A Prologue to Math 231

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1 Introduction

The purpose of this prologue is to give an informal discussion of the context and motivation behind the material that you will be studying in this course.

For reference, Math 231 is designed as a student's second exposure to the subject of algebraic topology, as a sequel to either Math 55B or Math 131. This prologue will thus be written with the assumption that the reader possesses this background.

2 Where we are

The central goal of topology is to understand the properties of geometric objects invariant under continuous deformation. In 55B or 131, we understood one of the formalisms of this goal to mean a classification of topological spaces up to homotopy equivalence, and learnt that studying a topological space's fundamental group simplified this goal.

For instance, imagine that one were equipped only with tools from point-set topology, and were tasked with showing that the two-sphere S^2 and the torus $S^1 \times S^1$, both compact connected 2-manifolds, were not homotopy equivalent. I don't know about you, but I would definitely be stumped trying to come up with a naïve argument to justify this (For those interested in learning in actually knowing how to do this, apparently the result uses the Jordan Curve Theorem, which to my knowledge is an absolute nightmare to prove using elementary methods). Armed with algebraic topology, however, our task is straightforward - we can simply observe that $\pi_1(S^2)$ is trivial, whereas $\pi_1(S^1 \times S^1)$ is $\mathbf{Z} \times \mathbf{Z}$.

The usefulness of the fundamental group did not stop there! We also saw it to be a useful tool to establish the non-existence of certain maps between topological spaces. Let us quickly recall the properties of the fundamental group that made this possible. (A quick disclaimer - there will be some use of categorical language in what follows; if you do not feel completely at home with that yet, that is totally fine - we'll have plenty of time to get comfortable with this during the year).

For example, in the proof of Brouwer's Fixed Point Theorem, we used the fact that if a retraction f from a 2-disk D^2 to its boundary circle S^1 were to exist, then the composition $f \circ i$, where $i : S^1 \to D^2$ is the inclusion of the boundary, would induce an isomorphism of fundamental groups $f_* \circ i_* : \pi_1(S^1) \to \pi_1(S^1)$ that factored through $\pi_1(D^2)$ - this is absurd as $\pi_1(S^1)$ is $\mathbf{Z}$, whereas $\pi_1(D^2)$ is trivial. The success of this argument depended on the fact that π_1 is *functorial*, meaning that it was able to capture not only information about the topological spaces upon which it acts but also translate maps between these spaces into algebraic information.

Another application of the fundamental group was in a proof of the fundamental theorem of algebra. Here, we used the fact that if a polynomial $f \in \mathbf{C}[x]$, say of degree d, were able to have no roots, then it would represent a (continuous) map $f : \mathbf{C} \to \mathbf{C}^*$ (where $\mathbf{C}^*$ is complex plane punctured at the origin, which we know to be homotopy equivalent to S^1), which must be nullhomotopic since $\mathbf{C}$ is contractible. A contradiction then arose from the observation that the restriction of f to a circle of radius sufficiently large for z^d to dominate would have to represent a map $S^1 \to S^1$ that was simultaneously nullhomotopic (and hence of degree 0) and of degree d. Here, we took advantage not only of functoriality, but also of *homotopy invariance* - the fact that maps that were unique up to homotopy induced the same homomorphism of fundamental groups.

Despite all that it has done for us, the fundamental group is still very far from being a tool sufficient for getting anywhere close to our central goal of classifying topological spaces up to homotopy equivalence. In the hope of making further progress, a natural way to proceed would be to construct more algebraic invariants that come equipped with desirable properties inspired by those of the fundamental group, and to investigate both the computation and the application of these new invariants - this turns out to lead to a rich and successful theory, and will thus serve as our main goal in this course.

3 What lies ahead

Let us start putting this into formal language.

The basic object of study in our course will be an **algebraic invariant** of a topological space. This will associate to each topological space X a group $H(X)$ such that:

- the association is functorial. H will be a functor from the category (Top) of topological spaces to the category (Grp) of groups
- the association is homotopy invariant. H, as a functor, will factor through the naïve homotopy category $(hTop)$. (Essentially, this is the category with the same objects as (Top) but whose morphisms are homotopy equivalence classes of maps).

Notice that the association $X \mapsto \pi_1(X, x_0)$ satisfies these desired properties of an algebraic invariant, and therefore a natural way to begin looking for more of these will be simply to generalise this notion.

3.1 Higher Homotopy Groups

Let us start lazy, and generalise our old notion in the most obvious way possible.

Recall the definition of $\pi_1(X, x_0)$ as the set of homotopy classes of based maps $[S^1, X]_*$ (equivalently, the set of homotopy classes of maps $(I, \partial I) \to (X, x_0)$). This forms a group under concatenation.

Here goes nothing:

Definition 3.1.1: The **nth homotopy group** of a topological space X, denoted $\pi_n(X, *)$, is the set of homotopy classes of based maps $[S^n, X]_*$ with law of composition given by concatenation.

As with the case of π_1, here we can think of $\pi_n(X, *)$ as being the set of equivalence classes of maps $(I^n, \partial I^n) \to (X, *)$ with concatenation defined as follows:

$$f * g(t_1, ..., t_n) = \begin{cases} f(2t_1, ..., t_n) & t_1 \in [0, \frac{1}{2}] \\ g(2t_1 - 1, ..., t_n) & t_1 \in [\frac{1}{2}, 1] \end{cases}$$

This just corresponds intuitively to taking the n-cubes representing the domains of f and g, squishing them each in half respectively, and reassembling them into one n-cube representing the domain of $f * g$.

Well that was hard.

In all seriousness though, for such minimal effort put into defining them, the higher homotopy groups retain and expand upon a lot of the good properties of the fundamental group. I'll leave the task of verifying that π_n is indeed a functor $(Top) \to (Grp)$ factoring through $(hTop)$ to you, but here are a couple nice surprises:

Claim 3.1.2: For all $n \geq 2$, $\pi_n(X, *)$ is always abelian. Furthermore, if X is connected, then $\pi_n(X, *)$ does not depend on the choice of basepoint (and can thus be referred to simply as $\pi_n(X)$).

So in some sense, the higher homotopy groups are theoretically better behaved than π_1 - this will be a common theme later in our discussion of homotopy theory.

Recall in 55B or 131, we had an extensive discussion of coverings of a topological space, and saw that understanding these coverings gave us valuable information about a space's fundamental group and vice versa. We are fortunate that this is yet another continuing theme - fibre bundles, which are the generalisation of covering spaces in the sense that we allow the covering map $\pi : E \to B$ to have any topological space F as its fibre, will feature heavily in our discussion of the higher homotopy groups. For example, here is a result that directly generalises the 'covering space "exact sequence"':

Claim 3.1.3: (Homotopy Fibre Sequence) Let $\pi : E \to B$ be a fibre bundle with fibre F. We have a long 'exact sequence' as follows:

$$... \xrightarrow{\pi_*} \pi_{n+1}(B) \xrightarrow{\delta} \pi_n(F) \xrightarrow{\iota_*} \pi_n(E) \xrightarrow{\pi_*} \pi_n(B) \xrightarrow{\delta} \pi_{n-1}(F) \xrightarrow{\iota_*} ...$$

Best of all, for an important class of topological spaces known as CW complexes (to which most of the geometric objects one would apply topology to belong), we shall see that the homotopy groups actually do succeed at being a complete set of invariants, which as we mentioned earlier, is not a task accomplished by the fundamental group alone (for example, S^4 and $\mathbf{CP}^2$ are both 4-manifolds with trivial π_1 - the fact that they are not homotopy equivalent is only known through computing their respective π_2):

Claim 3.1.4: (Whitehead's Theorem) Let X and Y be two connected CW complexes. If there exists an $f : X \to Y$ such that for each $n \geq 1$, the induced map $f_n^* : \pi_n(X) \to \pi_n(Y)$ is an isomorphism, then X and Y are homotopy equivalent.

This is all great news. In principle, we now have a set of algebraic invariants for CW complexes that can be used to distinguish between them. All that's left for us to do, in theory, is given a CW complex, compute all of its homotopy groups...

And here, my friends, is where things start to unravel.

When encountering an unfamiliar topological space, a natural way to understand it better is to first express it as the union of open subsets that are each homotopy equivalent to familiar spaces. We saw in 55B or 131 that the Seifert van Kampen Theorem allows us to study an unfamilar space's fundamental group in this fashion. This turns out to be pivotal for the fundamental group's success as a practical tool - it is to our dismay that van Kampen does not generalise to the higher homotopy groups, which substantially affects our ability to compute them in practice.

To understand the extent of this problem, one need not look further than the homotopy groups of spheres. Curiously, for $n \geq 2$, $\pi_k(S^n)$ for $k > n$ is not only highly nontrivial but also highly unpredictable. A computational algorithm for determining these groups has not yet been found, and as a result, progress on filling this table is ongoing, slow and laborious.

It is thus clear that we are in need of a more practical alternative. Here is what mathematicians came up with:

3.2 Homology

We again start by unpacking the definition of the fundamental group, and see how we can generalise it.

We make an observation: To say that a map $f : S^1 \to X$ belongs to the trivial equivalence class in $\pi_1(X, *)$ is to say that it is the restriction to the boundary ∂D^2 of a map $\phi : D^2 \to X$. The nontrivial elements of $\pi_1(X, *)$, of interest to us, are the maps for which this does not hold.

Great, now let's attempt a generalisation: A map from a boundary-less m-dimensional 'thing' M into X, which we will call an **m-cycle**, is to-be-considered trivial if it is *itself* an **m-boundary** - the restriction to ∂N of a map from an (m+1)-dimensional 'thing' N into X. We will interest ourselves in finding maps $M \to X$ for which this is not the case.

You might be wondering - what exactly is a 'thing' and how on earth are we going to make that precise? Well, homology turns out to be an invariant with several candidate *theories* (by this we just mean a means of computation) that essentially differ on how they each define a 'thing'. It really is a miracle that somehow these all manage to coincide for (most of) the spaces we care about!

We shall have plenty of time during the course to familiarise ourselves with a few of these theories and what exactly makes them valid definitions of 'homology' - for now, let us first focus on getting one of these to work.

Let us discuss *singular* homology. In this theory, our 'things' of choice will take the form of spaces that can be realised as unions of the following building block:

Definition 3.2.1: The **standard n-simplex** Δ^n is defined as the locus

$$\Delta^n := \{(x_0, ..., x_n) \in \mathbf{R}^{n+1} \mid \sum x_i = 1 \text{ and } x_i \geq 0 \ \forall i\}$$

Sketching a few low-dimensional cases will hopefully convince you that this is essentially just an n-dimensional analogue of a triangle.

Notice that within Δ^n, the locus where one of the $x_i = 0$ is just a standard (n-1)-simplex. Formally, we can identify Δ^{n-1} with Δ^n via a **face map** - an inclusion given by $d^i : (x_0, ..., x_{n-1}) \mapsto (x_0, ..., x_i, 0, x_{i+1}, ..., x_n)$.

At this point, it is tempting for us to note that if we possess enough knowledge about our space X to *triangulate* it - express X as a *specific* union along faces of maps $\Delta^n \hookrightarrow X$ - then our task just amounts to finding the unions of simplices within this *simplicial complex structure* of X which do form interesting cycles. This is useful computationally, and is in fact the story of *simplicial homology* (which is not discussed in Miller's text, though a discussion of the simplicial homology of Δ-complexes is offered in section 2.1 of Hatcher's text if you are interested). However, this extra structure that we are imposing on X presents an unnecessary theoretical obstacle. It is not clear that the homology of X, computed in this way, is independent of the *choice* of triangulation. To bypass this type of issue, it will prove more efficient to do theoretical work with singular homology, whose construction does not require (and thus does not depend on) such a choice. For this we will want to be able to 'reach around in the dark' with arbitrary maps $\Delta^n \to X$:

Definition 3.2.2: Let X be a topological space. By a **singular n-simplex** in X we mean a map $\sigma : \Delta^n \to X$, and we will denote the set of singular n-simplices in X by $Sin_n(X)$

In order to realise the 'things' of interest as unions of images of these maps, we would first need a notion of 'adding' simplices. We correspondingly make the following construction:

Definition 3.2.3: The group of **singular n-chains**, $S_n(X)$, is the free abelian group generated by $Sin_n(X)$.

This enables us to encode an m-dimensional 'thing' algebraically as a **Z**-linear combination of singular m-simplices, and also gives us a means to relate such a 'thing' to its boundary as follows:

Definition 3.2.4: Let $d^i : \Delta^{n-1} \hookrightarrow \Delta^n$ for $0 \leq i \leq n$ be the face maps of Δ^n. We define the **boundary operator** $d_n : S_n(X) \to S_{n-1}(X)$ by setting

$$d_n\sigma := \sum_{i=0}^{n} (-1)^i \ \sigma \circ d^i$$

for any singular n-simplex σ in X and extending by linearity.

With all of our pieces in place, we can now make our desired definition in one fell swoop:

Definition 3.2.5: We say that an element γ of $S_m(X)$ is an **m-cycle** if it has no boundary - i.e. if $d_m\gamma = 0$. The set of m-cycles is the subgroup $ker(d_m)$ in $S_m(X)$ and we will denote it by $Z_m(X)$.

Definition 3.2.6: We say that an element β of $S_m(X)$ is an **m-boundary** if it is the boundary of some singular (m+1)-chain α - i.e. if $d_{m+1}\alpha = \beta$. The set of m-boundaries is the subgroup $im(d_{m+1})$ and we will denote it by $B_m(X)$.

We make a key observation: for any non-negative integer m, we always have that $d_m \circ d_{m+1} = 0$ (Please do try verifying this yourself). This means that $B_m(X)$ is always a subgroup of $Z_m(X)$, which guarantees the existence of:

Definition 3.2.7: The **mth singular homology group** of a topological space X is the quotient

$$H_m(X) := \frac{Z_m(X)}{B_m(X)}$$

Now, let us take a step back from that monster of a definition and see where we are. Unfortunately, we still find ourselves in a bit of a mess - recall that to convince ourselves that we really have succeeded in finding an algebraic invariant of a topological space, we would like to show that H_m does actually satisfy the properties we wanted it to have! Showing that functoriality holds will turn out not give us too much of an issue, though it is definitely not a piori clear (and highly nontrivial to prove) that H_m is homotopy invariant...

Anyhow, for the sake of boring you further with too much unfamiliar terminology, I shall omit the details of this verification and instead go on to show you why we worked so hard to define homology by discussing the properties of homology that make it effectively computable.

We said earlier that a key fault of the higher homotopy groups was the inability for us to understand them 'additively'. Here are two commonly exploited facts which will hopefully convince you that homology does not suffer from the same defect.

This first result relates the homology of a topological space to that of one of its subspaces:

Claim 3.2.8: (Long Exact Sequence of a Pair) Let X be any topological space and $A \subset X$ be any subspace. Setting $S_n(X, A) := \frac{S_n(X)}{S_n(A)}$, we have for each n a short exact sequence

$$0 \longrightarrow S_n(A) \longrightarrow S_n(X) \longrightarrow S_n(X, A) \longrightarrow 0$$

which in turn induces the following long exact sequence:

$$... \longrightarrow H_{n+1}(X,A) \xrightarrow{\delta} H_n(A) \longrightarrow H_n(X) \longrightarrow H_n(X,A) \xrightarrow{\delta} H_{n-1}(A) \longrightarrow ...$$

The power of this long exact sequence is especially evident when $H_*(X,A) \to H_*(X/A,*)$ is an isomorphism. For example, this holds for the case $(X,A) = (D^m, \partial D^m)$ for $m \geq 2$, which together with the long exact sequence gives us that the homology groups of spheres must be:

$$H_n(S^{m-1}) = \begin{cases} \mathbf{Z} & n = 0, m-1 \\ 0 & otherwise \end{cases}$$

Note the great contrast between this and the irregularity of the homotopy groups of spheres! Big sigh of relief.

The second result's usefulness is self-explanatory - it is essentially a homological analogue of the Seifert van Kampen Theorem on steroids:

Claim 3.2.9: (Mayer-Vietoris Sequence) Let X be a topological space, and $\{A, B\}$ be an open cover of X. We have the following long exact sequence:

$$\begin{tikzcd}
 & ... \arrow[r] & H_{n+1}(X) \arrow[dll, "\delta"] \\
H_n(A \cap B) \arrow[r, "\alpha"] & H_n(A) \oplus H_n(B) \arrow[r, "\beta"] & H_n(X) \arrow[dll, "\delta"] \\
H_{n-1}(A \cap B) \arrow[r] & ... &
\end{tikzcd}$$

To give a quick application, one can use the Mayer-Vietoris sequence to immediately deduce the homology of a wedge sum of spheres, which has powerful consequences for the homology theory of CW complexes...

The moral of the story is - homology is quite awkward to define, and establishing even its basic properties as an algebraic invariant will take quite a bit of work and new language. All this effort, though, will be worth it as you really do end up with a much more effectively computable alternative to higher homotopy.

4 How we will proceed

The pedagogy of algebraic topology is still a very debated subject matter in the mathematical scene today. Second courses on the subject can look very different depending on the choice of textbook and the whims of the instructor. Some prefer the practically-driven approach (for example, that presented in Hatcher's text) that emphasises the geometric style of argument, whereas others might prefer the more conceptual progression that emphasises the presentation of the modern categorical abstractions of the subject (given for example by May's text). I personally have found reason to be in agreement with both approaches, and believe that having good command of both the categorical and homological-algebraic language and the geometric intuition is essential to have a true grasp of the subject.

With that said, here is an overview of our course.

- Math 231A, instructed by Professor Ye Fan, will focus on homology theory. We will be following the Lecture Notes of Professor Haynes Miller, who takes a formal and practically-driven point of view.
  - The course will open with the construction of *singular homology* and the establishment of its key properties as an algebraic invariant. We will use this as an opportunity to review and expand upon some of the categorical language introduced in Math 55 or 131, as well as introduce the formulation of singular homology in terms of *chain complexes*, which will provide us with a neater language in which to state and prove the theory's *excisive* and *additive* properties. The *Eilenberg-Steenrod axioms* will arise as a distillation of the key aspects of the singular theory, freeing the subject of homology from our choice of implementation.
  - The establishment of the homology groups of spheres in the preceding unit will enable us to develop the theory of *cellular* homology for CW complexes. A wish to reformulate homology into the language of vector spaces over field coefficients (which we know to be algebraically simpler to manipulate than the language of general $\mathbf{Z}$-modules) will lead us to a discussion of the theory of tensor products of chain complexes. This desire will (eventually) be fulfilled in the presentation of the *Universal Coefficient Theorem*, which will (as a pleasant surprise) generalise quite naturally to the *Künneth formula* - a result enabling us to compute the homology of a product of topological spaces.
  - We will then turn to the study of *cohomology*, the algebraic dual of homology. A lot of the basic properties of cohomology will turn out to mimic their homological counterparts, though we will find a nice added bonus - the ability for us to equip cohomology with a '*cup product*' gives it the additional structure of a ring. As a capstone to our discussion, we will discuss the homology theory of manifolds, which will reveal a hidden symmetry between the homology and the cohomology of such spaces in the form of the beautiful result of *Poincaré duality.*
- Having not yet discussed how Math 231BR will be run with Professor Senger who will be teaching it in the spring, I am hesitant to make concrete

conclusions about how it will proceed - my prediction is that it will be a course on homotopy theory and fibre bundle theory. Please remind me to update this when appropriate if you are planning on continuing on to 231BR!

In any case, I hope that this prologue has given you a better idea of what we will be doing in Math 231 and how we will be doing it. Please feel free to email me at gregwong@college.harvard.edu with any questions you might have after reading this, and I look forward to discussing the material with you all this year.

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