MATH 230A Assignment 6

Problem 1: Recall that there is a 1-1 correspondence between vector field v on a smooth manifold M and derivation $\mathcal{L}_v : C^\infty(M;\mathbb{R}) \to C^\infty(M;\mathbb{R})$. For two vector fields v, w, define $[v,w]$ be the vector field corresponding to the derivation

$$\mathcal{L}_{[v,w]}(f) = \mathcal{L}_v \circ \mathcal{L}_w(f) - \mathcal{L}_w \circ \mathcal{L}_v(f),$$

where $f \in C^\infty(M;\mathbb{R})$. This is called the **Lie bracket** on vector fields. Prove the following.

1. For $a, b \in \mathbb{R}$, v, w, u vector fields, $[av + bw, u] = a[v,u] + b[w,u]$ and $[v, aw + bu] = a[v,w] + b[v,u]$.
2. $[v,v] = 0$ and $[v,w] = -[w,v]$.
3. $[v,[w,u]] + [u,[v,w]] + [w,[u,v]] = 0$ (Jabobi identity).

Problem 2: Suppose G is a Lie group. Given $g \in G$, define $L_g : G \to G$ and $R_g : G \to G$ by $L_g(h) = gh$ and $R_g(h) = hg$. A vector field v is called **left-invariant** if $(L_g)_* v = v$, **right-invariant** if $(R_g)_* v = v$. Let $\mathfrak{g}$ be the set of left-invariant vector fields. Prove the following.

1. $\mathfrak{g}$ can be identified with the tangent space $T_e G$ of G at the identity e.
2. $\mathfrak{g}$ can be identified with the set of right-invariant vector fields.
3. If $v, w \in \mathfrak{g}$, then the Lie bracket $[v,w]$ is also in $\mathfrak{g}$.

Problem 3: Let $\pi : P \to \mathbb{R}$ be a principal G-bundle over $\mathbb{R}$. Prove P is isomorphic to the product principal bundle $\mathbb{R} \times G$.

Problem 4: Suppose G is a subgroup of $GL(n,\mathbb{R})$ and let $P = S^1 \times G$ is a product principal bundle over S^1. Let $A_0 \in C^\infty(P; \mathfrak{g} \otimes T^*P)$ be the $\mathfrak{g}$-valued 1-form which at $(t,g) \in S^1 \times G$ has the form $g^{-1}dg$. Prove the following.

1. A_0 is a connection on P.
2. any other connection A on P must have the form $A = g^{-1}dg + g^{-1}\alpha g \cdot dt$, where α is a map from S^1 to $\mathfrak{g}$.

Problem 5: Read Chapter 19 in Cliff's book about Hodge star. You don't need to write down anything for this problem.