want to use the results in your answers of previous submissions, please rewrite them and put them in your answer for the final exam.

We always suppose M is a smooth manifold.

Problem 1: Suppose g is a metric on M. Suppose $f : M \to \mathbb{R}$ is a smooth function. Describe the definition of the gradient of f as a vector field on M and prove it is well-defined. (Hint: first define it locally on a chart and use a partition of unity to define it globally, then prove it is independent of the chart.)

Problem 2: Let $T \subset \mathbb{R}^3$ be obtained by rotating the circle

$$\{(x, y, z) \Big| \; y = 0, z^2 + (x-2)^2 = 1\}$$

about the z-axis. Let T be parameterized by the coordinates $(\theta, \phi) \in [0, 2\pi] \times [0, 2\pi]$

$$f(\theta, \phi) = ((2 + \sin\phi)\cos\theta, (2 + \sin\phi)\sin\theta, \cos\phi).$$

Define the Riemannian metric g_0 on T by the induced metric from $\mathbb{R}^3$. Define the metric g on $S^1 \times S^1$ by the pull-back metric $f^* g_0$.

1. Compute the inner product $g(\frac{\partial}{\partial\theta}, \frac{\partial}{\partial\theta}), g(\frac{\partial}{\partial\theta}, \frac{\partial}{\partial\phi}), g(\frac{\partial}{\partial\phi}, \frac{\partial}{\partial\phi})$.
2. Let $x^1 = \theta, x^2 = \phi$. Compute the **Christoffel symbol**

   $$\Gamma^i_{jk} = \frac{1}{2} g^{il} (\partial_j g_{lk} + \partial_k g_{jl} - \partial_l g_{jk}),$$

   for all $i, j, k \in \{1, 2\}$, where $\{g^{il}\}_{i,l \in [1,n]}$ is the inverse matrix of $\{g_{il}\}_{i,l \in \{1,2\}}$.
3. Guess all geodesics from $p = (2, 0, 1)$ to $q = (2, 0, -1)$ in T. Show that they satisfy the geodesic equation.

Problem 3: For integer $n > 1$, suppose $B^n \subset \mathbb{R}^n$ is the unit ball and 0 is the origin point. Compute the de Rham cohomology $H^k_{dR}(B^n \backslash \{0\})$ for $k = 0$ and n. (Hint: you may need to compute $H^k_{dR}(S^{n-1})$ for $k = 0, n$.)

Problem 4: Let $E = M \times \mathbb{R}^n$ be the product vector bundle over M and let $C^\infty(M; E)$ be the space of smooth sections $s : M \to E$. Let $\nabla = \nabla^0 + \alpha$ be a covariant derivative on E with $\alpha \in C^\infty(End(E) \otimes T^*M)$, where ∇^0 is the standard covariant derivative from partial derivatives in $\mathbb{R}^n$.

1. Describe the definitions of the induced connections ∇^* and $\nabla^\otimes$ on the vector bundle E^* and $E \otimes E$. (You don't need to prove the uniqueness).
2. We write $\nabla^* = \nabla^{*0} + \beta$ with $\beta \in C^\infty(End(E^*) \otimes T^*M)$, where ∇^{*0} is the standard covariant derivative from partial derivatives in $(\mathbb{R}^n)^*$. Describe and prove the relation between α and β.

Problem 5: Recall that there is a 1-1 correspondence between vector field v on a smooth manifold M and derivation $\mathcal{L}_v : C^\infty(M;\mathbb{R}) \to C^\infty(M;\mathbb{R})$. For two vector fields v, w, define $[v, w]$ to be the vector field corresponding to the derivation

$$\mathcal{L}_{[v,w]}(f) = \mathcal{L}_v \circ \mathcal{L}_w(f) - \mathcal{L}_w \circ \mathcal{L}_v(f),$$

where $f \in C^\infty(M;\mathbb{R})$. Suppose on a local chart U, we write

$$v = \sum_i v^i \frac{\partial}{\partial x^i},\ w = \sum_j w^j \frac{\partial}{\partial x^j},\ \text{and } [v, w] = \sum_k u^k \frac{\partial}{\partial x^k}.$$

Compute the formula of u^k using v^i and w^j.