MATH 230A Assignment 4

Problem 1: Let M be a smooth manifold of dimension n and let $\Omega_k = C^\infty(M; \bigwedge^k T^*M)$ be the space of k-forms on M. On a chart, let $dx^1, \ldots, dx^n$ be a basis of 1-forms obtained by pull-back from $\mathbb{R}^n$. Suppose $f : M \to \mathbb{R}$ is a smooth map. Define the **exterior derivative** $d : \Omega_k \to \Omega_{k+1}$ by

$$d(f \cdot dx^{i_1} \wedge \cdots \wedge dx^{i_k}) = df \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k} = \sum_{m \neq \{i_1, \ldots, i_k\}} \frac{\partial f}{\partial x^m} \cdot dx^m \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

and then extending linearly for any general k-form.

1. Compute $d(x^1 x^2 x^3 dx^1)$. Note that x^i is the coordinate functions, not the i-th power.
2. Show $d^2 = 0$ in the chart about $x^1, \ldots, x^n$.
3. For any k-form ω and l-form η, prove

   $$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$$

   in the chart about $x^1, \ldots, x^n$.
4. The definition is independent of charts, *i.e.*, if we have another chart with coordinates $y^1, \ldots, y^n$ as smooth functions $y^i = y^i(x^1, \ldots, x^n)$, and a k-form is represented locally as

   $$\sum_{i_1 < \cdots < i_k} f_{i_1, \ldots, i_2} \cdot dx^{i_1} \wedge \cdots \wedge dx^{i_k} = \sum_{i'_1 < \cdots < i'_k} f'_{i_1, \ldots, i_2} \cdot dy^{i'_1} \wedge \cdots \wedge dy^{i'_k},$$

   then the definition of d on the left is the same as d on the right.

Problem 2: Following the notation in Problem 1, define the **de Rham cohomology** of M by

$$H^k_{dR}(M) = \ker(d : \Omega_k \to \Omega_{k+1}) / \operatorname{Im}(d : \Omega_{k-1} \to \Omega_k).$$

1. Prove $H^k(\mathbb{R}^n) = 0$ for $k > 0$.
2. Prove $H^k(M) = 0$ for $k > \dim M$.
3. If M is connected, prove $H^0(M) \cong \mathbb{R}$. If this is hard, you may assume M is path connected (though these conditions are equivalent for manifolds).

Problem 3: Assume the following proposition: if $\psi, \phi : M \to N$ are homotopic maps between two smooth manifolds, then

$$\psi^* = \phi^* : H^k_{dR}(N) \to H^k_{dR}(M)$$

for any k. Suppose p and q are the north and south poles of S^n. Prove

$$H^k_{dR}(S^n \backslash \{p, q\}) \cong H^k_{dR}(S^{n-1})$$

for $n \geq 2$.

Problem 4: Suppose $\pi : E \to M$ is a vector bundle and $\nabla : C^\infty(M; E) \to C^\infty(M; E \otimes T^*M)$ is a covariant derivative, *i.e.* for any $f \in C^\infty(M; \mathbb{R})$ and $s \in C^\infty(M; E)$, we have

$$\nabla(f \cdot s) = s \otimes df + f \cdot \nabla s.$$

1. Prove there is a unique covariant derivative $\nabla^* : C^\infty(M; E^*) \to C^\infty(M; E^* \otimes T^*M)$ determined by
$$d\langle s, s^* \rangle = \langle \nabla s, s^* \rangle + \langle s, \nabla^* s^* \rangle$$
for a given $s^* \in C^\infty(M; E^*)$ and any $s \in C^\infty(M; E)$, where $\langle \cdot, \cdot \rangle$ is the canonical pairing between E and E^*.

2. Prove there is a unique covariant derivative $\nabla^\otimes : C^\infty(M; E \otimes E) \to C^\infty(M; E \otimes E \otimes T^*M)$ determined by
$$\nabla^\otimes(s_1 \otimes s_2) = \nabla s_1 \otimes s_2 + s_1 \otimes \nabla s_2$$
for any $s_1, s_2 \in C^\infty(M; E)$.

3. The vector bundle $E^* \otimes E^*$ is (canonically) isomorphic to $(E \otimes E)^*$. Prove that the induced covariant derivatives on $E^* \otimes E^*$ (first take the dual and then take the tensor product) and $(E \otimes E)^*$ (first take the tensor product and then take the dual) are the same under the isomorphism.

Problem 5: Read Section 12.7 in Cliff's book about commutators and the alternative definition of the curvature. You don't need to write down anything for this problem.