

MATH 230A ASSIGNMENT 5

Problem 1: Suppose $\nabla : C^\infty(M; E) \rightarrow C^\infty(M; E \otimes T^*M)$ is a covariant derivative. It induces a covariant derivative (which we still denote by ∇) $C^\infty(M; E \otimes E) \rightarrow C^\infty(M; E \otimes E \otimes T^*M)$ by Problem 4 of assignment 4. It is called **compatible** with a metric $g \in C^\infty(M; E \otimes E)$ if

$$\nabla g = 0.$$

Suppose ∇ and ∇' are two covariant derivative compatible with g . Suppose $\nabla' = \nabla + \alpha$, where $\alpha \in C^\infty(M; \text{End}(E) \otimes T^*M)$ is a matrix-valued 1-form (the standard notation is $\mathbf{a}$ but I use α in the class for writing simplicity). Prove

$$\alpha^T \cdot g = -g \cdot \alpha$$

Problem 2: For simplicity, let $M = \mathbb{R}^n$. Let $dx^1, \dots, dx^n$ be the basis of sections of $T^*\mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n$. Let

$$\mathbb{A} : \bigotimes_{i=1}^k T^*M \rightarrow \bigwedge^k T^*M$$

be the anti-symmetrization map that sends

$$dx^{i_1} \otimes \dots \otimes dx^{i_k}$$

to

$$dx^{i_1} \wedge \dots \wedge dx^{i_k}$$

and extends linearly over $C^\infty(M; \mathbb{R})$. This map also extend to a map

$$\mathbb{A} : \bigwedge^k T^*M \otimes T^*M \rightarrow \bigwedge^{k+1} T^*M.$$

Let $\nabla : C^\infty(M; T^*M) \rightarrow \nabla : C^\infty(M; T^*M \otimes T^*M)$ be a covariant derivative and define the induced covariant derivative on

$$\nabla : C^\infty(M; \bigwedge^k T^*M) \rightarrow C^\infty(M; \bigwedge^k T^*M \otimes T^*M)$$

inductively by

$$\nabla(\omega \wedge \eta) = (\nabla\omega \wedge \eta) + (-1)^k \omega \wedge (\nabla\eta)$$

for any k -form ω and l -form η and then extending linearly. Prove the following.

1. For any k -form ω and l -form η ,

$$\mathbb{A}(\nabla(\omega \wedge \eta)) = \mathbb{A}(\nabla\omega \wedge \eta) + (-1)^k \omega \wedge \mathbb{A}(\nabla\eta).$$

2. For any k -form ω , define the **torsion tensor** $T_\nabla\omega = \mathbb{A}(\nabla\omega) - d\omega$. We say ∇ is **torsion-free** if $T_\nabla\omega = 0$ for any 1-form ω . If ∇ is torsion-free, prove $T_\nabla\omega = 0$ for any k -form ω .

Problem 3: Following Section 16.1.1 in Cliff's book, suppose $S^n \subset \mathbb{R}^{n+1}$ is the sphere of radius $\rho > 0$, prove the curvature 2-form at $y = 0$ specified by

$$R_{abcd} = \rho^{-2} \delta_{ac} \delta_{bd} - \delta_{ad} \delta_{bc}.$$

For simplicity, you may suppose $n = 2$. You have to clarify the sketch in the section from the definitions (of Γ_{bc}^a, R_{ab} , etc) and add details in the proof.

Problem 4: Following Section 16.1.3 in Cliff's book, suppose $M \subset \mathbb{R} \times \mathbb{R}^n$ is the manifold defined by $|x|^2 - t^2 = -\rho^2$ and $t > 0$, where t is the coordinate of $\mathbb{R}$, x is the coordinate of $\mathbb{R}^n$, and $\rho > 0$ is fixed. Prove the curvature 2-form at $(t = \rho, x = 0)$ is specified by

$$R_{abcd} = -\rho^{-2}\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}.$$

For simplicity, you may suppose $n = 2$. You have to clarify the sketch in the section from the definitions (of Γ_{bc}^a, R_{ab} , etc) and add details in the proof.

Problem 5: Read Section 16 in Cliff's book about more examples of Riemannian curvature tensor. You don't need to write down anything for this problem.