Math 231A
Solution-Set Collage for Problem Set 1

Students of Math 231A
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1 Compiler's Note

This collage presents a set of solutions for Problem Set 1 of the Fall 2024 Section of Math 231A, with input from the compiler providing light exposition on the goal and contextualisation of each problem.

The compiler would like to thank each and every contributor to this collage and remind the reader that the existence of this collage testament to their excellent work.

2 The Problems

This first problem set familiarises students with the building blocks of singular homology. The emphasis was on understanding what the group of singular chains and the singular homology groups were as constructions, and on being able to bridge intuitive 'visual' reasoning in the abstract sense with rigorous symbolic and algebraic argument. Attached is the list of assigned problems:

Problem 1 (Exercise 1.8):

1. Let $[n]$ denote the totally ordered set $\{0, 1, \ldots, n\}$. Let $\phi : [m] \to [n]$ be an order preserving function (so that if $i \leq j$ then $\phi(i) \leq \phi(j)$). Identifying the elements of $[n]$ with the vertices of the standard simplex Δ^n, ϕ extends to an affine map $\Delta^m \to \Delta^n$ that we also denote by ϕ. Give a formula for this map in terms of barycentric coordinates: If we write $\phi(s_0, \ldots, s_m) = (t_0, \ldots, t_n)$, what is t_j as a function of $(s_0, \ldots, s_m)$?

2. Write $d^j : [n-1] \to [n]$ for the order preserving injection that omits j as a value. Show that an order preserving injection $\phi : [n-k] \to [n]$ is uniquely a composition of the form $d^{j_k} d^{j_{k-1}} \cdots d^{j_1}$, with $0 \leq j_1 < j_2 < \cdots < j_k \leq n$. Do this by describing the integers $j_1, \ldots, j_k$ directly in terms of ϕ, and then verify the straightening rule
$$d^i d^j = d^{j+1} d^i \quad \text{for} \quad i \leq j.$$

3. Use the relations among the d_i's to prove that
$$d^2 = 0 : S_n(X) \to S_{n-2}(X).$$

Problem 2 (Example 2.1): Suppose X is a topological space and $\sigma : \Delta^1 \to X$ is a continuous map. Define $\phi : \Delta^1 \to \Delta^1$ by sending $(t, 1-t)$ to $(1-t, t)$. Precomposing σ with ϕ gives another singular simplex $\overline{\sigma}$ which reverses the orientation of σ. Show that it is not true that $\overline{\sigma} = -\sigma$ in $S_1(X)$ but $\overline{\sigma} \equiv -\sigma \bmod B_1(X)$. (This means that there is a 2-chain in X whose boundary is $\overline{\sigma} + \sigma$. If $d_0\sigma = d_1\sigma$, so that $\sigma \in Z_1(X)$, then $\overline{\sigma}$ and $-\sigma$ are homologous cycles, so that $[\overline{\sigma}] = -[\sigma]$ in $H_1(X)$.)

Problem 3 (Exercise 2.3): Construct an isomorphism
$$H_n(X) \oplus H_n(Y) \to H_n(X \coprod Y).$$

Problem 4 (Exercise 3.7): Write $\pi_0(X)$ for the set of path-components of a space X. Construct an isomorphism
$$\mathbb{Z}\pi_0(X) \to H_0(X).$$

3 Problem 1

The first part of this problem focuses on the rigorous details for which we can store information about affine maps between standard simplices simply by order-preserving functions between ordered sets, which intuitively dictate where the vertices of the standard simplices are to be sent. Presented here is the submission by Tasuku Ono:

We first consider how ϕ maps veritices in Δ^m to those in Δ^n in terms of barycentric coordinates. For an arbitrary non-negative integer k, we assign label $\ell \in \{0, 1, \cdots, k\}$ to the vertex with coordinates $\underbrace{(0, \cdots, 0, 1, 0, \cdots, 0)}_{k+1}$, where they are all zero except the ℓth coordinate. Now the map $\phi : [m] \to [n]$ maps the vertex ℓ with coordinates $\underbrace{(0, \cdots, 0, 1, 0, \cdots, 0)}_{\ell\text{th element is 1}}$ in Δ^m to the vertex $\phi(\ell)$ with coordinates $\underbrace{(0, \cdots, 0, 1, 0, \cdots, 0)}_{\phi(\ell)\text{th element is 1}}$ in Δ^n, which can also be written as

$$(t_j = \delta(j, \phi(\ell)))_{j \in \{0,1,\cdots,n\}}. \tag{01.1}$$

Then extending this to an affine map $\Delta^m \to \Delta^n$ gives

$$t_j = \sum_{i=0}^{m} s_i \delta(j, \phi(i)). \tag{01.2}$$

Here, the fact that we are wroking on the barycentric coordinate system allows the coordinates s_i's to be carried through in an affine map. One can also check that the resulting t_i's are well-defined in that they still satisfy the condition $\sum_j t_j = 1$:

$$\sum_{j=0}^{n} t_j = \sum_{j=0}^{n} \sum_{i=0}^{m} s_i \delta(j, \phi(i)) = \sum_{i=0}^{m} s_i \underbrace{\left(\sum_{j=0}^{n} \delta(j, \phi(i)) \right)}_{1} = \sum_{i=0}^{m} s_i = 1. \tag{01.3}$$

The second part of this problem explores further the consequences of this correspondance by allowing us to draw a parallel between non-degenerate face maps between standard simplices and order-preserving injections between ordered sets. The 'straightening rule' verified here is a very handy lemma, which not only provides intuition for and verifying the existence and uniqueness of the factorisation of such injections, but also gives a tool for the next part of the problem. Here is the solution by Georgi Ivanov:

Solution - Let us first start by describing d_j. Observe that for all values less than j, i.e $i < j$, we have that d_j preserves their value, i.e $d_j(i) = i$. On the other hand, for all values $i \geq j$, we have that $d_j(i) = i + 1$, by accounting for that shift by one, i.e the function is defined as:

$$d_j(i) = \begin{cases} i & \text{if } i < j \\ i+1 & \text{if } i \geq j \end{cases}$$

Observe that by definition, $j_1, \cdots j_k$ are the integers that are being 'skipped', i.e in this case, they are just defined as the complement of pre-image of our map with their explicit ordering:

$$S = \{j_1 < j_2 < \cdots < j_k\} = [n] \backslash \mathrm{Im}(\phi)$$

Let us now verify the straightening rule. By the definition we wrote above for d_j, we can compute the

composition $d_{j+1}d_j$ directly by using the fact $i \leq j$, i.e their composition arises naturally by inclusion:

$$d_{j+1}d_i(k) = \begin{cases} k & \text{if } k < i \\ k+1 & \text{if } i < k < j+1 \\ k+2 & \text{if } j+1 < k \end{cases}$$

Now, let us study the composition of $d_i d_j$ by breaking it down into steps: First, for d_j we have $d_j(k) = k$ for $k < j$ and $d_j(k) = k + 1$ for $j \leq k$. Now, for all $k < i \leq j$, we fall under the first case, which remains unchanged, i.e that $d_i(d_j(k)) = k$. However, for $i \leq k$, all of the remaining components are going to get shifted by 1, including the upper cutoff index itself (since the insertion happens at index i, which is before index j), i.e:

$$1) \quad d_j(k) = k \text{ for } i \leq k < j \Rightarrow d_i(d_j(k)) = k+1 \text{ for } i \leq k < j+1$$

$$2) \quad d_j(k) = k+1 \text{ for } j < k \Rightarrow d_i(d_j(k)) = k+2 \text{ for } j+1 \leq k$$

With this we have in fact confirmed for all three cases that $d_{j+1}d_i(k) = d_i d_j(k)$. Now, let us prove using induction the an order-preserving injection is indeed of the form $d_{j_k}d_{j_k\ 1}\cdots d_{j_1}$. Starting with the base case, for a map $[n-1] \to [n]$, we clearly have that it is equal to d_j for whichever index j is being omitted. Now, for an inductive step, let us assume that any order-preserving injection $\psi : [n-k+1] \to [n]$, for indexes $j_1 < j_2 < \cdots < j_{k-1}$ is indeed of the form $d_{j_k}d_{j_{k-1}}\cdots d_{j_1}$. Now, consider an arbitrary order-preserving and injective map $\phi : [n-k] \to [n]$. Let $S = \{j_1 < j_2 < \cdots < j_k\} = [n] \backslash \mathrm{Im}(\phi)$ (by injectivity and the order-preserving property). Observe that per the inductive hypothesis, we can construct an order preserving map ψ omitting $S \backslash \{j_k\}$ with a unique expression $\psi = d_{j_{k-1}}d_{j_{k-1}}\cdots d_{j_1}$, s.t. clearly $\phi = d_{j_k} \circ \psi = d_{j_k}d_{j_{k-1}}\cdots d_{j_1}$ since $j_k > j_{k\ 1} > \cdots > j_1$, i.e d_{j_k} is the identity on the image of ψ. Now, to prove it is unique, let us instead remove an arbitrary index j_p with $1 \leq p < k$ and define by the inductive hypothesis a map omitting $S \backslash \{j_p\}$. Now, in this case, observe that since we are removing an intermediary index, all of the subsequent values will be shifted by one back, i.e $j'_q = j_q - 1$ for all $k \geq q > p$, i.e let:

$$\psi_p = d_{j'_k}d_{j'_{k-1}}\cdots d_{j'_{p+1}}d_{j_p\ 1}\cdots d_{j_1} = d_{j_k-1}d_{j_k\ 1-1}\cdots d_{j_{p+1}-1}d_{d_p\ 1}\cdots d_{j_1} \Rightarrow$$

s.t. the composition can simply be calculated by swapping pairwise until it gets to the correct index:

$$\Rightarrow d_{j_p} \circ \psi_p = d_{j_p} \circ d_{j_k\ 1}d_{j_{k-1}\ 1}\cdots d_{j_{p+1}\ 1}d_{j_{p-1}}\cdots d_{j_1} = d_{j_k}d_{j_p}d_{j_{k-1}\ 1}\cdots d_{j_{p+1}\ 1}d_{j_{p-1}}\cdots d_{j_1} =$$

$$= d_{j_k}d_{j_k\ 1}d_{j_p}\cdots d_{j_{p+1}\ 1}d_{j_p\ 1}\cdots d_{j_1} = d_{j_k}d_{j_k\ 1}\cdots d_{j_{p+1}}d_{j_p}d_{j_p\ 1}\cdots d_{j_1} = d_{j_k} \circ \psi = \phi$$

And since p in this case was arbitrary, we can perform the aforementioned composition for any index that has been removed, and as such, it follows that ϕ has indeed a unique decomposition as the one we demonstrated above. With this, we have in fact completed the inductive step, and as such, we have verified that it is true for all injective order-preserving maps.

The importance of the third part of the problem needs no introduction - it is precisely because this result holds that allows for singular homology to be defined in the first place! For those who have studied differential topology, recall that the exterior derivative acting on differential forms on a smooth manifold satisfies a similar identity - we shall understand how this relates to our current study of singular homology later on in the course. In any case, here is the solution by Adi Raj:

Proof. We want to use the relations among d^i's to show that

$$d_{n-1} \circ d_{n-2} = 0 : S_n(X) \to S_{n-2}(X)$$

We have that the boundary operator is given by

$$d_n \sigma = \sum_{i=0}^{n} (-1)^i d_i \sigma$$

which extends to a homomorphism by additivity. We can now use this definition:

$$d_{n-1} \circ d_{n-2} = d_{n-1} \left(\sum_{i=0}^{n} (-1)^i d_i \sigma \right) = \sum_{i=0}^{n} (-1)^i d_{n-1} (\sigma \circ d^i)$$

$$= \sum_{i=0}^{n} (-1)^i \sum_{j=0}^{n-1} (-1)^j (\sigma \circ d^i \circ d^j)$$

We have thus expanded out using the boundary operators, and we want to manipulate this expression:

$$\sum_{i=0}^{n} (-1)^i \sum_{j=0}^{n-1} (-1)^j (\sigma \circ d^i \circ d^j) = \sum_{0 \leqslant j < i \leqslant n} (-1)^{i+j} (\sigma \circ d^i \circ d^j) + \sum_{0 \leqslant j < i \leqslant n-1} (-1)^{i+j} (\sigma \circ d^i \circ d^j)$$

Here, we can use the relations among d^i's by referencing the straightening rule, which gives that

$$d^i d^j = d^{j+1} d^i$$

We also recognize that we can change the labels of the indices in this sum to combine more easily; we see a natural change of indices given by $i \to j' + 1$ and $j \to i'$. We now rewrite the previous expression in terms of $(j+1)$ and i, using the straightening rule on the first sum and modifying the indices on the second:

$$\sum_{0 \leqslant j < i \leqslant n} (-1)^{i+j} (\sigma \circ d^{j+1} \circ d^i) + \sum_{0 \leqslant j' < i' \leqslant n} (-1)^{i'+j'-1} (\sigma \circ d^{j'+1} \circ d^{i'})$$

We can see that these sums cover identical indices, so we can simply standardize these in terms of k and m:

$$\left(\sum_{0 \leqslant k < m \leqslant n} (-1)^{m+k} (\sigma \circ d^{k+1} \circ d^m) \right) - \left(\sum_{0 \leqslant k < m \leqslant n} (-1)^{m+k} (\sigma \circ d^{k+1} \circ d^m) \right) = 0$$

We have thus shown that $d^2 = 0 : S_n(X) \to S_{n-2}(X)$.

□

4 Problem 2

The first part of the problem verifies the reader's understanding of what the group of singular chains is as a construction. A key point here was ensuring comprehension of the fact that the 'sum' here referred to formal linear combinations and not any addition of the actual maps. Attached is Luke Zhu's solution:

We begin by showing that it is not necessarily true that $\overline{\sigma} = -\sigma$ in $S_1(X)$. For example, take the space $X = S^1$ embedded in the complex plane and the singular 1-simplex $\sigma : \Delta^1 \to X$ that sends $t \mapsto e^{2\pi i t}$ where $t \in [0, 1]$. Then, to find the reversed simplex $\overline{\sigma}$, we define the orientation-reversal map $\phi(t) = 1 - t$ from $\Delta^1 \to \Delta^1$, and precomposing yields $\overline{\sigma} = \sigma \circ \phi$ which means

$$\overline{\sigma}(t) = \sigma(1 - t) = e^{2\pi i(1-t)} = e^{-2\pi i t}$$

From this we see that σ traverses the unit circle once in an anticlockwise fashion, while the reversal $\overline{\sigma}$ traverses the unit circle once in a clockwise direction. Importantly, we clearly have $\sigma \neq \overline{\sigma}$ as they only meet when $t = 1/2$ because $e^{\pi i} = e^{-\pi i}$. Because they are not identical as maps, they are distinct generators in the free abelian group $S_1(X)$ (which is generated by n-simplices $\mathbb{Z}\mathrm{Sin}_1(X)$). The formal inverse $-\sigma$ of σ in $S_1(X)$ then has no relation with $\overline{\sigma}$: in particular, it is not the case that $\overline{\sigma} = -\sigma$, because this would imply that $\overline{\sigma}$ is the formal inverse of σ which is not necessarily true.

The second part of the problem both gives the intuition that the 'path' represented by a 1-chain $\sigma + \overline{\sigma}$ could be seen as the boundary of a 'degenerate' map of a triangle into a space, and also ensures that the reader understands that translating this intuition into rigorous language could involve some nontrivial changes to their desired construction (in this case, the need to 'cancel out' the extraneous degenerate 1-simplex $c^1_{\sigma(0)}$). The following writeup is brought to us by Ari Krishna:

For the second part, we furnish a 2-chain τ with the property that $d\tau = \sigma + \overline{\sigma}$. To this end, consider the projection map $\pi : \Delta^2 \to \Delta^1$ which is defined to be the affine extension of the map sending both e_0, e_2 to e_0, and mapping e_1 to e_1.

Let us write this map π in barycentric coordinates. By definition we have that

$$\pi(e_0) = e_0, \quad \pi(e_1) = e_1, \quad \pi(e_2) = e_0$$

so writing $x = s_0e_0 + s_1e_1 + s_2e_2 \in \Delta^2$, we have the expression

$$\pi(x) = s_0\pi(e_0) + s_1\pi(e_1) + s_2\pi(e_2) = s_0e_0 + s_1e_1 + s_2e_0.$$

that is,

$$\pi(x) = (s_0 + s_2)e_0 + s_1e_1.$$

Thus, in barycentric coordinates, the map π sends (s_0, s_1, s_2) in Δ^2 to (t_0, t_1) in Δ^1, where:

$$t_0 = s_0 + s_2, \quad t_1 = s_1.$$

We claim that $\tau = \sigma \circ \pi$, so pulling back by the projection constructs the desired 2-chain.

To this end, let $c^n_x : \Delta^n \to X$ denote the constant map with value $x \in X$. We then have

$$d(\sigma \circ \pi) = \sigma\pi d^0 - \sigma\pi d^1 + \sigma\pi d^2 = \overline{\sigma} - \sigma^1_{\sigma(0)} + \sigma.$$

Then, consider the constant 2-simplex which maps to the point $\sigma(0)$ *i.e.* the map $c^2_{\sigma(0)}$. Let us apply the boundary map to this 2-simplex: we calculate

$$d(c^2_{\sigma(0)}) = c^1_{\sigma(0)} - c^1_{\sigma(0)} + c^2_{\sigma(0)} = 0$$

which shows that

$$d(\sigma\phi + c^2_{\sigma(0)}) = \sigma + \overline{\sigma}$$

so $\sigma = \overline{\sigma}$ as elements of $H_1(X)$. □

5 Problem 3

This problem formalises the key intuition that homology groups are in some sense 'additive', making it much more effectively computable than the more theoretically simple higher homotopy groups. This turns out to be a special case of the Mayer-Vietoris sequence, in which we allow spaces X and Y to have nontrivial intersection; though the proof of this is more involved and does not generalise very readily. A key moral of this problem is that when proving analogous facts about homology groups that rely on topological facts, it is generally more useful to first work at the level of $S_*(X)$ and then deduce the required fact by putting things back together. Working directly with homology involves more care in keeping track of conditions that one needs to verify. Presented below is the solution by Carl Scandelius, whose submission demonstrates both of these approaches.

Singular simplices here are continuous maps $\Delta^n \to X \coprod Y$. Since X, Y disjoint in $X \coprod Y$, the image of any continuous simplex σ must lie entirely within either X or Y. Thus, each singular simplex in $X \coprod Y$ is a simplex in one of X or Y. We thus have

$$S_n(X \coprod Y) = S_n(X) \oplus S_n(Y).$$

This holds also for the boundary operator, which acts independently on X and Y components of a chain in the coproduct space.

We define our isomorphism $\phi : H_n(X) \oplus H_n(Y) \to H_n(X \coprod Y)$ by sending homology classes

$([\alpha], [\beta]) \mapsto [\alpha+\beta] \in H_n(X \coprod Y)$, by $\phi([\alpha], [\beta]) = \iota_{X*}([\alpha]) + \iota_{Y*}([\beta]) = [\iota_X(\alpha) + \iota_Y(\beta)]$ for inclusion maps

$$\iota_X : X \hookrightarrow X \coprod Y \text{ and } \iota_Y : Y \hookrightarrow X \coprod Y.$$

and with induced ι_{X*}, ι_{Y*}. We observe ϕ to be well-defined and linear:

$$\phi([a] + [a'], [b] + [b']) = [\iota_X(a + a') + \iota_Y(b + b')]$$

Moreover, ϕ is injective. Suppose $\phi([\alpha], [\beta]) = 0$ in $H_n(X \coprod Y)$, then $[\alpha + \beta] = 0$ s.t. $\alpha + \beta$ is a boundary in $X \coprod Y$. Since X, Y disjoint, α, β must be boundaries in their respective spaces X, Y. i.e. we have $[\alpha] = 0$ in $H_n(X)$ and $[\beta] = 0$ in $H_n(Y)$ and thus $([\alpha], [\beta]) = (0, 0)$. We claim ϕ is also surjective. Any homology class $[\omega]$ in $H_n(X \coprod Y)$ can be represented by a chain $\omega = \alpha + \beta$ where α is some chain in X and β some chain in Y. Then we have $[\omega] = [\iota_X(\alpha) + \iota_Y(\beta)] = \phi([\alpha], [\beta])$. Thus ϕ is also surjective.

Note, for the proofs of both injectivity and surjectivity, we rely on the continuity of singular simplices: In $X \coprod Y$, any continuous simplex must map entirely into either X or Y alone, since X, Y are disjoint and open. Thus each singular simplex in $S_n(X \coprod Y)$ is confined either to $S_n(X)$ or $S_n(Y)$ exclusively. That is, if a chain maps to zero in $S_n(X \coprod Y)$, its components in $S_n(X)$ and $S_n(Y)$ must individually be boundaries which guarantees that the original elements were zero (we use this for injectivity). Similarly, every homology class in $H_n(X \coprod Y)$ comes from chains entirely in $S_n(X)$ or entirely in $S_n(Y)$ i.e. we can represent any class as a sum from $H_n(X)$ and $H_n(Y)$ (this we use for the surjectivity proof).

Thus we have isomorphism $\phi : H_n(X) \oplus H_n(Y) \to H_n(X \coprod Y)$.

6 Problem 4

This problem explains the reason for which we could consider the zeroeth singular homology group $H_0(X)$ as an invariant that stores information about the number of path-connected components of a space X. The key here was that if two points x and y are path-connected, then the chain $x - y$ is the boundary of a simplex representing a path connecting them, which allows us to conclude that x and y are homologous 0-simplices. Note that nothing in this problems solution has any requirements on the base ring being $\mathbf{Z}$, which gives us that this intuition of $H_0(X)$ representing path-connected components is valid also when we consider homology based in other rings (well, maybe not the trivial ring but we never get anything useful out of that anyway). Sebastian Attlan's solution is chosen here.

Proof. Let's unpack these objects: $H_0(X) = \frac{\ker(d:S_0(X)\to S_{-1}(X))}{\operatorname{im}(d:S_1(X)\to S_0(X))}$. By definition, $S_{-1}(X)$ is empty so the numerator is just $S_0(X) = \mathbb{Z}\{x \in X\}$, while the denominator $\operatorname{im}(d : S_1(X) \to S_0(X))$ consists of formal elements that are the boundary of Δ^1s, or segments. Thus, $H_0(X)$ is precisely the formal sums of points mod points connected by line.

For $\mathbb{Z}\pi_0(X) \cong H_0(X)$, this is the formal sum of path components in X.

By description, these should be equivalent. For an explicit isomorphism, take the morphism $f : \mathbb{Z}\pi_0(X) \to H_0(X)$ carrying

$$\sum a_i A_i,$$

where A_i is a path component, to

$$\sum a_i \sigma_i,$$

where $\sigma_i \in H_0$ is some map from Δ^0 into A_i. We should verify that this is well defined in that our choice in $\sigma_i \in H_0$ doesn't matter. Indeed, if $\sigma_i : * \to p$ and $\sigma_i' : * \to p'$ with $p, p' \in A_i$, by path connectivity of A_i, we have some $\sigma : \Delta^1 \to X$ with $\sigma(0) = p$ and $\sigma(1) = p'$. Thus, $\sigma_i' - \sigma_i$ is in the image of $d : S_1(X) \to S_0(X)$, so they are in the same quotient class.

We exhibit the inverse as f^{-1} taking

$$\sum a_i \sigma_i,$$

to

$$\sum a_i A_i,$$

where each A_i contains the image of σ_i, which is well defined because the image is a point. By construction, $f \circ f^{-1} = \mathrm{id}$ and $f^{-1} \circ f = \mathrm{id}$. Thus, f is an isomorphism, and

$$\mathbb{Z}\pi_0(X) \cong H_0(X).$$

□

As a point of alternative perspective, here is a solution by Ignasi Segura Vicente, who used a nice lemma in his work:

Let $[C] \in \pi_0(X)$ denote the element of $\pi_0(X)$ corresponding to the path-connected component C of X, and let $p_C \in S_0(X)$ be the 0-chain corresponding to the constant map from Δ^0 to a fixed point on C: note that $[p_C] \in S_0(X)/B_0(X)$ is the same regardless of what point we choose on C, since for any $p, q \in C$ there exists a map $f\colon \Delta^1 \to C$ with $f(0) = p, f(1) = q$, so $df = q - p$ and hence the constant maps from Δ^0 to p or q are the same modulo boundaries. Furthermore, every 0-chain is a cycle, so $S_0(X)/B_0(X) = H_0(X)$, and thus $[p_C]$ is an element of $H_0(X)$, for every C. Then, define $\varphi\colon \mathbb{Z}\pi_0(X) \to H_0(X)$ by $[C] \mapsto [p_C]$. We now show that this is an isomorphism.

This boils down to observing that the free group on a set $X/{\sim}$, where $\sim$ is an equivalence relation, equals the quotitent of $\mathbb{Z}X$ by the free group generated by the elements $x_1 - x_2$ for all $x_1 \sim x_2$; letting H be the subgroup of $\mathbb{Z}X$ generated by the elements of the form $x_1 - x_2$ for $x_1 \sim x_2$, we want to show

$$\mathbb{Z}(X/\sim) \cong \frac{\mathbb{Z}X}{H}.$$

To show this, note that there is a surjective map $\mathbb{Z}X \to \mathbb{Z}(X/\sim)$ given by $x \mapsto [x]$ and extending by linearity. Every element of the form $x_1 - x_2$ where $x_1 \sim x_2$ is in the kernel of this surjection. Conversely, if $\sum_{x \in X} a_x x$ is in the kernel of the surjection, it means that for each equivalence class $c \in X/\sim$, $\sum_{x \in c} a_x = 0$. Then, for every $c \in X/\sim$ such that there is some $y \in c$ with $a_y \neq 0$, we know that

$$a_y = -\sum_{x \in [c], x \neq y} a_x$$

and hence

$$\sum_{x \in [c]} a_x = \sum_{x \in [c], x \neq y} a_x(x - y)$$

which is clearly in the ideal generated by the differences $x_1 - x_2$ with $x_1 \sim x_2$. In our case, the equivalence relation is given by letting two points be equivalent if they are in the same path component, so that $X/\sim = \pi_0$ and $\mathbb{Z}X = S_0X$ and $H = \langle x_1 - x_2 | x_1 \sim x_2 \rangle = B_0X$ so that $\frac{\mathbb{Z}X}{H} = H_0(X)$. The group-theoretical proof we gave shows that the map $H_0(X) \to \mathbb{Z}\pi_0$ given by taking the class $[p]$ of any point $p \in X$ and mapping it to the connected component that p is in, is an isomorphism. This map is clearly the inverse of the map φ defined above, concluding the proof that φ is an isomorphism.

7 Acknowledgements

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