MATH 230A Assignment 5

Problem 1: Suppose $\nabla : C^\infty(M;E) \to C^\infty(M; E\otimes T^*M)$ is a covariant derivative. It induces a covariant derivative (which we still denote by ∇) $C^\infty(M; E\otimes E) \to C^\infty(M; E\otimes E\otimes T^*M)$ by Problem 4 of assignment 4. It is called **compatible** with a metric $g \in C^\infty(M; E\otimes E)$ if

$$\nabla g = 0.$$

Suppose ∇ and ∇' are two covariant derivative compatible with g. Suppose $\nabla' = \nabla + \alpha$, where $\alpha \in C^\infty(M; End(E)\otimes T^*M)$ is a matrix-valued 1-form (the standard notation is $\mathfrak{a}$ but I use α in the class for writing simplicity). Prove

$$\alpha^T \cdot g = -g\cdot \alpha$$

Problem 2: For simplicity, let $M = \mathbb{R}^n$. Let $dx^1, \cdots, dx^n$ be the basis of sections of $T^*\mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n$. Let

$$\mathbb{A} : \bigotimes_{i=1}^{k} T^*M \to \bigwedge^{k} T^*M$$

be the anti-symmetrization map that sends

$$dx^{i_1} \otimes \cdots \otimes dx^{i_k}$$

to

$$dx^{i^1} \wedge \cdots \wedge dx^{i_k}$$

and extends linearly over $C^\infty(M;\mathbb{R})$. Let $\nabla : C^\infty(M; T^*M) \to \nabla : C^\infty(M; T^*M\otimes T^*M)$ be a covariant derivative and define the **exterior covariant derivative**

$$d_\nabla : C^\infty(M; \bigwedge^{k} T^*M) \to C^\infty(M; \bigwedge^{k} T^*M \otimes T^*M)$$

by

$$d_\nabla(s\otimes \omega) = \nabla s \wedge \omega + s\otimes d\omega$$

for any $s \in C^\infty(M; T^*M), \omega \in C^\infty(M; \bigwedge^k T^*M)$ and then extending linearly, *i.e.*

$$d_\nabla(\sum_i s_i \otimes \omega_i) = \sum_i d_\nabla(s_i\otimes\omega_i).$$

Prove the following.

1. For any k-form ω and l-form η,

$$\mathbb{A}(d_\nabla(\omega\wedge\eta)) = \mathbb{A}(d_\nabla\omega\wedge\eta) + (-1)^k\omega\wedge\mathbb{A}(d_\nabla\eta).$$

2. For any k-form ω, define the **torsion tensor** $T_\nabla\omega = \mathbb{A}(d_\nabla\omega) - d\omega$. We say ∇ is **torsion-free** if $T_\nabla\omega = 0$ for any 1-form ω. If ∇ is torsion-free, prove $T_\nabla\omega = 0$ for any k-form ω.

Problem 3: Following Secton 16.1.1 in Cliff's book, suppose $S^n \subset \mathbb{R}^{n+1}$ is the sphere of radius $\rho > 0$, prove the curvature 2-form at $y = 0$ specified by

$$R_{abcd} = \rho^{-2}\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}.$$

For simplicity, you may suppose $n = 2$. You have to clarify the sketch in the section from the definitions (of Γ^a_{bc}, R_{ab}, *etc*) and add details in the proof.

Problem 4: Following Secton 16.1.3 in Cliff's book, suppose $M \subset \mathbb{R} \times \mathbb{R}^n$ is the manifold defined by $|x|^2 - t^2 = -\rho^2$ and $t > 0$, where t is the coordinate of $\mathbb{R}$, x is the coordinate of $\mathbb{R}^n$, and $\rho > 0$ is fixed. Prove the curvature 2-form at $(t = \rho, x = 0)$ is specified by

$$R_{abcd} = -\rho^{-2}\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}.$$

For simplicity, you may suppose $n = 2$. You have to clarify the sketch in the section from the definitions (of Γ^a_{bc}, R_{ab}, *etc*) and add details in the proof.

Problem 5: Read Section 16 in Cliff's book about more examples of Riemannian curvature tensor. You don't need to write down anything for this problem.