MATH 230A Assignment 1

Problem 1: Use the definition of the topological manifold to show the following topological spaces are not topological manifolds. The topology comes from the induced topology or the quotient topology.

1. The union of all rays with irrational slope from the origin in $\mathbb{R}^2$.
2. The Hawaiian earring: $\bigcup_{n\in\mathbb{N}}\{(x,y)|(x-\frac{1}{n})^2+y^2=\frac{1}{n^2}\}$.
3. The bug-eyed line: $\mathbb{R}\cup\mathbb{R}/x\sim y$ if $x=y\neq 0$.
4. The line with an irrational slope in $\mathbb{R}^2/(x,y)\sim(x+m,y+n)$ for any $m,n\in\mathbb{Z}$.

Problem 2: Use the implicit function theorem and the definition of a submanifold to prove the following.

Let $f: N\to M$ be a smooth map between smooth manifolds. Let $p\in N$. If for any $q\in f^{-1}(p)$, any chart $U_q\subset N$ with diffeomorphism $\phi_q: U_q\to\mathbb{R}^n$, any chart $U_p\subset M$ with diffeomorphism $\phi_p: U_p\to\mathbb{R}^m$, the Jacobian of

$$\phi_p\circ f\circ\phi_q^{-1}:\mathbb{R}^n\to\mathbb{R}^m$$

at $\phi_q(q)$ is surjective, then $f^{-1}(p)$ is a smooth submanifold of N.

Problem 3: Prove that $O(n), SO(n), U(n), SU(n)$ are closed and bounded in $\mathbb{R}^{n^2}$ and $\mathbb{R}^{2n^2}$. (Then Bolzano-Weierstrass theorem implies that they are compact.)

Problem 4: Let $\mathbb{H}$ be a 4-dimensional vector space generated by $1,i,j,k$ (the set of quaternions). Define the multiplication m on $\mathbb{H}$ by

$$1\cdot l=l\cdot 1=l \text{ for } l\in\{i,j,k\}$$
$$i\cdot i=j\cdot j=k\cdot k=-1$$
$$i\cdot j=-j\cdot i=k,\quad j\cdot k=-k\cdot j=i,\quad k\cdot i=-i\cdot k=j,$$

and extend it linearly. Let S^3 be the unit sphere in H. Prove the following.

1. The manifold S^3 with the multiplication m is a Lie group.
2. There is a diffeomorphism $\phi: S^3\to SU(2)$ that preserves the multiplication and the inverse operation.

Problem 5: Read the construction of the partition of unity in Appendix 1.2 of Cliff's book. Suppose M is a smooth manifold and let $\{(U_1,\phi_1),\cdots,(U_N,\phi_N)\}$ be a finite atlas of M. A partition of unity $\{\chi_\alpha\}_{\alpha\in N}$ is a set of smooth functions on M such that χ_α has support in U_α and $\sum_{\alpha=1}^N\chi_\alpha=1$ at each point. You don't need to write down anything for this problem.