Categories via Algebraic Topology

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1 Author's Note

This article will seek to provide an account of and to expand upon review sections, led by the author, on the fundamentals of category theory given to students of the Fall 2024 season of Harvard's Math 231A. The discussion will focus on the constructions that have the most direct relevance to the study of algebraic topology.

In an attempt to make this a useful reference throughout the course, examples will be included in an order that does not directly resemble that of Math 231. As such it is not required for one to engage with all the presented examples in a linear fashion upon first reading.

2 Motivation

In one's study of different types of mathematical objects, it is often the case that one comes across different versions of the same 'type' of structure. Though the individual definitions of these versions may differ depending on the specific nature of the object in question and of the way in which *maps* between these objects are defined, it is often the case that these distinctions disappear when one analyses just their underlying behaviour with respect to composition (i.e. when maps in or out of such a construction arise and what do these maps do). Category theory provides a formalism for the study of this behaviour by representing mathematical systems as diagrams of arrows - which simplifies their study to just that of their underlying compositional structure.

Now a universal language in modern mathematics, category theory actually originally arose to provide a more succinct and illuminating language to construct the foundations of algebraic topology. It is thus my hope that through introducing the concepts of category theory via the lens of the content Math 231, you would gain a better sense of the *true nature* of each categorical construction we will introduce.

3 Categories

We start, of course, by restricting our study of 'diagrams of arrows' to just those that could be representative of how a mathematical system would actually behave. This idea is formalised in the concept of a category.

Definition 3.1: A **category** $\mathscr{C}$ consists of the following data:

- A collection of **objects**, denoted $Ob(\mathscr{C})$.
- For every pair of objects $A, B \in \mathscr{C}$, a collection of **morphisms**, denoted $Mor_{\mathscr{C}}(A, B)$.
- A rule of **composition**: For every triple $A, B, C \in \mathscr{C}$, a map

  $\phi_{ABC} : Mor_{\mathscr{C}}(A, B) \times Mor_{\mathscr{C}}(B, C) \to Mor_{\mathscr{C}}(A, C)$.

  For $\eta \in Mor_{\mathscr{C}}(A, B)$ and $\xi \in Mor_{\mathscr{C}}(B, C)$, We denote $\phi_{ABC}(\eta, \xi)$ by $\xi \circ \eta$.

We require these to satisfy two axioms:

1. (**Identity**) For every $A \in Ob(\mathscr{C})$, there exists an element $id_A \in Mor_{\mathscr{C}}(A, A)$ such that for any $A, B \in Ob(\mathscr{C})$ and any $f \in Mor_{\mathscr{C}}(A, B)$, we have the commutation of the following diagram:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "id_A"'] \arrow[dr, "f"] & B \arrow[d, "id_B"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

2. (**Associativity**) For any objects $A, B, C, D \in Ob(\mathscr{C})$, we have the commutation of the following diagram:

$$\begin{tikzcd} Mor_{\mathscr{C}}(A,B) \times Mor_{\mathscr{C}}(B,C) \times Mor_{\mathscr{C}}(C,D) \arrow[r, "\phi_{ABC} \times id_{Mor_{\mathscr{C}}(C,D)}"] \arrow[d, "id_{Mor_{\mathscr{C}}(A,B)} \times \phi_{BCD}"] & Mor_{\mathscr{C}}(A,C) \times Mor_{\mathscr{C}}(C,D) \arrow[d, "\phi_{ACD}"'] \\ Mor_{\mathscr{C}}(A,B) \times Mor_{\mathscr{C}}(B,D) \arrow[r, "\phi_{ABD}"] & Mor_{\mathscr{C}}(A,D) \end{tikzcd}$$

Note: The use of the word 'collection' here is deliberate to avoid logical issues that arise when one attempts to refer to things such as the 'set of all sets'. The rigorous notion here used to describe collections such as $Ob(\mathscr{C})$ and $Mor_{\mathscr{C}}(A, B)$ is known as a *proper class* if you are interested in reading up on this.

Let's begin acquiring a vocabulary for our new language by giving some examples of categories. Please do convince yourself that in each of the following cases all the aforementioned axioms are indeed satisfied.

Example 3.2: Here is a list of categories commonly used in algebraic topology:

1. The category (Set), whose objects are sets and morphisms are maps of sets.
2. The category (Top), whose objects are topological spaces and whose morphisms consist of (continuous) maps between them.

   Note: A key subtlety - in practice, the default setting of algebraic topology is often taken implicitly to be the category $(kTop)$ of *compactly-generated* topological spaces, which is a *full subcategory* of (Top). The key intuition for doing this is to ensure that for any two topological spaces X and Y in $(kTop)$, the space Y^X of maps $X \to Y$ is equipped with a well-behaved topology, which allows $(kTop)$ to be Cartesian-closed. Consult either Chapter 5 of May's text or Chapter 40 of Miller's Notes to see a detailed discussion of this - I shall not digress further in order to not get sidetracked from our discussion.

3. A couple important variations of (Top):
   - The category (Top^*) of based topological spaces, with $Ob((Top^*))$ consisting of pairs $\{(X, *_X)\}$, where X is a topological space and $*_X \in X$ is a distinguished base point, and $Mor_{(Top^*)}(X, Y)$ consisting of maps $f : (X, *_X) \to (Y, *_Y)$. This is important for talking about homotopy groups rigorously.
   - As a generalisation of (Top^*), the category (Top_2) of pairs of topological spaces (X, A) where A is a subspace of X. This is the setting of relative homotopy and relative homology. We similarly have the category (Top_3) of triples (X, A, B) with $B \subset A \subset X$, and the category (Top_3^*) of based triples - both of these arise in homotopy theory.
   - The homotopy category $(hTop)$. The objects of this category are the same as those of (Top), but whose morphisms are homotopy equivalence classes of maps. This captures the notion of homotopy invariance, and consequently is the category through which we would like algebraic invariants to factor. The categories $(hTop^*)$, $(hTop_2)$ and so on are similarly defined.
   - Let B be a topological space. We define the category (Top_B) of spaces over B as follows:
     - We set $Ob(Top_B)$ to be the collection of pairs (E, π_E) where $\pi_E : E \to B$ is a map.

– We set $Mor_{Top_B}(E, E')$ to be collection of maps $f : E \to E'$ which enable the commutation of the following diagram:

$$\begin{tikzcd} E \arrow[rr, "f"] \arrow[dr, "\pi_E"'] & & E' \arrow[dl, "\pi_{E'}"] \\ & B & \end{tikzcd}$$

This is the general setting for talking about fibre bundles over B, including covering spaces and principal G-bundles over B and so forth.

4. The category (Grp), whose objects are groups, and whose corresponding morphisms consist of homomorphisms between them. The category of abelian groups (Ab) is a full subcategory of (Grp).

5. Other important algebraic objects that will arise in our discussion:

   - The category $(CRing)$ of commutative rings, whose morphisms are ring homomorphisms.
   - The category $(Field)$ of fields, whose morphisms are homomorphisms of fields, forms a full subcategory of $(CRing)$.
   - Fix a commutative ring R. We have the category (Mod_R) of modules over R, whose morphisms are R-linear maps. If R happens to be a field k, then this is equal to the category $(Vect_k)$ of vector spaces over k.
   - We will often work not with R-modules by themselves but with chain complexes of R-modules. These also form a category $(Chain_R)$, whose morphisms are chain maps.

Example 3.3: And here are some other examples that represent a fuller scope of what a category can represent:

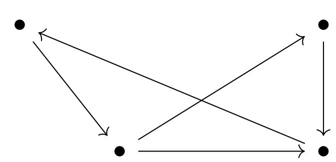
1. Notice above that all the categories we formed earlier come in the form of collections of decorated sets - sets with added structure. This does *not* have to be the case. One example of this is the category $\mathscr{N}$, whose objects are the natural numbers, and a morphism $a \to b$ exists iff a divides b.

2. As a generalisation of this idea, any preordered set $(P, \leq)$ has itself the structure of a category $\mathscr{P}$ in a similar fashion.

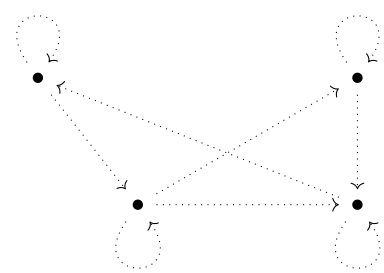
3. Any group G may be regarded as itself a 1-object category, with morphisms given by $\{$left actions of $g \in G$ on $G\}$. The usefulness of this idea will become apparent when we come to discuss the theory of *classifying spaces* later on in the year.

4. Note that a group as a category has the property that every morphism is an isomorphism (i.e. for each g in $Mor_G(A, B)$ there is an h in $Mor_G(B, A)$ such that $g \circ h = id_B$ and $h \circ g = id_A$). Categories that have this property are known as **groupoids**. A fun example of such a construction (for our purposes) is the *fundamental groupoid* $\Pi(X)$ of a topological space X, which gives another way to talk about the fundamental group of X. This has $Ob(\Pi(X)) = X$ and $Mor_{\Pi(X)}(x, y) = \{$homotopy equivalence classes of paths from x to $y\}$ (so then $Mor_{\Pi(X)}(x, x)$ is in fact just $\pi_1(X, x)$). See Chapter 3 of May's text for an alternative formulation of the theory of covering spaces using this language if you are interested.

5. Finally, any *directed graph* J generates a category $\mathscr{J}$, which we will call the **free category generated by** J. We can do this by taking $Ob(\mathscr{J})$ to be the vertices of J, and designating any finite directed path from k to l (for k, $l \in J$) to be an element of $Mor_{\mathscr{J}}(k, l)$. We will take the set of directed vertices of J, augmented with 'identity' paths when required by the axioms of a category, to generate the full set of possible paths by finite applications of the law of composition. Here is an example:



J



$\mathscr{J}$

The importance of this construction will become evident when we go on to discuss *limits* and *colimits* in full generality.

4 Products and Sums

We mentioned earlier that one of the purposes of category theory is to unify the study of constructions that are similar up to how they arise compositionally. We have already developed sufficient language to begin on this goal by discussing products and sums.

Definition 4.1: Let $\mathscr{C}$ be a category, and let $A, B \in Ob(\mathscr{C})$. The **product** of A and B, if it exists, is an object $A \times B \in Ob(\mathscr{C})$ together with a pair of morphisms $\pi_A : A \times B \to A$ and $\pi_B : A \times B \to B$ satisfying the following property:

For all objects $Z \in Ob(\mathscr{C})$ with morphisms $\alpha : Z \to A$ and $\beta : Z \to B$, there exists a morphism $\phi_{\alpha\beta} : Z \to A \times B$ such that the following diagram commutes:

$$\begin{tikzcd} & Z \arrow[dl, "\alpha"'] \arrow[d, dashed, "\phi_{\alpha\beta}"'] \arrow[dr, "\beta"] & \\ A & A \times B \arrow[l, "\pi_A"'] \arrow[r, "\pi_B"] & B \end{tikzcd}$$

A couple immediate observations:

- Firstly, if such an object exists, it is unique up to isomorphism (to clarify, an isomorphism is simply a morphism with both left and right inverses).
- Secondly, one can extend this notion to a product of a family of objects indexed by an arbitrary set I, provided that such an object does indeed exist (we will see that this is in fact not guaranteed). It is important to note here that I does not need to be of finite cardinality.
- Lastly, a quick word on nomenclature - we will say that a category *has all products* if a product exists for every indexed family of objects.

All of these will also apply to sums, which we will now define.

Definition 4.2: Let $\mathscr{C}$ be a category, and let $A, B \in Ob(\mathscr{C})$. The **sum** or **coproduct** of A and B, if it exists, is an object $A \oplus B \in Ob(\mathscr{C})$ together with a pair of morphisms $\iota_A : A \to A \oplus B$ and $\iota_B : B \to A \oplus B$ satisfying the following property:

For all objects $Z \in Ob(\mathscr{C})$ with morphisms $\alpha : A \to Z$ and $\beta : B \to Z$, there exists a morphism $\psi_{\alpha\beta} : A \oplus B \to Z$ such that the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "\iota_A"] \arrow[dr, "\alpha"'] & A \oplus B \arrow[d, dashed, "\psi_{\alpha\beta}"'] & B \arrow[l, "\iota_B"'] \arrow[dl, "\beta"] \\ & Z & \end{tikzcd}$$

Again, I would like to emphasise that existence of sums, like the existence of products, is not guaranteed for an arbitrary category. Let us now see what product and sum objects look like in some of the categories we introduced above.

Example 4.3: As promised. In verifying each of the following, I suggest a constructive approach - pay attention especially to the results in bold.

1. In (Set), products are Cartesian products of sets, and coproducts are disjoint unions.

2. In (Top), products are Cartesian products equipped with the *product topology* (**not** the box topology - this distinction is important when referring to infinite products), and coproducts are disjoint unions (equipped with the disjoint union topology).

3. In (Top^*), the product of $(X, *_x)$ and $(Y, *_y)$ is $(X \times Y, (*_X, *_Y))$, and their coproduct is their **wedge product** $(X \vee Y, *_{X \vee Y}) = (X \sqcup Y / *_X \sim *_Y, *_{X \vee Y} = *_X = *_Y)$.

4. In (Top_B), coproducts are disjoint unions as in (Top). Products in this category though are more subtle - we will come back to this later when we discuss the notion of a pullback. Why is the product of two objects E and E' in this category **not** just $E \times E'$?

5. In (Ab), products are direct products of abelian groups, and coproducts are given by direct sums of abelian groups. Note that although these notions coincide for a finite number of abelian groups, this is **not** the case for an infinite number of objects - can you work out what the distinction might be?

6. As a direct generalisation of (Ab), (Mod_R) (and also $(Vect_k)$) also have direct products as products and direct sums as coproducts.

7. In (Grp), products are direct products of groups, and a coproduct of groups G and H is their **free sum** $G * H$. (To see why the direct sum here does not work, explain why taking $Z = G * H$ would render taking $G \times H$ as a coproduct impossible).

8. $(CRing)$ also has products and coproducts. The products here are again given by direct products, but the coproduct of two objects R and S is their **tensor** product $R \otimes_{\mathbf{Z}} S$. (To see that why the direct sum $R \times S$ does not work, pay attention to whether the inclusions ι_R and ι_S could send 0_R, 0_S, 1_R and 1_S to where they need to go).

9. The category $(\mathscr{N})$ gives a cute demonstration. Here, the product of numbers a and b is their greatest common denominator $gcd(a, b)$, and their coproduct is their lowest common multiple $lcm(a, b)$.

Example 4.4: We claim that $(Field)$ is an example of a category that has neither all products nor all coproducts.

Proof: Let K and L be fields of different characteristic. If K and L were to have a product P, then the existence of maps $\pi_K : P \to K$ and $\pi_L : P \to L$ would mean that $char(P) = char(K)$ and $char(P) = char(L)$ which is a contradiction. The proof for non existence of a coproduct of K and L is analogous.

Other examples of nonexistence of products and coproducts abound, and importantly these two conditions are independent of each other. Try for example to come up with a directed graph whose corresponding generated free category that has all products but not all coproducts, and vice versa.

5 Functors

In our study of algebraic topology, we see that the usefulness of the group structure of homotopy, homology and cohomology is augmented by the fact that the associations π_n, H_n and H^n not only map topological spaces to groups, but also translate maps between topological spaces into algebraic information. This idea of a map between categories preserving the source category's morphism structure will be captured more generally by the notion of a *functor*.

There are two variants of this concept. Both will be relevant to our work ahead.

Definition 5.1: Let $\mathscr{C}$ and $\mathscr{D}$ be categories. A **covariant functor** $F : \mathscr{C} \to \mathscr{D}$ consists of the following data:

- A map $F : Ob(\mathscr{C}) \to Ob(\mathscr{D})$. That is, to each $A \in Ob(\mathscr{C})$, we assign an object $F(A) \in Ob(\mathscr{D})$.
- To each pair $A, B \in Ob((C))$, a map $Mor_{\mathscr{C}}(A, B) \to Mor_{\mathscr{D}}(F(A), F(B))$ assigning to each $\psi : A \to B$ a morphism $F(\psi) : F(A) \to F(B)$.
- The assignment of morphisms must be done in a way that satisfies the two following axioms:
  - For each $A \in Ob(\mathscr{C})$, we have $F(id_A) = id_{F(A)}$
  - For each $A, B, C \in Ob(\mathscr{C})$, the commutation of:

$$\begin{tikzcd} Mor_{\mathscr{C}}(A,B) \times Mor_{\mathscr{C}}(B,C) \arrow[r, "F \times F"] \arrow[d, "\phi_{ABC}"] & Mor_{\mathscr{D}}(F(A),F(B)) \times Mor_{\mathscr{D}}(F(B),F(C)) \arrow[d, "\phi_{F(A)F(B)F(C)}"'] \\ Mor_{\mathscr{C}}(A,C) \arrow[r, "F"] & Mor_{\mathscr{D}}(F(A),F(C)) \end{tikzcd}$$

Analogously,

Definition 5.2: A **contravariant functor** F from $\mathscr{C} \to \mathscr{D}$ is defined similarly, except that the map of morphisms instead takes $Mor_{\mathscr{C}}(A, B)$ to $Mor_{\mathscr{D}}(F(B), F(A))$, with the second axiom in the third bullet point above altered accordingly.

Note: A quick aside. For any category $\mathscr{C}$, we can form its *opposite category* $\mathscr{C}^{op}$ by reversing the direction of all its morphisms. One can then see that a contravariant functor $\mathscr{C} \to \mathscr{D}$ is in fact just a covariant functor $\mathscr{C}^{op} \to \mathscr{D}$. This allows us to henceforth implicitly assume covariance when referring to a functor $\mathscr{C} \to \mathscr{D}$, and indicate contravariance where necessary - this is a convention commonly used in the literature.

We can see that, for example, π_n and H_n are covariant functors $(Top) \to (Grp)$, whereas $H^n : (Top)^{op} \to (Grp)$ is contravariant.

Example 5.3: Here are a couple more occurrences of functors that will be prevalent in our discussion.

1. We observe that for any group G and any *abelian* group H, any homomorphism $\phi : G \to H$ must send its commutator subgroup $C(G)$ to the identity in H by definition. This is to say that any homomorphism from G to H factors through its *abelianisation* $G/C(G)$, which lets us give this association a notion of functoriality as follows:

   Definition: The **abelianisation functor** $(-)^{ab} : (Grp) \to (Ab)$ will send each $G \in Ob((Grp))$ to $G^{ab} = G/C(G)$, and each $\phi \in Mor_{(Grp)}(G, H)$ to a $\phi^{ab} \in Mor_{(Ab)}(G^{ab}, H^{ab})$ where ϕ^{ab} is induced as follows:

$$\begin{tikzcd} & H \arrow[dr, dotted, "F"] & \\ G \arrow[ur, dotted, "\phi"] \arrow[rr, "F\circ\phi"'] \arrow[dr, "F"'] & & H^{ab} \\ & G^{ab} \arrow[ur, dashed, "\phi^{ab}"'] & \end{tikzcd}$$

   This will come into play in our discussion of the relationship between the fundamental group and the first homology group of a connected topological space.

2. Given a set S, we can define the free group F_S generated by S. This association defines the 'free' functor $F_{(-)} : (Set) \to (Grp)$, which sends a set map $f : A \to B$ to the unique group homomorphism ϕ_f extending the map $\iota_B \circ f$, where $\iota_B : B \to F_B$ is the canonical inclusion. The association $\mathbf{Z}[-] : (Set) \to (Ab)$ sending S to the free abelian group $\mathbf{Z}[S]$ generated by S is similarly functorial, and in fact we can see that $\mathbf{Z}[-]$ is the same

as $(-)^{ab} \circ F_{-}$. This construction will be important in the computational side of our work, for example in cellular homology.

3. We can take a categorical viewpoint on the following constructions in singular homology:

   - We can view the **standard simplex** as a functor $\Delta^* : \Delta \to (Top)$ (where Δ is the simplex category with objects being the set of $[n] = \{0, 1, ..., n\}$ and morphisms being order-preserving maps).
   - Fix a topological space X. We can define a functor $\Delta^{op} \to (Set)$ sending $[n] \mapsto \{\text{maps: } \Delta^n \to X\}$. Observe that the images of the morphisms in Δ are compositions of the face and degeneracy maps between the images of simplices in X.
   - More generally, a functor $\Delta^{op} \to (Set)$ is known as a **simplicial set**.
   - We shall see that the collection of simplicial sets forms a category $(sSet)$ (observe that the morphisms in $(sSet)$ would be *maps between functors* - start thinking about how we might make this precise). This thus makes the association $Sin_*(-) : (Top) \to (sSet)$ sending $X \mapsto Sin_*(X)$ functorial.
   - Combining this component-wise with the 'free' functor $\mathbf{Z}[-]$ defined above gives us the functor $S_*(-) : (Top) \to (sSet)$.
   - It turns out that there exists a sort of 'inverse' map to S_* in the following way. To each simplicial set K_* we can associate it functorially to its **geometric realisation** $|K_*| = \sqcup_{n \geq 0} K_n \times \Delta^n / \sim$, where the equivalence relation ensures that the face and degeneracy maps in K_* get taken to appropriate face and degeneracy maps between simplices. When we take K_* to be the singular chain $S_*(X)$ of a space X, it turns out that there exists a weak equivalence between the resulting space $|S_*(X)|$ and X, which gives us a bridge between singular and cellular homology.

4. Fix a commutative ring R. Observe that for any R-module M, we can define functors $(-) \otimes_R M : (Mod_R) \to (Mod_R)$ and $Hom_R(-, M) : (Mod_R)^{op} \to (Mod_R)$. An important part of the subject of homological algebra, whose tools we will be using to develop homology using coefficients other than $\mathbf{Z}$, will be to investigate how these two functors behave with respect to exact sequences of R-modules.

   Note: More generally, an analogue of the former construction works in any *monoidal category*. As for the latter construction, note that we do not in general have that $Mor_{\mathscr{C}}(-, A)$ is an object in $\mathscr{C}$ of course, but for categories whose collections of morphisms form sets, the 'Hom' functors $Mor_{\mathscr{C}}(A, -) : \mathscr{C} \to (Set)$ and $Mor_{\mathscr{C}}(-, A) : \mathscr{C}^{op} \to (Set)$ are indeed well defined. More about this later when we talk about Yoneda's lemma.

5. Think of a category $\mathscr{J}$ as an index category (for example, let this be the free category generated by a directed graph J). A **diagram** of shape $\mathscr{J}$ in $\mathscr{C}$ is a functor $D_{\mathscr{J}\to\mathscr{C}} : \mathscr{J} \to \mathscr{C}$. Here is an example:

$$\begin{tikzcd}
\bullet \arrow[loop above, dashed] \arrow[dr, dashed] & & \bullet \arrow[loop above, dashed] \arrow[d, dashed] \\
& \bullet \arrow[loop below, dashed] \arrow[ur, dashed] \arrow[r, dashed] & \bullet \arrow[loop below, dashed] \arrow[ull, dashed]
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
A \arrow[loop above, "id_A"] \arrow[dr, "\phi_{AC}"'] & & B \arrow[loop above, "id_B"] \arrow[d, "\phi_{BD}"] \\
& C \arrow[loop below, "id_C"] \arrow[urr, "\phi_{CB}"] \arrow[r, "\phi_{CD}"] & D \arrow[loop below, "id_D"] \arrow[ull, "\phi_{AD}"]
\end{tikzcd}$$

$\mathscr{J}$ $\qquad\qquad$ $Im(D_{\mathscr{J}\to\mathscr{C}})$

6 Pushouts and Pullbacks

In topology, it is often a useful tactic to study a topological space X by realising it as a union of two simpler spaces U_1 and U_2 glued along a common subspace. For example, the Seifert van Kampen theorem tells us that in the case where U_1 and U_2 are nonempty and path-connected, we have that $\pi_1(X, *)$ is isomorphic to the *amalgamated free sum* $\pi_1(U_1, *) *_{\pi_1(U_1\cap U_2, *)} \pi_1(U_2, *)$, where $*$ is a point in $U_1 \cap U_2$. Curiously, these two structures arise the same way compositionally and thus admit a common categorical generalisation.

Definition 6.1: Let $\mathscr{C}$ be a category, let A, B, $C \in Ob(\mathscr{C})$, and let $f \in Mor_{\mathscr{C}}(C, A)$, $g \in Mor_{\mathscr{C}}(C, B)$. The **pushout** of A and B along f and g, if it exists, is an object P together with morphisms $\iota_A \in Mor_{\mathscr{C}}(A, P)$, $\iota_B \in Mor_{\mathscr{C}}(B, P)$ such that the following diagram

$$\begin{tikzcd}
P & A \arrow[l, "\iota_A"'] \\
B \arrow[u, "\iota_B"] & C \arrow[l, "g"] \arrow[u, "f"']
\end{tikzcd}$$

commutes, and is also universal in the following sense - for all triples (Q, j_A, j_B) that give rise to the commutation of the following diagram, there must exist a unique induced $j_P \in Mor_{\mathscr{C}}(P, Q)$ also making it commute:

$$\begin{tikzcd}
Q & & \\
& P \arrow[ul, dashed, "j_P"] & A \arrow[l, "\iota_A"'] \arrow[ull, "j_A"'] \\
& B \arrow[u, "\iota_B"] \arrow[uul, "j_B"] & C \arrow[l, "g"] \arrow[u, "f"']
\end{tikzcd}$$

Like in the case of products and sums, I would like to remind you that existence of pushouts of guaranteed in an arbitrary category (this will be investigated in more depth later), though if a pushout exists it would be again unique up to isomorphism.

Example 6.2: The definition of a pushout is difficult one to wrap your head around at first, but as you will see through this worked example, when the category in question has objects of the form of 'decorated sets', it is generally a good starting place to have an initial view of a pushout as 'a coproduct of two objects identified along the respective images of a subobject'.

We shall demonstrate that in (Grp), the pushout of G and H along ϕ and ψ, where $\phi : K \to G$ and $\psi : K \to H$ are group homomorphisms, is the amalgamated free product $G *_K H = G * H/\phi(k) \sim \psi(k)$ for $k \in K$.

Proof: Suppose J is a group for which we have the commutation of:

$$\begin{tikzcd}
J & H \arrow[l, "\alpha_H"'] \\
G \arrow[u, "\alpha_G"] & K \arrow[l, "\phi"] \arrow[u, "\psi"']
\end{tikzcd}$$

Let $G *_K H$ be defined as above with $\iota_G : G \to G *_K H$ and $\iota_H : H \to G *_K H$ be induced from the canonical inclusions into $G * H$ from G and H respectively. Thus by definition we have that $\iota_G \circ \phi(k) = \iota_H \circ \psi(k)$.

Defining $\alpha_{G*_K H} : G *_K H \to J$ by letting $\alpha_{G*_K H}$ send $g \mapsto \alpha_G(g)$ for each $g \in G$ and $h \mapsto \alpha_H(h)$ for each $h \in H$, and extending by concatenation gives that $\alpha_{G*_K H} \circ \iota_G = \alpha_G$ and $\alpha_{G*_K H} \circ \iota_G = \alpha_H$.

Example 6.3: Here are a couple more examples of pushouts:

1. By completely analogous logic as above, we come to the conclusion that pushouts in (Top) have the form of disjoint unions identified along subspaces with a common preimage (and they similarly exist for the important variants of (Top) we have mentioned). A particularly important case of this is in the construction of a **CW-complex** X, where for each n, the nth-skeleton X_n of X can be regarded as a pushout obtained from X_{n-1} as follows:

$$\begin{tikzcd} \sqcup_{J_n} S^{n-1} \arrow[r, "\alpha_{J_n}"] \arrow[d, hook, "\iota_{J_n}"] & X_{n-1} \arrow[d, dashed] \\ \sqcup_{J_n} D^n \arrow[r, dashed] & X_n \end{tikzcd}$$

Here, ι_{J_n} is the inclusion of each S^{n-1} in J_n (the set indexing the attaching maps) to the boundary of the appropriate D^n, and α_{J_n} specifies where in X_{n-1} these attaching maps are to take place.

2. In the category $(CRing)$, the pushout of A and B along $f : C \to A$ and $g : C \to B$ is the tensor product $A \otimes_C B$, where A (resp. B) is recognised as a C-module with 'scalar multiplication' given by $(c, a) \mapsto f(c)a$ (resp. $(c, b) \mapsto g(c)b$).

Like in the case of products and coproducts, we again have a dual notion for the pushout, known as the pullback.

Definition 6.4: Let $\mathscr{C}$ be a category, and A, B, C be objects and $f \in Mor_{\mathscr{C}}(A, C)$, $g \in Mor_{\mathscr{C}}(B, C)$. The **pullback** of A and B along f and g is an object R together with morphisms $\pi_A \in Mor_{\mathscr{C}}(R, A)$ and $\pi_B \in Mor_{\mathscr{C}}(R, B)$ such that

$$\begin{tikzcd} R \arrow[r, "\pi_A"] \arrow[d, "\pi_B"'] & A \arrow[d, "f"] \\ B \arrow[r, "g"'] & C \end{tikzcd}$$

commutes, and is universal amongst such diagrams. That is, for all triples (S, ρ_A, ρ_B) making the following diagram commute, there exists a unique induced $\rho_R \in Mor_{\mathscr{C}}(S, R)$ also ensuring its commutation:

$$
\begin{tikzcd}
S \arrow[dr, dashed, "\rho_R"] \arrow[drr, "\rho_A"] \arrow[ddr, "\rho_B"'] & & \\
& R \arrow[r, "\pi_A"] \arrow[d, "\pi_B"'] & A \arrow[d, "f"] \\
& B \arrow[r, "g"'] & C
\end{tikzcd}
$$

Again, the definition here is initially difficult to grasp, but the intuition of a pullback as a 'restriction of a product to tuples whose components lie in the respective fibres over a common element' serves as a good prototype in the case where the objects of the category in question are 'decorated sets'.

Example 6.5: Here are a couple examples of pullbacks that illustrate this:

1. We asked the question earlier of what products in the category (Top_B) look like. We can now see that the product of (E, π_E) and $(E', \pi_{E'})$ is just the pullback $E \times_B E' = \{(e, e') \in E \times E'$ such that $\pi_E(e) = \pi_{E'}(e') = b\}$ of E and E' along π_E and $\pi_{E'}$, with map $\pi_{E \times_B E'} : E \times_B E' \to B$ sending $(e, e') \mapsto b$.

2. More importantly, if (E, π_E) is a fibre bundle over B and X is any space with a map $\chi : X \to B$, then the pullback $X \times_B E$ of X and E along χ and π_E is a fibre bundle over X with fibre homeomorphic to $\pi_E^{-1}(b)$ for any $b \in B$. This fact is crucial in the study of vector bundles, and more generally principal bundles, where we can demonstrate that every G-bundle over a space X is the pullback of a 'universal' G-bundle $EG \to BG$.

3. Pullbacks exist in (Grp), $(CRing)$ and (Mod_R) and all take the form of 'fibred products'. That is, the structure

$$A \times_C B = \{(a, b) \in A \times B \text{ such that } f(a) = g(b) = c\}$$

7 Limits and Colimits

Notice that products, sums, pullbacks and pushouts all have something in common - they are all 'universal factoring stations' for their own respective 'diagram shapes'. This, of course, means that they admit a common categorical generalisation in the form of the notions of *limits* and *colimits*.

To give some insight as to how on earth such a vague-sounding descriptor might be generalised, let us first take an alternate perspective on the definition of a pushout.

Definition 7.1: Let J_{push} be the directed graph shaped as '$\cdot \leftarrow \cdot \rightarrow \cdot$', and let $\mathscr{J}_{push}$ be the free category generated by J_{push}. Let $D_{\mathscr{J}_{push} \to \mathscr{C}}$ be a $\mathscr{J}_{push}$-shaped diagram in $\mathscr{C}$, whose image we shall denote $A \leftarrow C \rightarrow B$ with identity morphisms implied.

We define a co-cone of $D_{\mathscr{J}_{push} \to \mathscr{C}}$ to be an object $Q \in Ob(\mathscr{C})$ together with a family of morphisms $\psi_X : D_{\mathscr{J}_{push} \to \mathscr{C}}(X) \to Q$ for each $X \in Ob(\mathscr{J}_{push})$, such that for each morphism $f \in Mor_{\mathscr{J}_{push}}(X, Y)$, we have that $\psi_X = \psi_Y \circ D_{\mathscr{J}_{push} \to \mathscr{C}}(f)$. In other words, we have that the following diagram commutes.

$$
\begin{tikzcd}
C \arrow[r] \arrow[d] \arrow[ddrr, dashed] & A \arrow[ddr, dashed] & \\
B \arrow[drr, dashed] & & \\
 & & Q
\end{tikzcd}
$$

A **pushout** is then the colimit of $D_{\mathscr{J}_{push} \to \mathscr{C}}$, in other words, a co-cone (P, ϕ) such that for all co-cones (Q, ψ) as above, there exists a $u \in Mor_{\mathscr{C}}(P, Q)$ such that $u \circ \phi_X = \psi_X$ for all X in $\mathscr{J}_{push}$. In other words, we have that the following diagram commutes:

$$
\begin{tikzcd}
C \arrow[rr] \arrow[d] \arrow[dr, dashed] \arrow[ddrrr] & & A \arrow[dl, dashed] \arrow[ddr] & \\
B \arrow[r, dashed] \arrow[drrr] & P \arrow[drr, dotted, "u"] & & \\
 & & & Q
\end{tikzcd}
$$

Please do verify that a coproduct can be defined in exactly the same way, except with J_{push} replaced with J_{coprod}, which is just the directed graph given by a collection of points with no arrows between them. It is hopefully thus clear that this is our desired way to generalise these two constructions, letting us define the colimit of a $\mathscr{J}$-shaped diagram for any index category $\mathscr{J}$.

I have stepped all over the punchline here, but for the sake of clarity, here is the formal definition of a colimit.

Definition 7.2: Let $\mathscr{J}$ be an index category, and let $D_{\mathscr{J} \to \mathscr{C}}$ be a $\mathscr{J}$-shaped diagram in a category $\mathscr{C}$.

A co-cone of $D_{\mathscr{J}\to\mathscr{C}}$ is an object $M \in Ob(\mathscr{C})$ together with a family of morphisms $\psi_K : D_{\mathscr{J}\to\mathscr{C}}(K) \to M$ for each $K \in \mathscr{J}$, such that for each $f \in Mor_{\mathscr{J}}(K, L)$ we have that $\psi_K = \psi_L \circ D_{\mathscr{J}\to\mathscr{C}}(f)$.

The **colimit** of $D_{\mathscr{J}\to\mathscr{C}}$ then, if it exists, is a co-cone (P, ϕ) of $D_{\mathscr{J}\to\mathscr{C}}$ that is universal amongst such co-cones. That is, for any (M, ψ) as above, there exists a $\iota \in Mor_{\mathscr{C}}(P, M)$ such that $\psi(K) = \iota \circ \phi(K)$ for all K in $\mathscr{J}$.

As the name might suggest, we also, in completely analogous fashion, have the dual notion of a limit, which generalises constructions such as products and pullbacks.

Definition 7.3: Let $\mathscr{J}$ be an index category, and let $D_{\mathscr{J}\to\mathscr{C}}$ be a $\mathscr{J}$-shaped diagram in a category $\mathscr{C}$.

A cone to $D_{\mathscr{J}\to\mathscr{C}}$ is an object $S \in Ob(\mathscr{C})$ together with a family of morphisms $\psi_K : S \to D_{\mathscr{J}\to\mathscr{C}}(K)$ for each $K \in \mathscr{J}$, such that for each $f \in Mor_{\mathscr{J}}(K, L)$ we have that $D_{\mathscr{J}\to\mathscr{C}}(f) \circ \psi_K = \psi_L$.

The **limit** of $D_{\mathscr{J}\to\mathscr{C}}$ then, if it exists, is a cone (R, ϕ) of $D_{\mathscr{J}\to\mathscr{C}}$ that is universal amongst such cones. That is, for any (S, ψ) as above, there exists a $\pi \in Mor_{\mathscr{C}}(P, M)$ such that $\psi(K) = \phi(K) \circ \pi$ for all K in $\mathscr{J}$.

Doing a quick sanity check, we can see that the pullback is the limit of $D_{\mathscr{J}_{pull}\to\mathscr{C}}$, where $\mathscr{J}_{pull}$ is generated by J_{pull} shaped as '$\cdot \to \cdot \leftarrow \cdot$', and the product is the limit of $D_{\mathscr{J}_{prod}\to\mathscr{C}}$, where $\mathscr{J}_{prod}$ is generated by J_{prod}, which is again just a collection of points with no arrows between them.

Great, so what is the point of all of this? Well, it turns out that algebraic topology is abound with *limit-preserving* and *colimit-preserving* functors - in other words, covariant functors $\mathscr{C} \to \mathscr{D}$ which take limits in $\mathscr{C}$ to limits in $\mathscr{D}$ and colimits in $\mathscr{C}$ to colimits in $\mathscr{D}$ respectively - which is a big part of the reason why topological spaces that admit a nice description in this language, such as CW complexes, are well-suited to being studied by algebraic means. A plethora of examples of limit-preserving and colimit-preserving functors will arise when we discuss *adjoint pairs* in due course; for now, let us familiarise ourselves further with the language of limits and colimits.

We first introduce initial and terminal objects. These seem like mere formalities initially in a vacuum, but in fact will give us another point on view on how the notions of colimits and limits are defined.

Example 7.4: Let J_ϕ be the empty directed graph, $\mathscr{J}_\phi$ the corresponding empty free category generated by J_ϕ and $D_{\mathscr{J}_\phi\to\mathscr{C}}$ be the (unique) empty diagram in $\mathscr{C}$. We define the **initial object** of $\mathscr{C}$ as $colim(D_{\mathscr{J}_\phi \to \mathscr{C}})$, and the **terminal** or **final object** in $\mathscr{C}$ as the diagram's limit.

Spelling out this pair of rather strange definitions, it could be seen that (given existence), a category's initial object take the form of an object I(unique up to isomorphism) equipped with a unique morphism $I \to X$ to every object X in the category, whilst the terminal object of a category is an object T together with a unique morphism $Y \to T$ from each object Y in $\mathscr{C}$. Note that there are instances where these coincide - for example in (Grp), where the trivial group acts as both the initial and the terminal object (we say in which case that this is a **zero object** of the category) - but they generally do not, for example in $(Ring)$, whose initial object is $\mathbf{Z}$ and whose terminal object is the trivial ring.

We now use these notions to present alternative definitions of colimits and limits. Given a directed graph J, its corresponding free category $\mathscr{J}$ and a diagram $D_{\mathscr{J} \to \mathscr{C}}$, we define the category $(Cocone_{D_{\mathscr{J} \to \mathscr{C}}})$, whose objects are co-cones to $D_{\mathscr{J} \to \mathscr{C}}$ and whose morphisms are morphisms of co-cones - that is, given co-cones (P, ϕ) and (Q, ψ), a morphism $\gamma : P \to Q$ such that $\psi(K) = \gamma \circ \phi(K))$ for all K in $\mathscr{J}$. Given existence, the colimit of $D_{\mathscr{J} \to \mathscr{C}}$ is then the initial object of $(Cocone_{D_{\mathscr{J} \to \mathscr{C}}})$. Likewise, a limit of $D_{\mathscr{J} \to \mathscr{C}}$ is the terminal object of $(Cone_{D_{\mathscr{J} \to \mathscr{C}}})$, defined analogously. It could be seen from this that there typically are various ways to define categorical notions, and that justifying to oneself that these different ways agree is a worthwhile exercise in building intuition for understanding these constructions.

We move on to introducing coequalisers and equalisers, which are examples of colimits and limits respectively. The notion of a coequaliser generalises 'the quotient of a set by an equivalence relation', whereas the notion of an equaliser generalises 'the difference kernel'.

Example 7.5: Let J_{eq} be the directed graph shaped as '$\cdot \rightrightarrows \cdot$', and let $\mathscr{J}_{eq}$ and $D_{\mathscr{J}_{eq} \to \mathscr{C}}$ with image $A \rightrightarrows^f_g B$ be the corresponding free category and diagram in $\mathscr{C}$. Provided that their respective existences hold, the colimit of $D_{\mathscr{J}_{eq} \to \mathscr{C}}$ is known as the **coequaliser** of f and g, whereas its limit is known as the **equaliser** of f and g.

1. In (Top), a coequaliser $colim(X \rightrightarrows^f_g Y)$ is simply the space $Y/f(x) \sim g(x)$, whereas an equaliser $lim(X \rightrightarrows^f_g Y)$ is simply the subspace $B \subset X$ such that $B := \{x \in X | f(x) = g(x)\}$.

2. In (Grp), a coequaliser $colim(A \rightrightarrows^\psi_\phi B)$ is simply the group $Y/N_{(\phi,\psi)}$, where $N_{(\phi,\psi)}$ is the normal subgroup generated by elements of the form $\phi(a)\psi(a)^{-1}$ for $a \in A$; an equaliser $lim(A \rightrightarrows^\psi_\phi B)$ is the subgroup $M_{(\phi,\psi)} \subset A$ consisting of elements whose images under ϕ and ψ are equal. Similar constructions hold for (Ab), $(Ring)$ and (Mod_R).

3. As a special case of above, if ψ is taken to be the zero map, then $lim(A \rightrightarrows_{\phi}^{\psi} B)$ is simply $ker(\phi)$. Likewise, $colim(A \rightrightarrows_{\phi}^{\psi} B)$ is simply $coker(\phi)$.

Our last example is the direct limit. This is the generalisation of the concept of a CW-complex X, which we can think of as a colimit of $D_{(\mathbf{N},\leq)\to(Top)}$, the functor sending $0 \to 1 \to 2 \to ...$ to $X_0 \to X_1 \to X_2 \to ...$ where X_k is the k-th skeleton of X, to arbitrary directed posets and categories (given existence).

Example 7.6: Let $\mathscr{K}$ be a directed partially-ordered set viewed as a category with morphisms given by the partial order, and let $D_{\mathscr{K}\to C}$ be an $\mathscr{K}$-shaped diagram in $\mathscr{C}$. [1] The colimit of $D_{\mathscr{K}\to C}$ is known as its **direct limit**. There is also the analogous concept of an inverse limit, which is the limit of a diagram shaped like $\mathscr{K}^{op}$.

It is hopefully now clear that many 'universal' constructions in mathematics can be phrased in terms of limits and colimits. This is very helpful conceptually, but in order for the notions to be of practical use in a given category we would want an efficient way to quickly guarantee their existence for any shape of diagram if it does hold. The following result shows that we are in luck:

Claim 7.7: A category is cocomplete (i.e. it has all colimits) if it has all coproducts and coequalisers. Likewise, a category is complete if it has all products and equalisers.

Proof: Let $\mathscr{C}$ be a category with all coproducts and coequalisers. Let $\mathscr{J}$ be any index category and $D_{\mathscr{J}\to\mathscr{C}}$ be a $\mathscr{J}$-shaped diagram in $\mathscr{C}$. Consider the coequaliser of maps between coproducts $B_{D_{\mathscr{J}\to\mathscr{C}}} \rightrightarrows_{K_{D_{\mathscr{J}\to\mathscr{C}}}}^{L_{D_{\mathscr{J}\to\mathscr{C}}}} T_{D_{\mathscr{J}\to\mathscr{C}}}$, where $B_{D_{\mathscr{J}\to\mathscr{C}}} := \sqcup_{f\in Mor_{\mathscr{J}}(i,j)} D_{\mathscr{J}\to\mathscr{C}}(i)$ is the coproduct of all the source objects of all the morphisms in the diagram, $T_{D_{\mathscr{J}\to\mathscr{C}}} := \sqcup_{k\in Ob(\mathscr{J})} D_{\mathscr{J}\to\mathscr{C}}(k)$ is the coproduct of all the objects in the diagram, $L_{D_{\mathscr{J}\to\mathscr{C}}}$ sends $x \in D_{\mathscr{J}\to\mathscr{C}}(i)$ to the x in the $D_{\mathscr{J}\to\mathscr{C}}(i)$ component of $T_{D_{\mathscr{J}\to\mathscr{C}}}$, and $K_{D_{\mathscr{J}\to\mathscr{C}}}$ sends $x \in D_{\mathscr{J}\to\mathscr{C}}(i)$ to $D_{\mathscr{J}\to\mathscr{C}}(f)(x)$ in the $D_{\mathscr{J}\to\mathscr{C}}(j)$ component of $T_{D_{\mathscr{J}\to\mathscr{C}}}$.

The verification that $colim(B_{D_{\mathscr{J}\to\mathscr{C}}} \rightrightarrows_{K_{D_{\mathscr{J}\to\mathscr{C}}}}^{L_{D_{\mathscr{J}\to\mathscr{C}}}} T_{D_{\mathscr{J}\to\mathscr{C}}})$ is isomorphic to $colim(D_{\mathscr{J}\to\mathscr{C}})$ involves three steps that will be left as an exercise to the reader:

1. Show that for every $i \in Ob(\mathscr{J})$, there exists a map $\iota_i : D_{\mathscr{J}\to\mathscr{C}}(i) \to colim(B_{D_{\mathscr{J}\to\mathscr{C}}} \rightrightarrows_{K_{D_{\mathscr{J}\to\mathscr{C}}}}^{L_{D_{\mathscr{J}\to\mathscr{C}}}} T_{D_{\mathscr{J}\to\mathscr{C}}})$.
2. Show that for $f \in Mor_{\mathscr{J}}(i,j)$, we have $\iota_j \circ D_{\mathscr{J}\to\mathscr{C}}(f) = \iota_i$.

[1] Miller calls this an $\mathscr{K}$-directed system.

3. Show that if there is an object Y in $\mathscr{C}$ with $q_i : D_{\mathscr{J}\to\mathscr{C}}(i) \to Y$ and $q_j \circ D_{\mathscr{J}\to\mathscr{C}}(f) = q_i$ for any $f \in Mor_{\mathscr{J}}(i,j)$, then there is a unique map $\tilde{q} : colim(B_{D_{\mathscr{J}\to\mathscr{C}}} \rightrightarrows^{L_{D_{\mathscr{J}\to\mathscr{C}}}}_{K_{D_{\mathscr{J}\to\mathscr{C}}}} T_{D_{\mathscr{J}\to\mathscr{C}}}) \to Y$.

The completeness characterisation comes from the fact that if a category has all equalisers and products, then its opposite category has all coequalisers and coproducts, and hence all colimits, so then the original category has all limits.

There are also other possible characterisations of completeness and cocompleteness - for example, one can show that if a category has all coproducts and pushouts then it has all coequalisers, which by what we just proved means that it has all colimits.

8 Natural Transformations and Adjunctions

We have done quite a bit on investigating constructions we can make within a category, but not so much on relating maps between functors and see how they come into the picture. This is about to change in our investigation of natural transformations and adjoint pairs.

The former of these relationships sort of acts analogously to how you would imagine a 'homotopy between functors' to function - which is actually an intuition we could formalise as you are about to see.

Definition 8.1: Let $\mathscr{C}$ and $\mathscr{D}$ be categories, and $F, G : \mathscr{C} \to \mathscr{D}$ be (covariant) functors. A **natural transformation** $\eta : F \to G$ consists of the following data:

1. To each object X in $\mathscr{C}$, η associates a morphism $\eta_X : F(X) \to G(X)$, known as the component of η at X.

2. For every morphism $f \in Mor_{\mathscr{C}}(X,Y)$, we have the commutation of the following diagram [2]:

$$\begin{tikzcd} F(X) \arrow[r, "\eta_X"] \arrow[d, "F(f)"] & G(X) \arrow[d, "G(f)"] \\ F(Y) \arrow[r, "\eta_Y"] & G(Y) \end{tikzcd}$$

[2] If the functors are instead contravariant, then the vertical arrows in the diagram flip. Also, note that a natural transformation can only take place between functors of the same type.

As promised, here is the 'homotopy' definition - prove that the two definitions are equivalent as an exercise. Two things to bear in mind about this analogy: firstly, unlike a homotopy, a natural transformation may have no inverse (so the existence of $\eta : F \to G$ does not mean that there is a natural transformation in the other direction); secondly, the 'unit interval' category only has '$t = 0'$ and '$t = 1'$ states and is not continuously vaying.

Let I be the directed graph '$0 \to 1'$ and let $\mathscr{I}$ be the free category generated by I. Let $\mathscr{C}$, $\mathscr{D}$ be categories, and $F, G : \mathscr{C} \to \mathscr{D}$ be functors. A natural transformation from F to G is a functor $\eta : \mathscr{C} \times \mathscr{I} \to \mathscr{D}$ [3] such that $\eta|_{X\times 0} = F$ and $\eta|_{X\times 1} = G$.

Natural transformations arise everywhere in algebraic topology, mainly as a way to transfer information stored by different algebraic invariants in a compatible way.

Example 8.2: Here are a few examples:

1. In one's first encounter of linear algebra, one is often told that 'a finite-dimensional vector space is naturally isomorphic to its double dual' followed by some mumbling about how this is because 'it does not depend on a choice of basis', as opposed to the isomorphism between a vector space and is dual. What is actually meant by naturality in that phrase, of course, is more rigorously phrased as follows - the map taking a finite-dimensional vector space V to its double dual V^{**} defines a natural transformation between the identity and double dual functors, and such a transformation is known as a natural isomorphism because each component is an isomorphism. On the other hand, recall that because the functor taking a vector space to its dual is contravariant, it is not possible to define a natural transformation from the identity functor - which is covariant, to the dual functor.

   Ok, maybe that wasn't exactly the most satisfying explanation - let us pretend that this isn't just a problem with the terminology we are using and act like a natural transformation from a covariant to a contravariant functor could be defined [4]. In the present case, the commutative diagram to-be-satisfied would be as follows:

[3] In case it is not immediately clear how a product of two categories should be defined, google a definition.

[4] These are known as dinatural transformations for those interested in reading further.

$$\begin{tikzcd} V \arrow[r, "\eta_V"'] \arrow[d, "f"] & V^* \\ W \arrow[r, "\eta_W"] & W^* \arrow[u, "f^T"] \end{tikzcd}$$

But then notice that, for example, by taking f to be the zero map, that then the only way we could get the components to work out was if they were all zero maps, which are definitely not isomorphisms.

2. Another illustrative example is as follows. Let $(Top^*_{2\times})$ denote the category with objects being products of two based topological spaces and morphisms being maps between the respective components preserving base points. Then the functors $F, G : (Top^*_{2\times}) \to (Grp)$, with F sending $(X, Y) \mapsto \pi_n((X, x) \times (Y, y))$ and G sending $(X, Y) \mapsto \pi_n(X, x) \times \pi_n(Y, y)$ are naturally isomorphic via the canonical projection and inclusion maps.

   If treated as above, for example, the fundamental group of the 2-torus T^2, a priori recognised as $S^1_x \times S^1_y$, splits naturally as $\pi_1(S^1_x, x) \times \pi_1(S^1_y, y)$. However, without having defined this extra structure, the splitting of $\pi_1(T^2, t)$ as $\pi_1(S^1, x) \times \pi_1(S^1, y)$ would not be natural.

   The lack of naturality in the second case may seem like another deficiency in our terminology, so let us look at the concrete topological implications of this. Let us consider T^2 as the quotient space R^2/Z^2 and consider the self homeomorphism $f : T \to T$ given by $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, then the decomposition of T^2 as a product $S^1_x \times S^1_y$ where S^1_x is the path $(0,0) \mapsto (0,1)$ and S^1_y is the path $(0,0) \mapsto (1,0)$ would not be preserved. In other words, if we gave $\pi(T^2, (0,0))$ a splitting as $\langle x \rangle \times \langle y \rangle$, where x represented the homotopy class of a one traversal of S^1_x, and similarly for y and S^1_y, then $\pi_*(f)$ would not preserve the splitting as it would have taken x to $x * y$ which fails to preserve the splitting. On the other hand, if we a priori considered T^2 as $S^1_x \times S^1_y$, then any pair of maps $f : S^1_x \to S^1_x, g : S^1_y \times S^1_y$ would have left the splitting intact.

3. We previously left open the question of what morphisms in $(sSet)$ are - recall that this is the category whose objects are functors $\Delta^{op} \to (Set)$. We are now in a position to address this - a morphism $\eta \in Mor_{(sSet)}(F, G)$ is simply a natural transformation from F to G. Notice that we could make an analogous statement about the category of $\mathscr{K}$-directed systems for any directed partially-ordered set $\mathscr{K}$. Even more generally, for any categories $\mathscr{C}$ and $\mathscr{D}$, one could define the category $\mathscr{F}(\mathscr{C}, \mathscr{D})$ whose objects

are functors $\mathscr{C} \to \mathscr{D}$ and whose morphisms are natural transformations between such functors.

The fact that morphisms in $(sSet)$ can be composed in this example gives an illustration of the fact that natural transformations can be **composed horizontally** - namely, if there were a natural transformation η from F to G for $F, G : \mathscr{C} \to \mathscr{D}$ and another natural transformation ξ from J to K for $J, K : \mathscr{D} \to \mathscr{E}$, then the composition of functors would allow for a natural transformation $\xi * \eta$ from $J \circ F$ to $K \circ G$ with components $(\xi * \eta)_X = \xi_{G_X} \circ J\eta_X$. Other types of composition of natural transformations also exist - **vertical composition** (namely, if $\eta : F \to G$, $\xi : G \to H$ are natural transformations with $F, G, H : \mathscr{C} \to \mathscr{D}$, then $\xi \circ \eta : F \to H$ is a natural transformation) and **whiskering** (if $\eta : F \to G$ is a natural transformation of functors $F, G : \mathscr{C} \to \mathscr{D}$ and $H : \mathscr{D} \to \mathscr{E}$ is a functor, then we have a natural transformation $H\eta : H \circ F \to H \circ G$).

We make a quick second observation from this example. We started our discussion with categories, which has objects and morphisms. We can form categories, for example, the category (Cat) of small categories, whose objects are themselves categories and their morphisms functors. We now know that we could form categories such as $(sSet)$ whose objects are functors and whose morphisms are natural transformations. We can then form categories whose objects are natural transformations and morphisms are ... (you get the point). This idea of maps of maps of maps of ... of categories can be extended n-fold to any arbitrary finite n, and even beyond via the notion of **quasi-categories** or **infinity categories**. It turns out that many ideas in algebraic topology can again be given a different perspective from the point of view of this resulting theory of higher categories, and that studying them via this route brings sheds light on new phenomena.

Other important examples of naturality that you will encounter in our course include the abelianisation map, the Hurewicz homomorphism $h : \pi_n(-) \to H_n(-)$, the Brown representability map $F(-) \to Mor_{Hot}(-, C)$ characteristic classes $Bun_G(-) \to H^n(-, A)$, and the Steenrod operations $Sq^k : H^n(-) \to H^{n+k}(-)$ (amongst others - we shall not discuss these at length here as the importance of their naturality will become self-evident when you gain working knowledge of each of these).

Before we leave the subject of natural transformations, I would like to make a mention to a nice result that will occur, albeit slightly infrequently, in our study of algebraic topology; more importantly, the proof of this lemma is an instructive way to become comfortable with the formal manipulation of natural transformations. In the same way we study an unfamiliar group's structure by considering its representations (i. e. its actions on a vector space), the following lemma allows us to study a locally-small category $\mathscr{C}$ via functors from $\mathscr{C}$ into (Set).

Claim 8.3: (Yoneda's Lemma) Let $\mathscr{C}$ be a locally small category (a category in which all collections of morphisms are sets). Then for each object $A \in Ob(\mathscr{C})$, $Mor_{\mathscr{C}}(A, -)$ defines a functor $\mathscr{C} \to (Set)$. Let $F : \mathscr{C} \to (Set)$ be a functor. Then we have a bijection between $Nat(Mor_{\mathscr{C}}(A, -), F)$ and $F(A)$.

Proof Sketch: The verification of the details in this proof will be left as an exercise to the reader - including them here will render this part of the text completely illegible.

The construction of the maps are as follows. Given an element $\alpha \in F(A)$, the association $F(A) \to Nat(Mor_{\mathscr{C}}(A, -), F)$ sends $\alpha \to \eta_\alpha$, where η_α is the natural transformation $Mor_{\mathscr{C}}(A, -) \to F$ whose components $\eta_{\alpha_{Mor_{\mathscr{C}}(A,B)}}$ take a morphism $g : A \to B$ to the image of α under the map of sets $F(g) : F(A) \to F(B)$. The verification of the fact that η_α is indeed a natural transformation will be left as an exercise. Conversely, given a natural transformation $\eta \in Nat(Mor_{\mathscr{C}}(A, -), F)$, the association $Nat(Mor_{\mathscr{C}}(A, -), F) \to F(A)$ sends η to the image $\eta_{Mor_{\mathscr{C}}(A,A)}(id_A)$ of the identity map at A under the component of η at $Mor_{\mathscr{C}}(A, A)$. Try verifying yourself that both associations are injective and surjective, and that they are inverses of each other.

Let us now move on to discussing adjunctions, our second example of relationships between functors. Adjunctions could be thought of as a sort of 'equivalence' between two categories, allowing you to understand certain morphisms within a category via morphisms in another. Here is a formal definition:

Definition 8.4: Let $\mathscr{C}$ and $\mathscr{D}$ be categories. An **adjunction** between $\mathscr{C}$ and $\mathscr{D}$ is a pair of functors $F : \mathscr{D} \to \mathscr{C}$ and $G : \mathscr{C} \to \mathscr{D}$ such that for all objects X in $Ob(\mathscr{C})$ and Y in $Ob(\mathscr{D})$, we have a bijection between $Mor_{\mathscr{C}}(F(Y), X)$ and $Mor_{\mathscr{D}}(X, G(Y))$. In this case, F is said to be **left-adjoint** to G, G is said to be **right-adjoint** to F, and F and G are said to form an **adjoint pair**.

Example 8.5: Here are a few examples:

1. One attribute of adjoint pairs is their ability to allow us to study a morphism set via a morphism set in a different category that is better understood or easier to work with. A good demonstration of this comes from the fact that the free group functor $F : (Set) \to (Group)$, sending Y to the free group generated by Y, is left adjoint to the forgetful functor $G : (Grp) \to (Set)$, sending a group to its underlying set. A completely analogous construction also holds in $(CRing)$.

2. Often when wanting to establish truths regarding vector spaces over $\mathbf{R}$ (and especially in algebraic topology, for $\mathbf{R}$-vector bundles), it turns out that first studying an analogous truth for vector spaces over $\mathbf{C}$ then trying to deduce the real case from the complex case is commonly a viable strategy. A useful fact that enables us to do this is the fact that the complexification functor $(- \otimes_{\mathbf{R}} \mathbf{C} : (Vect_{\mathbf{R}}) \to (Vect_{\mathbf{C}})$ is left-adjoint to the restriction of scalars $(Vect_{\mathbf{C}}) \to (Vect_{\mathbf{R}})$ that 'forgets' multiplication by complex numbers in general and 'remembers' only how to multiply by real numbers. An analogous construction holds for any commutative rings R, S with a homomorphism $\rho : R \to S$ via which S could be regarded as an R-module.

3. Various other algebraic examples abound. For example, the inclusion functor $(Ab) \to (Grp)$ is right adjoint to the abelianisation functor $(Grp) \to (Ab)$. Another instructive example is Frobenius reciprocity from representation theory over the complex numbers - the fact that for a group G and a subgroup $H \subset G$, the induction functor $Ind_H^G : (Rep_H) \to (Rep_G)$ is left adjoint to the restriction functor $Res_H^G : (Rep_G) \to (Rep_H)$.

4. In terms of topological examples, the most important is the suspension and loop space adjunction. Consider the suspension functor $\Sigma : (Hot^*) \to (Hot^*)$, which takes a space $X \mapsto (X \times I)/(X \times \{0\})/(X \times \{1\})$ and a map $f : X \to Y$ to the induced map $\Sigma(f) : \Sigma X \to \Sigma Y$ such that $\Sigma(f) : (x,t) \to (f(x), t)$. This turns out to be left adjoint to the loop space functor, which sends a space X to the space ΩX of loops $S^1 \to X$ equipped with the compact-open topology and a map $f : X \to Y$ to the induced map on loops $\Omega f : \Omega X \to \Omega Y$ such that $\Omega f(\gamma) = f(\gamma)$. This adjunction is pivotal in homotopy theory, in particular the derivation of the much-loved homotopy fibre sequence.

5. The final example of an adjunction we shall discuss is possibly the most important fact in homological algebra - the tensor-hom adjunction. Fix a commutative ring R and a module M over it. Observe that for any module N over R, both $N \otimes_R M$ and $Hom_R(M, N)$ have the structure of modules over R. Furthermore, the tensor functor $(- \otimes_R M) : (Mod_R) \to (Mod_R)$ is left adjoint to the 'hom' functor $Hom_R(M, -) : (Mod_R) \to (Mod_R)$. As we know from homological algebra, the tensor functor is right exact and not left exact, and the hom functor is left exact but not right exact - this turns out to be a demonstration of the more general relationship between (co)limits and adjoint pairs, which we shall now address:

Claim 8.6: With obvious existence conditions satisfied, left adjoints preserve colimits and right adjoints preserve limits.

The proof of this is essentially just a spelling out of definitions, but it ends up being quite a deep result that saves us from having to do loads of these types of verifications (and might even give us unexpected results) within every new category that we come across.

9 Conclusion and Acknowledgements

I hope that you have found this an insightful first introduction to the tools of category theory that are relevant to our course. I would just like to mention that category theory is enormous and still-growing, helping us unify and understand structural similarities across mathematics, so I hope that this first introduction has given you the necessary background and confidence to expand your knowledge of both the subject and its applications.

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