

MATH 230A ASSIGNMENT 3

This assignment is due at 11:59 pm on Thursday October 20, 2022.

Problem 1:

Let $H = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$ be the half-plane and define the Riemannian metric on H by

$$g_0 = \frac{dx \otimes dx + dy \otimes dy}{y^2}.$$

(This is usually called the **hyperbolic metric**. Let $D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$ and let $z = x + iy \in \mathbb{C} = \mathbb{R}^2$. Define the map $f : D \rightarrow H$ by

$$f_0(z) = i \cdot \frac{1 - z}{1 + z}.$$

1. Prove f_0 is a diffeomorphism between D and H .
2. Suppose $f : M \rightarrow N$ is a smooth map between manifolds and g is a metric on N . Suppose $f(p_0) = q_0$. Let $U \subset M$ be a chart of p_0 with $\phi_U : U \rightarrow \mathbb{R}^m$ and let $V \subset N$ be a chart of q_0 with $\phi_V : V \rightarrow \mathbb{R}^n$. In the chart V , let

$$g = \phi_U^*(g_{ij} \cdot dx^i \otimes dx^j),$$

where $g_{ij} : V \rightarrow \mathbb{R}$ is a smooth map. Compute the pull-back metric f^*g on the chart $f^{-1}(V) \cap U$.

3. Compute the pull-back metric $f_0^*g_0$ on H .

Problem 2: Let $T \subset \mathbb{R}^3$ be obtained by rotating the circle

$$\{(x, y, z) \mid y = 0, z^2 + (x - 2)^2 = 1\}$$

about the z -axis. Let T be parameterized by the coordinates $(\theta, \phi) \in [0, 2\pi] \times [0, 2\pi]$

$$f(\theta, \phi) = ((2 + \sin \phi) \cos \theta, (2 + \sin \phi) \sin \theta, \cos \phi).$$

Define the Riemannian metric g_0 on T by the induced metric from $\mathbb{R}^3$. Define the metric g on $S^1 \times S^1$ by the pull-back metric f^*g_0 .

1. Prove that f is 1-1 and onto as a map from $S^1 \times S^1$ to T such that its differential as map into $\mathbb{R}^3$ is injective. (This implies f is a diffeomorphism.)
2. Compute the inner product $g(\frac{\partial}{\partial \theta}, \frac{\partial}{\partial \theta}), g(\frac{\partial}{\partial \theta}, \frac{\partial}{\partial \phi}), g(\frac{\partial}{\partial \phi}, \frac{\partial}{\partial \phi})$, and the pull-back metric g .

Problem 3: Let $U \subset M$ be a chart with $\phi_U : U \rightarrow \mathbb{R}^n$. Suppose $\gamma : I = [0, 1] \rightarrow M$ is a smooth map (called a **path**). Let

$$\phi_U \circ \gamma : \gamma^{-1}(U) \subset I \rightarrow \mathbb{R}^n$$

be denoted by

$$(\gamma^1(t), \dots, \gamma^n(t))$$

and let

$$\dot{\gamma}^i = \frac{d\gamma^i}{dt}, \quad \ddot{\gamma}^i = \frac{d^2\gamma^i}{dt^2}.$$

The geodesic equation is

$$\ddot{\gamma}^i + \Gamma_{jk}^i \dot{\gamma}^j \dot{\gamma}^k = 0,$$

where the indice that appear twice should be taken a sum on $1, \dots, n$, *i.e.* $\Gamma_{jk}^i \dot{\gamma}^j \dot{\gamma}^k = \sum_{j,k=1}^n \Gamma_{jk}^i \dot{\gamma}^j \dot{\gamma}^k$. Note that the **Christoffel symbol**

$$\Gamma_{jk}^i = \frac{1}{2} g^{il} (\partial_j g_{lk} + \partial_k g_{jl} - \partial_l g_{jk}),$$

where $\{g^{il}\}_{i,l \in [1,n]}$ is the inverse matrix of $\{g_{il}\}_{i,l \in [1,n]}$.

Prove that the solution of the geodesic equation is independent of the chart, *i.e.* given another chart $V \subset M$ with $U \cap V \neq \emptyset$, the path satisfies the geodesic equation on U iff it satisfies the geodesic equation on V .

Problem 4: Let S^{n-1} be the sphere in $\mathbb{R}^n$ with radius 1. Let g_S be the induced metric from the standard metric on $\mathbb{R}^3$. (This is called the **round metric**). Let $\gamma : I \rightarrow S^{n-1} \rightarrow \mathbb{R}^n$ be denoted by

$$(x^1(t), \dots, x^n(t)).$$

1. Compute g_S and the Christoffel symbol Γ_{jk}^i
2. Show that the geodesic equation is

$$\ddot{x}^i + x^i |\ddot{x}|^2 = 0.$$

Problem 5: Read Appendix 8.1 in Cliff's book about the vector field theorem. You don't need to write down anything for this problem.