MATH 230A Assignment 1

This assignment is due at 11:59 pm on Thursday September 22, 2022.

Problem 1: Use the definition of the topological manifold to show the following topological spaces are not topological manifolds.

1. The union of all **lines** with irrational slopes from the origin in $\mathbb{R}^2$ **with either the induced topology from** $\mathbb{R}^2$**, or the the topology generated by open sets in lines**.
2. The Hawaiian earring: $\bigcup_{n\in\mathbb{N}}\{(x,y)|(x-\frac{1}{n})^2+y^2=\frac{1}{n^2}\}$ with **with either the induced topology from** $\mathbb{R}^2$**, or the topology generated by open sets in circles.**
3. The bug-eyed line: $\mathbb{R}\cup\mathbb{R}/x\sim y$ if $x=y\neq 0$ **with the quotient toplogy from** $\mathbb{R}\cup\mathbb{R}$.
4. The line with an irrational slope in $\mathbb{R}^2/(x,y)\sim(x+m,y+n)$ for any $m,n\in\mathbb{Z}$ **with the induced topology from** $\mathbb{R}^2$.

Problem 2: Use the implicit function theorem and the definition of a submanifold to prove the following.

Let $f:N\to M$ be a smooth map between smooth manifolds. Let $p\in N$. If for any $q\in f^{-1}(p)$, any chart $U_q\subset N$ with diffeomorphism $\phi_q:U_q\to\mathbb{R}^n$, any chart $U_p\subset M$ with diffeomorphism $\phi_p:U_p\to\mathbb{R}^m$, the Jacobian of

$$\phi_p\circ f\circ\phi_q^{-1}:\mathbb{R}^n\to\mathbb{R}^m$$

at $\phi_q(q)$ is surjective, then $f^{-1}(p)$ is a smooth submanifold of N.

Problem 3: Prove that $O(n),SO(n),U(n),SU(n)$ are closed and bounded in $\mathbb{R}^{n^2}$ and $\mathbb{R}^{2n^2}$. (Then Bolzano-Weierstrass theorem implies that they are compact.)

Problem 4: Let $\mathbb{H}$ be a 4-dimensional vector space generated by $1,i,j,k$ (the set of quaternions). Define the multiplication m on $\mathbb{H}$ by

$$1\cdot l=l\cdot 1=l \text{ for } l\in\{i,j,k\}$$

$$i\cdot i=j\cdot j=k\cdot k=-1$$

$$i\cdot j=-j\cdot i=k,\quad j\cdot k=-k\cdot j=i,\quad k\cdot i=-i\cdot k=j,$$

and extend it linearly. Let S^3 be the unit sphere in H. Prove the following.

1. The manifold S^3 with the multiplication m is a Lie group.
2. There is a diffeomorphism $\phi:S^3\to SU(2)$ that preserves the multiplication and the inverse operation.

Problem 5: Read the construction of the partition of unity in Appendix 1.2 of Cliff's book. Suppose M is a smooth manifold and let $\{(U_1,\phi_1),\cdots,(U_N,\phi_N)\}$ be a finite atlas of M. A partition of unity $\{\chi_\alpha\}_{\alpha\in N}$ is a set of smooth functions on M such that χ_α has support in U_α and $\sum_{\alpha=1}^N\chi_\alpha=1$ at each point. You don't need to write down anything for this problem.